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Parameterized Complexity and Inapproximability of Dominating Set Problem in Chordal and Near Chordal Graphs

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Abstract

In this paper, we study the parameterized complexity of Dominating Set problem in chordal graphs and near chordal graphs. We show the problem is W[2]-hard and cannot be solved in time $n^{o(k)}$ in chordal and s -chordal ($s > 3$) graphs unless W[1]=FPT. In addition, we obtain inapproximability results for computing a minimum dominating set in chordal and near chordal graphs. Our results prove that unless NP=P, the minimum dominating set in a chordal or s -chordal

($s > 3$) graph cannot be approximated within a ratio of $\frac{c}{3} \ln n$ in polynomial time, where n is the number of vertices in the graph and $0 < c < 1$ is the constant from the inapproximability of the minimum dominating set in general graphs. In other words, our results suggest that restricting to chordal or s -chordal graphs can improve the approximation ratio by no more than a factor of 3. We then extend our techniques to find similar results for the Independent Dominating Set problem and the Connected Dominating Set problem in chordal or near chordal graphs.

Keywords

Chordal graphs; Dominating set; Independent dominating set; Parameterized complexity

1 Introduction

Given a graph $G = (V, E)$, a vertex subset $D \subseteq V$ is a *dominating set* in G if every vertex in $V - D$ is connected to at least one vertex in D . A dominating set D is an *independent dominating set* if any two vertices in D are not connected in G . A *connected dominating set* is a dominating set that induces a connected subgraph in G . Dominating sets have practical applications in numerous areas [21] and in practice, it is often desirable to find dominating sets with sizes as small as possible. However, computing a minimum dominating set in a given graph has been shown to be NP-hard [13] and it is therefore highly unlikely to develop algorithms that can solve the problem in polynomial time.

Parameterized computation provides a potentially practical solution for some problems that are computationally intractable. In particular, one or a few parameters in some intractable problems can be identified, and parameterized computation studies whether efficient algorithms exist for these problems while all parameters are small. In particular, a parameterized problem may contain a few parameters $k_1, k_2, \dots, k_l$ and the problem is *fixed parameter tractable* if it can be solved in time $O(f(k_1, k_2, \dots, k_l)n^c)$, where f is a function of $k_1, k_2, \dots, k_l$, n is the size of the problem and c is a constant independent of all parameters. For example, the Vertex Cover problem is to determine whether a graph $G = (V, E)$ contains a vertex cover of size at most k or not. The problem is NP-complete. However, a

parameterized algorithm can solve the problem in time $O(1.2852^k + k|V|)$ [5]. In practice, this algorithm can be used to efficiently solve the Vertex Cover problem when the parameter k is fixed and small. On the other hand, some problems do not have known efficient parameterized solutions and are evidently parameterized intractable. Similar to the conventional complexity theory, a hierarchy of complexity classes has been constructed to describe the parameterized complexity of these problems [9]. For example, the Independent Set problem is to decide whether a graph contains an independent set of size k or not and has been shown to be $W[1]$ -hard [9]. It cannot be solved with an efficient parameterized algorithm unless all problems in $W[1]$ are fixed parameter tractable. A thorough investigation on these parameterized complexity classes is provided in [8].

The parameterized Dominating Set problem is to decide whether a graph contains a dominating set of a given size k or not. The problem is known to be $W[2]$ -hard for general graphs [8] and thus evidently cannot be solved with an efficient parameterized algorithm. However, the problem is fixed parameter tractable for graph families that contain certain structure features. For example, the problem is fixed parameter tractable on planar graphs [1], and techniques developed in [1, 3] can be employed to eliminate most of the vertices in a planar graph and the problem can then be solved on a graph whose size is at most a linear function of the parameter k . A graph is *chordal* if it does not contain chordless cycles of length at least 4. A graph is *s-chordal* if it contains at least one chordless cycle of length s and does not contain a chordless cycle of length at least $s+1$. A graph is *odd-chordal* if it is connected, not bipartite and does not contain a chordless cycle of an odd length larger than 3. A graph is *near chordal* if it is *s-chordal* or *odd-chordal*. Chordal and near chordal graphs contain rich structure features and these structure features can lead to efficient solutions for some computationally intractable problems. For example, a maximum independent set in a chordal graph can be found in time $O(n^2)$, where n is the number of vertices in the graph. A question therefore arises on whether the structure features of chordal or near chordal graphs can lead to efficient parameterized solutions for some NP-hard problems or not. Our previous work [17] has shown that the Dominating Set problem is $W[1]$ -hard in chordal graphs and cannot be solved in time $n^{o(\sqrt{k})}$ unless $W[1]=FPT$.

Approximation algorithms provide another practical approach to solving computationally intractable problems. Approximation algorithms can efficiently generate solutions that are close to the optimal ones. These solutions are often guaranteed to be within a ratio of the optimal solution and thus can be useful in practice. For example, the Minimum Vertex Cover problem can be approximated by a simple polynomial time algorithm within a ratio of 2. However, many intractable problems cannot be approximated within a certain ratio unless $NP=P$. For example, it has been shown that the Minimum Vertex Cover problem cannot be approximated within 1.36067 unless $NP=P$ [7]. The Minimum Dominating Set problem can be approximated in polynomial time within a ratio of $\ln n$ [16], where n is the number of vertices in the graph. On the other hand, it has been shown that it is NP-hard to approximate the minimum dominating set of a graph within a ratio of $c \ln n$ [20], where c is some constant less than 1.

Regardless of its inapproximability in general graphs, the minimum dominating set problem has a PTAS in planar graphs [2]. Certain structure features of a graph family can thus provide mechanisms that can lead to approximation algorithms with significantly improved approximation ratios. However, recent work [6] [22] also shows that restricting the underlying graph to certain graph families may not necessarily lead to significantly improved approximation ratios for the minimum dominating set problem or its variants. It is therefore natural to ask whether the minimum dominating set problem can be approximated within a constant ratio or even has a PTAS in chordal or near chordal graphs as it is in planar graphs.

Similar to the Dominating Set problem, the Independent Dominating Set and Connected Dominating Set problems are to determine whether a graph contains an independent dominating set or a connected dominating set of a given size k or not. These problems have been shown to be NP-complete for both general graphs and planar graphs [13]. In [12], it is shown that the Independent Dominating Set problem can be solved in polynomial time on chordal graphs. Although both problems are fixed parameter tractable in planar graphs [2, 18], it remains unknown whether efficient parameterized algorithms exist for the Connected Dominating Set problem when the underlying graph is a chordal graph and whether the parameterized Independent Dominating Set problem has efficient solutions on near chordal graphs.

In this paper, we study the parameterized complexity and inapproximability of the dominating set problem in chordal and s -chordal ($s > 3$) graphs. We show that the problem remains W[2]-hard on these graphs and cannot be solved in time $n^{\alpha(k)}$ unless W[1]=FPT. This improves the complexity and lower bound results we have obtained in our previous work [17]. For inapproximability, we show that it is NP-hard to approximate the minimum

dominating set in both chordal and s -chordal ($s \geq 4$) graphs within a ratio of $\frac{c}{3}$, where c is the constant in the inapproximability of minimum dominating set in general graphs. We then study the parameterized complexity and inapproximability of the independent dominating set and connected dominating set problems in chordal, s -chordal ($s > 3$) and odd-chordal graphs. Table 1 summarizes the results we have obtained.

2 Dominating Set in Chordal Graphs

Given a cycle in a graph, a *chord* is an edge in the graph that connects two vertices that are not adjacent in the cycle. A cycle is *chordless* if it does not contain a chord. A graph is a *chordal graph* if it does not contain a chordless cycle. A graph is *s-chordal* if it contains a chordless cycle with length s , but does not contain a chordless cycle with length at least $s + 1$. A graph is *odd-chordal* if it is connected, not bipartite and does not contain a chordless cycle of an odd length larger than 3. Given a chordal graph $G = (V, E)$ and a parameter k , the parameterized dominating set problem is to determine whether graph G contains a dominating set of size k or not.

We show that the Set Cover problem can be parametrically reduced to this problem in polynomial time. Given a parameter k , a set S and $L = \{S_1, S_2, \dots, S_m\}$ is a set of subsets of S , the Set Cover problem is to determine if there exists k subsets $S_{i_1}, S_{i_2}, \dots, S_{i_k} \in L$ such that $S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k} = S$. The parameterized Set Cover problem is known to be W[2]-hard. The following lemma on chordal graphs is needed to construct the reduction from the parameterized Set Cover problem.

Lemma 1

[11] Given a tree T and a set of subtrees $T_1, T_2, T_3, \dots, T_n$, where a subtree is an induced connected subgraph of a tree. An intersection graph H can be constructed as follows: each subtree $T_i, 1 \leq i \leq n$ is represented by a vertex v_i and two vertices v_s and v_t are connected by an edge if the corresponding subtrees T_s and T_t share at least an edge. Then H is a chordal graph.

Based on Lemma 1, we prove the following theorem on the parameterized complexity of the dominating set problem in chordal graphs.

Theorem 1

The parameterized dominating set problem is W[2]-hard in chordal graphs

Proof. We construct a reduction from the Set Cover problem. Given an instance of the Set Cover problem, where $S = \{a_1, a_2, \dots, a_n\}$, $L = \{S_1, S_2, \dots, S_m\}$, and $S_1, S_2, \dots, S_m$ are subsets S , the objective of the problem is to decide whether S can be covered by k subsets from L .

A tree T can be constructed from the set S . The root r of the tree has n children $v_1, \dots, v_n$, one for each element of S . For each child v_i , $3(k+1)$ additional vertices $u_{i,1}, u_{i,2}, \dots, u_{i,k+1}$, $s_{i,1}, s_{i,2}, \dots, s_{i,k+1}$, and $t_{i,1}, t_{i,2}, \dots, t_{i,k+1}$ are created. Vertices $v_i, u_{i,j}$ are connected with a tree edge for each j ($1 \leq j \leq k+1$). Similarly, vertices $v_i, s_{i,j}$ and $s_{i,j}, t_{i,j}$ are connected with tree edges for each j . Figure 1(a) provides an illustration of the tree T constructed from the set S . Vertices $u_{i,j}$'s and $s_{i,j}$'s are *auxiliary vertices* of vertex v_i . Vertices $t_{i,j}$'s are *leaf vertices* of v_i .

Subtrees in T can then be created based on the subsets in L . In particular, for each subset $S_l = \{a_{l_1}, a_{l_2}, \dots, a_{l_p}\}$, T_l is the subtree induced by vertices $v_{l_1}, v_{l_2}, \dots, v_{l_p}$ and their auxiliary vertices in T . Figure 1(b) shows the subtree T_l constructed from subset S_l . These subtrees correspond to the subsets in L and thus are *subset subtrees*. In addition to the m subset subtrees $T_1, T_2, \dots, T_m$, $n(k+1)$ *leaf subtrees* are created. Each leaf subtree is the subtree induced by three vertices $u_{i,j}, s_{i,j}$ and $t_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq k+1$) in T . The intersection graph of all created subtrees is a chordal graph H .

We show that graph H contains a dominating set of size K if and only if S can be covered by k subsets from L . First, we assume that S can be covered by k subsets $S_{q_1}, S_{q_2}, \dots, S_{q_k}$, then the vertices that correspond to subtrees $T_{q_1}, T_{q_2}, \dots, T_{q_k}$ in H form a dominating set of size k since every element in S is covered by at least one of the k subsets.

Now, we assume that H contains a dominating set D of size k . We denote the set of subtrees that correspond to the dominating set D as T_D . We then observe that each vertex v_i in tree T must be covered by at least one subset subtree in T_D since otherwise, the vertices that correspond to the $k+1$ leaf subtrees induced by vertices $u_{i,j}, s_{i,j}$ and $t_{i,j}$ ($1 \leq j \leq k+1$) must be included in D and thus D has a size at least $k+1$, which is contradictory to the assumption that D only contains k vertices. Now, we choose all subtrees in T_D that correspond to a subset in L and the subsets corresponding to these subtrees form a set cover of S with size at most k .

We thus have proved that the Set Cover problem can be reduced to the parameterized dominating set problem in chordal graphs in polynomial time. Since the Set Cover problem is known to be W[2]-hard, the parameterized dominating set problem in chordal graphs is also W[2]-hard.

Based on the proof of Theorem 1, we can obtain a parameterized lower bound of dominating set problem in chordal graphs. In particular, Theorem 5.1 in [4] shows that both the Set Cover problem and the Dominating Set problem cannot be solved in time $n^{\alpha(k)}$ unless W[1]=FPT. We have shown in the proof of theorem 1 that the Set Cover problem can be reduced to the dominating set problem in chordal graphs in polynomial time and the parameter k is not changed. Therefore by Theorem 5.1 of [4], we have the following theorem.

Theorem 2

Unless W[1]=FPT, there does not exist an algorithm that can decide whether a given chordal graph $G=(V, E)$ contains a dominating set of size k in time $n^{\alpha(k)}$.

In [20], it is shown that it is NP-hard to approximate the Minimum Dominating Set problem within a ratio $c \ln n$ in polynomial time, where c is some constant. Based on this inapproximability result, we can obtain the following inapproximability result on finding the minimum dominating set in chordal graphs.

Theorem 3

It is NP-hard to approximate the minimum dominating set in a chordal graph $G=(V, E)$

within a ratio $\frac{c}{3} \ln |V|$ in polynomial time, where c is the constant in the inapproximability result of the Minimum Dominating Set problem.

Proof. The Minimum Dominating set problem can be reduced to an instance of the Minimum Set cover problem where the number of subsets in L equals the number of elements in the set S . We denote this number by n , which is also the number of vertices in the graph of the Minimum Dominating Set problem. We then slightly change the reduction we have constructed in the proof of Theorem 1 to show the inapproximability of finding a minimum dominating set in a chordal graph.

In the tree T we have constructed in the proof of Theorem 1, each node v_i is associated with $2(k+1)$ auxiliary vertices and $k+1$ tree leaves. We replace the parameter k by $c \ln n$, where c is the constant in the inapproximability result of the Minimum Dominating Set problem. In other words, instead of having $3(k+1)$ additional vertices attached to each node v_i , $3n(c \ln n + 1)$ additional vertices are created for each v_i , where $2n(c \ln n + 1)$ of them are auxiliary vertices and the remaining $n(c \ln n + 1)$ vertices are tree leaves.

The resulting chordal graph H is the intersection graph of $n + n^2(c \ln n + 1)$ subtrees in T thus it contains $n + n^2(c \ln n + 1)$ vertices. We assume set S can be covered by a minimum of $OPT(I)$ subsets from L . Based on the same arguments in the proof of Theorem 1, the minimum dominating set in graph H contains at most $OPT(I)$ vertices.

We now assume that there exists an algorithm that can approximate the minimum

dominating set in a chordal graph $G=(V, E)$ within the ratio $\frac{c}{3} \ln |V|$ in polynomial time. We apply this algorithm to graph H and obtain an approximate solution A . We show that each element node in T must be covered by at least one subset in solution A , since if it is not the case, A contains at least $n(c \ln n + 1)$ vertices. The approximation ratio is thus at least

$\frac{n(c \ln n + 1)}{OPT(I)} \geq c \ln n + 1$, since $m = OPT(I)$. This is contradictory to the assumption that the

algorithm can approximate the optimal solution within a ratio of $\frac{c}{3} \ln |V|$, since

$$|V|=n+n^2(c \ln n+1) \quad (1)$$

$$< n^3 \quad (2)$$

for sufficiently large n .

We then consider all subset vertices in the solution A . The subsets corresponding to these vertices form a set cover of S , and we take $APX(I)$ to be the number of such subsets in A .

Since the algorithm can approximate the optimal solution within a ratio $\frac{c}{3} \ln |V|$, we have

$$\frac{APX(I)}{OPT(I)} \leq \frac{c}{3} \ln |V| \quad (3)$$

$$< \frac{c}{3} \ln n^3 \quad (4)$$

$$= c \ln n \quad (5)$$

This shows that $APX(I)$ is an approximate solution within a ratio of $c \ln n$. This shows that based on this algorithm, we can approximate the Minimum Dominating set problem within a ratio of $c \ln n$. However, such an algorithm does not exist unless $NP=P$. The theorem thus has been proved.

In the chordal graph H we have constructed to prove Theorems 1, we have shown that if H has a dominating set of size k , the vertices in this dominating set induce a clique in H since each of them corresponds to a subtree that represents a subset. The same argument can also be applied to the chordal graph constructed in the proof of Theorem 3. In addition, we can slightly change the chordal graph H into a s -chordal ($s > 3$) graph H' and extend our results on chordal graphs to s -chordal graphs. In particular, we can create a chordless cycle with s additional vertices and connect each of the s vertices with all subset vertices in H . It is not difficult to see that H' is a s -chordal graph and has a connected dominating set of size k if and only if S can be covered by k subsets. We thus can immediately obtain the following corollaries.

Corollary 1

The parameterized dominating set problem is $W[2]$ -hard in s -chordal graphs and cannot be solved in time $n^{\alpha(k)}$ unless $W[1]=FPT$. It is NP-hard to approximate a minimum dominating set in an s -chordal graph $G = (V, E)$ within a ratio of $\frac{c}{3} \ln |V|$, where c is the constant in the inapproximability result of the Minimum Dominating Set problem.

Corollary 2

The parameterized connected dominating set problem is $W[2]$ -hard in chordal and s -chordal graphs and cannot be solved in time $n^{\alpha(k)}$ unless $W[1]=FPT$. It is NP-hard to approximate a minimum connected dominating set in a chordal or s -chordal graph $G = (V, E)$ within a ratio of $\frac{c}{3} \ln |V|$, where c is the constant in the inapproximability result of the Minimum Dominating Set problem.

3 Independent Dominating Set in s -Chordal Graphs

We start this section with the parameterized independent dominating set problem in 4-chordal graphs. Given a 4-chordal graph $G = (V, E)$ and a parameter k , the objective of the problem is to determine whether G contains an independent dominating set of size k or not. We construct a reduction from the Dominating Set problem and show that this problem is $W[2]$ -hard.

Theorem 4

The parameterized independent dominating set problem is $W[2]$ -hard in 4-chordal graphs

Proof. We construct a reduction from the Dominating Set problem. Given a graph $G = (V, E)$ and a parameter k , we construct a 4-chordal graph H from graph G and show that H contains an independent dominating set of size k if and only if graph G contains a dominating set of size k .

We construct a *selector gadget* to select a vertex for the dominating set of the graph G . The gadget contains $|V|$ *dominating vertices* such that each dominating vertex corresponds to a vertex in the graph G . All dominating vertices in the gadget are connected into a clique. The gadget also contains $k + 1$ *individual vertices*. Each dominating vertex is connected to all individual vertices in the gadget. Figure 2 (a) shows a selector gadget. The graph H contains k selector gadgets in total.

We then generate $|V|$ *dominated vertices* in H . Each dominated vertex also corresponds to a vertex in G . All dominated vertices are connected into a clique. For each selector gadget in H , a dominating vertex in the gadget is connected to a dominated vertex if and only if the two vertices correspond to the same vertex or have an edge between them in G . Figure 2 (b) shows an example of the graph H .

We show that graph H contains a chordless 4-cycle. In particular, we may assume that G contains two vertices u and v that are not connected and both have a degree at least 1, since the Dominating Set problem remains W[2]-hard for graphs that contain 2 or more than 2 connected components. We then arbitrarily choose a selector gadget in H and there are two dominating vertices u' and v' that correspond to u and v in G in the selector gadget. Since both u and v have at least one neighbor, we can assume s and t are the neighbors of u and v in G respectively and u'' and v'' are the dominated vertices that correspond to s and t in H . u', v', u'' and v'' form a chordless 4-cycle since otherwise, both u' and v' are connected to one of u'' or v'' , which indicates that u and v are connected through s or t . This, however, contradicts with the assumption that u and v are not connected in G .

We then show that H contains no chordless cycle with length at least 5. First, a chordless cycle cannot contain an individual vertex since all dominating vertices in a selector gadget are connected into a clique. Second, the dominating vertices in a chordless cycle must be in the same selector gadget since otherwise at least two dominated vertices must be present and nonadjacent in the cycle. However, since all dominated vertices are connected into a clique, the cycle cannot be chordless. Based on the above two observations, a chordless cycle contains at most 2 dominated vertices and 2 dominating vertices and therefore cannot have a length larger than 4.

Now, we show that G has a dominating set of size k if and only if H contains an independent dominating set of size k . First, if G contains a dominating set of size k , we can assign each vertex in this dominating set to a selector gadget in H . Each selector gadget is assigned one and only one vertex. For each selector gadget in H , we can choose the dominating vertex that corresponds to the vertex assigned to it and it is not difficult to see that these k dominating vertices form an independent dominating set of size k in H . On the other hand, if H contains an independent dominating set D of size k , the k vertices must come from the dominating vertices of k different selector gadgets and each selector gadget contains exactly one of them. Since otherwise, the $k+1$ individual vertices in one selector gadget must all be included in the independent dominating set, which is contradictory to the fact that the set only contains k vertices. Now, the k vertices that correspond to the k dominating vertices in D form a dominating set in G since D does not contain dominated vertices and D is a dominating set in H . This shows that G contains a dominating set of size k if H contains an independent dominating set of size k .

The reduction can be constructed in polynomial time and does not change the value of the parameter. The theorem has been proved.

The reduction in the proof of theorem 4 can immediately lead to the following parameterized lower bound for the independent dominating set problem on 4-chordal graphs.

Theorem 5

Unless $W[1]=FPT$, there does not exist an algorithm that can decide if a 4-chordal graph contains an independent dominating set of size k in time $n^{O(k)}$.

The results in Theorems 4 and 5 can be extended to s -chordal ($s > 4$) graphs. We can slightly change the reduction in the proof of Theorem 4 to show this. In particular, we add a chordless s -cycle formed by s additional vertices. Each of these s additional vertices is connected to all dominating vertices in graph H . The resulting graph H' is a s -chordal graph and it has an independent dominating set of size k if and only if G has a dominating set of size k . We thus have the following corollary.

Corollary 3

The parameterized independent dominating set problem is $W[2]$ -hard for s -chordal ($s \geq 4$) graphs and cannot be solved in time $O(n^{O(k)})$ unless $W[1]=FPT$.

4 Independent Dominating Set in Odd-chordal Graphs

The parameterized complexity and lower bound of the independent dominating set problem in odd-chordal graphs can be immediately obtained from the proof of theorems 4 and 5, since the 4-chordal graph constructed in the proof of theorem 4 is also an odd-chordal graph. We thus have the following corollary

Corollary 4

The parameterized independent dominating set problem is $W[2]$ -hard for odd-chordal graphs and cannot be solved in time $O(n^{O(k)})$ unless $W[1]=FPT$.

We now study the inapproximability of computing a minimum independent dominating set in an odd-chordal graph. In [15], it is shown that unless $NP=P$, the minimum independent dominating set in a graph cannot be approximated within a ratio of $n^{1-\epsilon}$ in polynomial time, where n is the number of vertices in the graph and ϵ is any positive number. Based on this inapproximability result, we show that unless $NP=P$, the minimum independent dominating set in an odd-chordal graph cannot be approximated within a ratio of $n^{\frac{1}{2}-\epsilon}$ in polynomial time.

Theorem 6

It is NP-hard to approximate the minimum independent dominating set in an odd-chordal graph within a ratio of $n^{\frac{1}{2}-\epsilon}$ in polynomial time, where n is the number of vertices in the graph and ϵ is any positive number.

Proof. We construct a reduction from the Minimum Independent Dominating Set problem. Given a graph $G = (V, E)$ and a positive number ϵ , we construct an odd-chordal graph H from graph G . H contains two sides S_1 and S_2 . Each side contains $|V|$ representative vertices and each representative vertex represents a vertex in graph G . An edge is created between two representative vertices if they belong to different sides and they represent two vertices that are connected in G by an edge. For each side of representative vertices, we create

$\frac{|V|(|V| - 1)}{2}$ *exclusive vertices*. Each exclusive vertex is uniquely associated with a pair of representative vertices in the side and is connected to each vertex in the pair by an edge. H thus contains $|V|(|V| - 1)$ exclusive vertices in total. All these exclusive vertices are connected into clique and each exclusive vertex is connected to every representative vertex in the other side by an edge. We then create a vertex r and $2|V|^{2-\epsilon}$ *individual vertices*, where r is connected to every individual or exclusive vertex by an edge. Any pair of individual vertices are not connected in the graph. H contains in total $|V|^2 + |V| + 2|V|^{2-\epsilon} + 1$ vertices. Figure 3(a) and (b) show an example on the construction of graph H from a graph G .

We show that graph H is odd-chordal. It is not difficult to see that H is not bipartite since r and any two exclusive vertices form a cycle of length 3. We now assume H contains a chordless cycle C of odd length larger than 3. Then r is not in C since otherwise C contains at least one chord. C contains at least one exclusive vertex since representative vertices in H induce a bipartite subgraph and a bipartite graph does not contain a cycle of odd length. C cannot contain more than two exclusive vertices since all exclusive vertices are connected into a clique. Therefore, C contains one or two exclusive vertices.

If C contains only one exclusive vertex a , the two representative vertices r_1 and r_2 adjacent to a in C must come from different sides, since otherwise the length of C must be even, which is a contradiction. The remaining part of C is a path that connects r_1 and r_2 and only contains representative vertices. We assume r_1 is in the same side as a , then the path must contain at least one vertex b that is in the same side as that of r_2 . However there is an edge that connects a and b and therefore C contains at least one chord, which is contradictory to the assumption that C is chordless.

If C contains two exclusive vertices c and d , the two must be adjacent in C since otherwise the edge between them would be a chord in C . In addition, c and d must come from different sides since if c and d are in the same side, C must contain a representative vertex from the other side, both c and d are connected to this vertex by an edge, and C therefore must contain a chord, which is contradictory to the fact that C is chordless. Without loss of generality, we assume c and d are from S_1 and S_2 , respectively. The remaining part of C is a path that connects c and d and only contains representative vertices. To avoid creating chords in C , c must be adjacent to a representative vertex in S_2 and d must be adjacent to a representative vertex in S_1 . Since the representative vertices induce a bipartite subgraph in H , the length of this path must be odd, which is contradictory to the assumption that C is a cycle of odd length. Such a C thus does not exist, so H is odd-chordal.

We assume that a minimum independent dominating set D in G contains $OPT(G)$ vertices, graph H then contains an independent dominating set of size $2OPT(G) + 1$. In particular, we can include r and the representative vertices that correspond to the vertices in D in both S_1 and S_2 . It is not difficult to check that these $2OPT(G) + 1$ vertices form an independent dominating set in graph H .

Now, we assume that there exists a polynomial time algorithm that can approximate the minimum independent dominating set in an odd-chordal graph within a ratio of $\frac{1}{n^{\frac{1}{2}-\epsilon}}$. We then apply this algorithm to H and obtain an independent dominating set I . We show that I must contain r since otherwise the $2|V|^{2-\epsilon}$ individual vertices must be in I . Since $OPT(G) \leq |V|$, for sufficiently large V , the approximation ratio R satisfies

$$R \geq \frac{2|V|^{2-\epsilon}}{2OPT(G) + 1} \quad (6)$$

$$\geq \frac{2|V|^{2-\epsilon}}{3OPT(G)} \quad (7)$$

$$\geq |V|^{1-\frac{3}{2}\epsilon} \quad (8)$$

On the other hand, for sufficiently large $|V|$, the approximation ratio R must satisfy

$$R \leq \left(|V|^2 + |V| + 2|V|^{2-\epsilon} + 1\right)^{\frac{1}{2}-\epsilon} \quad (9)$$

$$\leq |V|^{(2+\epsilon)(\frac{1}{2}-\epsilon)} \quad (10)$$

$$\leq |V|^{1-\frac{3}{2}\epsilon-\epsilon^2} \quad (11)$$

$$\leq |V|^{1-\frac{3}{2}\epsilon} \quad (12)$$

which is a contradiction. This shows that I must contain r .

Since I is an independent dominating set and contains r , it does not contain exclusive vertices or individual vertices. The remaining vertices in I must come from S_1 or S_2 . We denote the intersection between I and S_1 by I_1 and that between I and S_2 by I_2 , and D_1 and D_2 are used to denote the corresponding vertex subsets of I_1 and I_2 in G . We show next that there exists a polynomial time algorithm that can compute an independent dominating set of size at most $|I|$ in G from I . We will need the following algorithm to change I , D_1 and D_2 such that $D_1 - D_2$ becomes empty and I remains an independent dominating set.

1. If $(D_1 - D_2)$ is empty, the algorithm returns; otherwise, let $C = (D_1 - D_2)$ and $U = \emptyset$; arbitrarily choose $u \in C$.
2. We denote the corresponding vertex of u in S_1 and S_2 by u_1 and u_2 respectively. I_2 is updated as $I_2 \cup \{u_2\}$ and D_2 is also updated accordingly.
3. We denote the set of all neighbors of u_2 in I_1 by $N(u_1)$. We update I_1 by $I_1 - N(u_1)$ and D_1 is also updated correspondingly. u is then marked as *served*.
4. Check if the removal of $N(u_1)$ from I_1 generates vertices that are not dominated in S_2 . If no such vertices exist, go back to step 1; otherwise, we assume these vertices form a set $U(u_1)$ and update U by $U \cup U(u_1)$;
5. If U is empty, go back to step 1; otherwise arbitrarily choose $v \in U$ and assume v' is the vertex in S_1 that represent the same vertex in G as v . We assume s_2 is a vertex in I_2 that dominates v' . s is the vertex in G represented by s_2 and s_1 is the vertex that represents s in S_1 . U is then updated by removing the vertices dominated by s_2 from it. Let $u = s$ and go back to step 2 (the roles of I_1 and I_2 and operations on them need to be interchanged in this iteration).

The algorithm terminates in polynomial time. In fact, s_1 is not in I_1 since otherwise v is still dominated after the removal of $N(u_1)$, which is a contradiction. Therefore s_1 is not marked as served when it is selected in step 5 since both representatives of a marked vertex are contained in I . This shows that the set of marked vertices is incremented by at least one after an iteration of the algorithm is completed. Since each operation in the algorithm can be

performed in polynomial time, the algorithm terminates in polynomial time. After the algorithm terminates, $D_1 - D_2$ becomes empty and I remains an independent dominating set.

In particular, the algorithm does not add new vertices to $D_2 - D_1$ while eliminating vertices in $D_1 - D_2$. To see this, we only need to notice that none of the vertices in $N(u_1)$ is in I_2 since otherwise I is not independent. We therefore can apply the algorithm twice to change D_1 and D_2 into two equal sets. We also claim that doing this will at most double the size of the independent dominating set since any vertex that is added to D_1 or D_2 by the algorithm is from I . Since D_1 and D_2 are two equal sets after the algorithm terminates, we have $|D_1| = |I|$.

We then use D_1 as an approximate solution for the minimum independent dominating set in G . Since A approximates the minimum independent set in H within a ratio of $n^{\frac{1}{2}-\epsilon}$, the

approximation ratio is at least $\frac{|I|}{2OPT(G)+1}$. For sufficiently large $|V|$, such that $|V|^{\epsilon/2+\epsilon^2} > 3$, we have

$$\frac{|I|}{OPT(G)} < \frac{3|I|}{2OPT(G)+1} \quad (13)$$

$$\leq 3(|V|^2 + |V| + 2|V|^{2-\epsilon} + 1)^{\frac{1}{2}-\epsilon} \quad (14)$$

$$\leq 3|V|^{(2+\epsilon)(\frac{1}{2}-\epsilon)} \quad (15)$$

$$\leq 3|V|^{1-\frac{3}{2}\epsilon-\epsilon^2} \quad (16)$$

$$\leq |V|^{1-\epsilon} \quad (17)$$

This shows that we can approximate the minimum independent dominating set within a ratio less than $|V|^{1-\epsilon}$ in polynomial time, which is contradictory to the known inapproximability result of this problem. The theorem has been proved.

In [19], a similar inapproximability result is obtained for $2 - P_3$ free perfect graphs. It is shown that the minimum independent dominating set problem cannot be approximated within a ratio of $n^{1-\epsilon}$ where ϵ is any positive number unless $P=NP$.

5 Conclusions

In this paper, we study the parameterized complexity and inapproximability of the dominating set problem on chordal and near chordal graphs. Our results show that the structural features of chordal or near chordal graphs cannot lead to efficient parameterized algorithms and significantly improved approximation ratios for this problem. In addition, we extend our techniques to study the parameterized complexity and inapproximability of the independent dominating set and connected dominating set problems in chordal or near chordal graphs. Our results show that, similar to the dominating set problem, having an underlying graph that is chordal or near chordal cannot make them significantly easier to be solved or approximated.

A graph is *perfect* if its chromatic number equals the size of the largest clique it contains. Perfect graphs have rich structural features and these features can lead to polynomial time solutions for many problems that are NP-hard in general graphs [14]. In [14], it is conjectured that an odd-chordal graph is perfect if its complement is also odd-chordal. It is therefore natural to ask if the structural features of odd-chordal graphs can make some NP-hard problems easier to solve or approximate. Our results, however, show that it is not the case for the Independent Dominating Set problem.

Our work also suggests that the independent dominating set problem is inherently different from the dominating set problem. For example, the minimum independent dominating set problem can be solved in polynomial time on chordal graphs. Although we have shown that the problem is W[2]-hard in s -chordal ($s > 3$) graphs, the inapproximability of this problem in these graphs remains open. This is a possible direction for future work.

It remains unknown whether the inapproximability results we have obtained are tight or not. For example, it remains open whether an algorithm exists that can approximate the minimum independent dominating set in an odd-chordal graph within a ratio of $n^{\frac{1}{2}}$ or not. Further investigations are needed to answer this question.

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Fig. 1.

(a) The tree T constructed from an instance of the Set Cover problem. To make the figure clear, only the auxiliary vertices ($u_{1,j}$'s and $s_{1,j}$'s) and leaf vertices ($t_{1,j}$'s) of v_1 are included;

(b) The subtree T_l constructed for a subset S_l that contains elements $a_{l1}, a_{l2}, \dots, a_{lq}$. To make the graph clear, only the auxiliary vertices ($u_{l1,j}$'s and $s_{l1,j}$'s) and leaf vertices ($t_{l1,j}$'s) of vertex v_{l1} are included.

Fig. 2.

(a) A graph G with four vertices a, b, c, d and a selector gadget constructed from G ; (b) The edges that connect the dominating vertices in a selector gadget (a', b', c', d') and the dominated vertices a'', b'', c'', d'' . Although not shown in the figure, a'', b'', c'', d'' are in fact connected into a clique. To make the graph clear, only one selector gadget is included and edges that connect dominating vertices and dominated ones are shown in dashed lines.

Fig. 3.

(a) A graph G , the representative vertices and the edges between them in H ; (b) u and u' are exclusive vertices for vertex pairs (a, b) and (a', b') . To make the graph clear, only two exclusive vertices are shown and edges that connect an exclusive vertex and the representative vertices from a different side are shown in dashed lines.

Table 1

The parameterized complexity, lower bounds and inapproximability results we have obtained in this paper. DS represents the dominating set problem, IDS represents the independent dominating set problem and CDS represents the connected dominating set problem; c is the constant in the inapproximability of minimum dominating set in general graphs.

	parameterized complexity	parameterized lower bound	inapproximability
DS in chordal and s -chordal graphs	W[2]-hard	$n^{O(k)}$	$\frac{c}{3} \ln n$
CDS in chordal and s -chordal graphs	W[2]-hard	$n^{O(k)}$	$\frac{c}{3} \ln n$
IDS in s -chordal graphs	W[2]-hard	$n^{O(k)}$	unknown
IDS in odd-chordal graphs	W[2]-hard	$n^{O(k)}$	$n^{\frac{1}{2}-\epsilon}$