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Calderón's method on an elliptical domain

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Abstract

One possible application for EIT is in medical imaging where lung and heart function may be monitored. One drawback of current algorithms is that they are implemented for use in a circular domain, but a human thorax is more elliptical than circular. In this paper, a reconstruction algorithm based on the work of A. Calderón on the inverse conductivity problem is derived for an elliptical domain. It is explained how this reconstruction algorithm uses a transformed Dirichlet-to-Neumann map. Experimental results from an elliptical tank are given to show how correct domain modelling reduces the artefacts produced by this version of Calderón's reconstruction algorithm.

Keywords

Calderón's method; electrical impedance tomography; reconstruction algorithm; elliptical domain

1. Introduction

Electrical impedance tomography (EIT) is an imaging modality in which electrodes are placed on the surface of a body to apply currents and measure the resulting voltages, or vice-versa. From these measurements, EIT seeks to determine the electrical properties within the body. EIT has many potential applications. One such application is its use as a medical imaging modality. In the medical field, EIT may be used to detect cancerous tumours in the breast (Boverman *et al* 2009) and prostate. EIT may also be used for heart and lung monitoring (Frerichs 2002). Outside of the medical field, EIT may be used for industrial monitoring and geophysical imaging.

Many advances have been made in EIT since Calderón's 1980 paper outlining a simple linearized reconstruction method. More reconstruction methods have been developed including direct and iterative methods. Among the direct methods, there have been studies continuing Calderón's original approach (Bikowski and Mueller 2008 and Isaacson and Isaacson 1989). Non-linear direct methods, in particular the complex geometrical optics methods, have been developed since A. Nachman's uniqueness proof in 1996 (Nachman 1996, Astala *et al* 2009, Isaacson *et al* 2004, Isaacson *et al* 2006 and Murphy and Mueller 2007).

Much of the original work in EIT considers general domains in n -dimensional space. However, practical algorithms and experimental results are frequently shown in two or three

dimensions for simple domains such as circles, rectangles and their three-dimensional counterparts. The main reason for these domain choices is the simplicity they provide in calculations. It is clear in DeAngelo and Mueller (2010) that assuming a circular geometry in the case of using human chest data causes the reconstructions to appear distorted from the expected physiology. Some reconstruction algorithms have recently been developed to account for unknown domain geometries. One such method by Nissinen *et al* (2011a) uses the approximation error approach to reconstruct the conductivity in an incorrect domain by compensating for the errors caused by the unknown domain. In a later paper (2011b), Nissinen *et al* go a step further and reconstruct both the domain boundary and internal conductivity. Another method utilizing Teichmüller maps has been developed in two and three dimensions by Kolehmainen *et al* (2007 and 2008). Experimental results from this theory have been published in two dimensions (Kolehmainen 2008).

The above-mentioned methods require some complicated procedures in order to achieve reasonable reconstructions in non-circular domains. Hemant Jain *et al* (1997) developed an iterative algorithm that accounts for known noncircular boundaries which relies on a least-squares error method. Jain's algorithm, however, requires accurate forward solver results. Murphy and Mueller (2007) investigated the effects of correctly modelling the domain boundary and accounting for its shape in the D-bar method itself. This work in the D-bar method retains the relative simplicity of the D-bar method while drastically improving reconstructions in non-circular domains when compared to the standard D-bar method assuming a circular domain. The D-bar method was implemented by Murphy and Mueller using experimental data from an elliptical tank at Rensselaer Polytechnic Institute (RPI). Murphy and Mueller (2009) continued this work and provided more quantitative results showing the benefit of correctly modelling the domain. Hamilton and Mueller (2013) go further and reconstruct on a thoracic model using similar expansions of functions with respect to the current patterns applied.

In this paper, we apply the concepts of Murphy and Mueller to Calderón's method. A general domain is assumed for the beginning of the derivation of the reconstruction method. The assumption of an elliptical domain is then used to finish the derivation. Our implementation of Calderón's method in an elliptical domain is tested using experimental data obtained from the ACT 4 system at RPI. Our reconstructions are also compared to those obtained using Calderón's method on a circular domain as found in Bikowski and Mueller (2008).

2. Forward model

The goal of EIT is to recover the conductivity inside a body given measurements made on the surface of the body. Let $u(\mathbf{x})$, $\gamma(\mathbf{x})$ and $j(\mathbf{x})$ represent the electric potential in the body, Ω , the conductivity in Ω and the current density applied on the surface of the body, $\partial\Omega$, respectively, where $\mathbf{x} \in \mathbb{R}^n$. Then

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega, \quad (1)$$

$$\gamma \frac{\partial u}{\partial \nu} = j \quad \text{on } \partial\Omega, \quad (2)$$

where ν is the outward-pointing unit normal to $\partial\Omega$. If a current is applied to the surface, then the resulting voltage, $u(\mathbf{x})|_{\partial\Omega}$ will be measured. In addition Isaacson and Isaacson (1989) states that j and $u(\mathbf{x})|_{\partial\Omega}$ must satisfy

$$\int_{\partial\Omega} j(\mathbf{x}) d\sigma = 0 \quad \text{and} \quad \int_{\partial\Omega} u(\mathbf{x}) d\sigma = 0 \quad (3)$$

in order to satisfy compatibility and existence/uniqueness conditions, respectively. These conditions need to be discretized when finitely many electrodes are placed on the surface of the body.

In this paper, we consider the current patterns

$$T_\ell^k = \begin{cases} M \cos(k\theta_\ell), & k=1, \dots, \frac{L}{2} \\ M \sin\left(\left(k - \frac{L}{2}\right)\theta_\ell\right), & k=\frac{L}{2}+1, \dots, L-1. \end{cases} \quad (4)$$

where M is the current amplitude, L is the number of electrodes on the surface of the body and $\theta_\ell = (2\pi/L)\ell$ is the angle of the ℓ th electrode. Let the normalized currents be $\mathbf{t}^k = \mathbf{T}^k /$

$\|\mathbf{T}^k\|_2$, where $\|\mathbf{T}^k\|_2 = \sqrt{\sum_{\ell=1}^L (T_\ell^k)^2}$ as in Isaacson *et al* (2004). Notice that $\mathbf{T}^k$ satisfy $\sum_{\ell=1}^L T_\ell^k = 0$, where T_ℓ^k is the current applied to the ℓ th electrode from the k th pattern, to satisfy the compatibility condition. We equivalently define V_ℓ^k to be the k th voltage on the ℓ th electrode. Again, to satisfy the existence and uniqueness condition, $\mathbf{V}^k$ is normalized so that $\sum_{\ell=1}^L V_\ell^k = 0$. The voltages are then normalized as $\mathbf{v}^k = \mathbf{V}^k / \|\mathbf{T}^k\|_2$

The data we collect when we apply currents and measure voltages can be used to calculate the Neumann-to-Dirichlet (or current-to-voltage) matrix, $R_\gamma^{m,n} = (\mathbf{t}^m, \mathbf{v}^n)_L$, where

$(u(\cdot), v(\cdot))_L \equiv \sum_{\ell=1}^L \bar{u}(\theta_\ell) v(\theta_\ell)$ is the L^2 inner product. However, the algorithm below calls for the Dirichlet-to-Neumann matrix, $\mathbf{L}_\gamma$, which is found easily by taking the inverse of the current density-to-voltage map, R_γ/A . So $\mathbf{L}_\gamma = (R_\gamma/A)^{-1}$, where A is the area of an electrode, which translates to the electrode length in two dimensions.

When considering a finite number of electrodes, there are a few different electrode models that can be used since there will be discrete boundary value information. These models modify (1), (2) and (3). Somersalo *et al* (1992) describe commonly used models. In this paper, the ave-gap model is used.

2.1. Ave-gap model

The ave-gap model is not the most accurate model for predicting a forward solution to the conductivity equation given a known conductivity, but it suffices for the approximations used below. The major flaw of this model is that it does not account for contact impedance of the electrodes. One benefit of the ave-gap model is that it does not alter the original equations above. This model specifies the j that is used. Specifically,

$$j(\mathbf{x}) = \begin{cases} \frac{I_\ell}{A}, & \mathbf{x} \in e_\ell, \quad \ell=1, \dots, L \\ 0 & \mathbf{x} \notin e_\ell, \quad \ell=1, \dots, L. \end{cases} \quad (5)$$

Also, assume

$$V_\ell \equiv \frac{1}{A} \int_{e_\ell} u(\mathbf{x}) d\sigma \quad (6)$$

is the measured voltage, where e_ℓ is the ℓ th electrode with area, A , and applied current I_ℓ

3. Calderón's method on a general domain

Calderón's method is a linear direct method for reconstructing conductivity in the EIT problem. The method assumes that the conductivity is a constant plus a small variation, $\gamma(\mathbf{x}) = \gamma_0 + \delta(\mathbf{x})$. This leads to the following equation derived in Calderón (1980):

$$\begin{aligned}\widehat{\gamma}(z) &= -\frac{1}{2\pi^2|z|^2} \int_{\partial\Omega} w_1 \Lambda_\gamma w_2 d\sigma + R(z) \\ &\equiv \widehat{F}(z) + R(z),\end{aligned}\quad (7)$$

where $\widehat{\gamma}(z)$ is the Fourier transform of the conductivity distribution $\gamma(\mathbf{x})$, $\widehat{F}(z)$ is the first term in (8), γ is the continuous Dirichlet-to-Neumann map for conductivity $\gamma(\mathbf{x})$, $R(z)$ is the remainder after deriving the first two terms and the w_j are unknown solutions to the conductivity equation which will be replaced by Calderón's harmonic solutions below. Calderón (1980) shows that $R(z)$ is negligible when $\delta(\mathbf{x})$ is small, which allows the following approximation to be made:

$$\widehat{\gamma}(z) \approx \widehat{F}(z). \quad (8)$$

Therefore, to reconstruct the conductivity, $\gamma(\mathbf{x})$, one simply needs to take the inverse Fourier transform of both sides of (8).

3.1. Approximating $\widehat{F}(z)$

Calderón assumed two solution functions, $u_1 = \exp\{\pi[\mathbf{a} \cdot \mathbf{x} + \mathbf{i}(\mathbf{z} \cdot \mathbf{x})]\}$ and $u_2 = \exp\{\pi[\mathbf{a} \cdot \mathbf{x} - \mathbf{i}(\mathbf{z} \cdot \mathbf{x})]\}$, where $\mathbf{a}$ and $\mathbf{z}$ satisfy $\mathbf{a} \cdot \mathbf{z} = 0$ and $|\mathbf{a}| = |\mathbf{z}|$. The importance of these functions is that $u_1|_{\Omega} = w_1|_{\Omega}$ and $u_2|_{\Omega} = w_2|_{\Omega}$ and the above integral is only over the boundary of the domain.

Traditionally with Calderón's method, the domain is assumed to be a circle and thus $\mathbf{x}|_{\Omega} = R(\cos(\theta); \sin(\theta))$, where R is the constant radius of the circle. We use the trigonometric basis because the trigonometric current patterns are optimal in the homogeneous circle case (Isaacson 1986). To include more general domains, we assume the domain boundary can be defined by the radius function $r(\theta)$ as was done in Murphy *et al* (2007) for the D-bar method. If the trigonometric basis is still used, $\mathbf{x}|_{\Omega} = r(\theta)(\cos(\theta); \sin(\theta))$. Let $\mathbf{z} = |\mathbf{z}|(\cos(\phi); \sin(\phi))$ and $\mathbf{a} = |\mathbf{a}|(\cos(\phi + \frac{\pi}{2}); \sin(\phi + \frac{\pi}{2}))$, $\mathbf{a}$ and $\mathbf{z}$ satisfy the conditions set forth by Calderón. Now, the exponents of the exponentials can be rewritten with these representations of $\mathbf{x}$, $\mathbf{z}$ and $\mathbf{a}$. First we compute $\mathbf{a} \cdot \mathbf{x}$ and $\mathbf{z} \cdot \mathbf{x}$:

$$\mathbf{a} \cdot \mathbf{x} = |\mathbf{z}|r(\theta) \cos\left(\phi + \frac{\pi}{2} - \theta\right) = |\mathbf{z}|r(\theta) \cos\left(\theta - \left(\phi + \frac{\pi}{2}\right)\right) \quad (9)$$

and

$$\mathbf{z} \cdot \mathbf{x} = |\mathbf{z}|r(\theta) \cos(\phi - \theta) = |\mathbf{z}|r(\theta) \cos(\theta - \phi). \quad (10)$$

Combining (9) and (10) in the exponents of u_1 and u_2 yields

$$\pi(\mathbf{a} \cdot \mathbf{x} \pm \mathbf{i}(\mathbf{z} \cdot \mathbf{x})) = \pi|\mathbf{z}|r(\theta) \text{ie}^{\mp \text{i}(\theta - \phi)}. \quad (11)$$

The goal is to expand $u_1|_{\Omega}$ and $u_2|_{\Omega}$ in terms of the basis functions $e^{\text{i}n\theta}$. Equation (11) allows the following expansion:

$$\begin{aligned}
 u_1|_{\partial\Omega} &= e^{\pi(a \cdot x + i(z \cdot x))}|_{\partial\Omega} = e^{\pi|z|r(\theta)i \exp(-i(\theta-\phi))} \\
 &= \sum_{j=0}^{\infty} [|z|i\pi r(\theta) \exp(-i(\theta-\phi))]^j (j!)^{-1} \\
 &= \sum_{j=0}^{\infty} \tilde{a}_j(z) r^j(\theta) e^{-ij\theta}
 \end{aligned} \quad (12)$$

and

$$u_2|_{\partial\Omega} = \sum_{k=0}^{\infty} \tilde{b}_k(z) r^k(\theta) e^{ik\theta}, \quad (13)$$

where

$$\tilde{a}_j(z) = \frac{(|z|i\pi e^{i\phi})^j}{j!} \quad \text{and} \quad \tilde{b}_k(z) = \frac{(|z|i\pi e^{-i\phi})^k}{k!}. \quad (14)$$

Note that this expansion simplifies to that given in Bikowski and Mueller (2008) when $r(\theta)$ is assumed to be a constant radius, R .

Since the boundary is defined by $\mathbf{x} = r(\theta)(\cos(\theta); \sin(\theta))$, we get $d\sigma = s(\theta)d\theta$ where $s(\theta) = \sqrt{r^2(\theta) + [r'(\theta)]^2}$ is the arclength factor. This yields

$$\widehat{F}(z) = - \left(2\pi^2 |z|^2 \right)^{-1} \int_0^{2\pi} u_1 \Lambda_\gamma u_2 s(\theta) d\theta. \quad (15)$$

For simplicity's sake, we will continue working only on the integral part of $\widehat{F}$, which we will call $I(\mathbf{z})$.

$$\begin{aligned}
 I(z) &= \int_0^{2\pi} u_1 \Lambda_\gamma u_2 s(\theta) d\theta \\
 &= \int_0^{2\pi} \sum_{j=0}^{\infty} \tilde{a}_j(z) r^j(\theta) e^{-ij\theta} \left(\Lambda_\gamma \sum_{k=0}^{\infty} \tilde{b}_k(z) r^k(\theta) e^{ik\theta} \right) s(\theta) d\theta \\
 &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_j(z) \tilde{b}_k(z) \int_0^{2\pi} s(\theta) r^j(\theta) e^{-ij\theta} \Lambda_\gamma (r^k(\theta) e^{ik\theta}) d\theta.
 \end{aligned}$$

If we define the continuous inner product to be $\langle f, g \rangle \equiv \int_0^{2\pi} \bar{f}(\theta) g(\theta) d\theta$, where $\bar{f}$ denotes the complex conjugate, then the above can be rewritten as

$$I(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_j(z) \tilde{b}_k(z) \langle s(\theta) r^j(\theta) e^{-ij\theta}, \Lambda_\gamma (r^k(\theta) e^{ik\theta}) \rangle. \quad (16)$$

We now focus on just the inner product, which will be called $P(\mathbf{z})$, noting that in the continuous case $\Lambda_\gamma = R_\gamma^{-1}$.

$$\begin{aligned}
 P(z) &= \langle s(\theta) r^j(\theta) e^{-ij\theta}, \Lambda_\gamma (r^k(\theta) e^{ik\theta}) \rangle \\
 &= \frac{1}{A} \langle s(\theta) r^j(\theta) e^{-ij\theta}, \left(\frac{R_\gamma}{A} \right)^{-1} (r^k(\theta) e^{ik\theta}) \rangle.
 \end{aligned} \quad (17)$$

Let $\Delta\theta \equiv 2\pi/L$, where L is the number of electrodes. When we discretize, we assume the ave-gap electrode model from above.

$$\begin{aligned}
P(z) &\approx \frac{\Delta\theta}{A} \left(s(\cdot) r^j(\cdot) e^{ij\cdot}, \left(\frac{R_y}{A}\right)^{-1} (r^k(\cdot) e^{ik\cdot}) \right)_L \\
&= \frac{\Delta\theta}{A} \left\{ \left(s(\cdot) r^j(\cdot) \cos(j\cdot), \left(\frac{R_y}{A}\right)^{-1} (r^k(\cdot) \cos(k\cdot)) \right)_L \right. \\
&\quad + i \left(s(\cdot) r^j(\cdot) \cos(j\cdot), \left(\frac{R_y}{A}\right)^{-1} (r^k(\cdot) \sin(k\cdot)) \right)_L \\
&\quad - i \left(s(\cdot) r^j(\cdot) \sin(j\cdot), \left(\frac{R_y}{A}\right)^{-1} (r^k(\cdot) \cos(k\cdot)) \right)_L \\
&\quad \left. + \left(s(\cdot) r^j(\cdot) \sin(j\cdot), \left(\frac{R_y}{A}\right)^{-1} (r^k(\cdot) \sin(k\cdot)) \right)_L \right\}. \quad (18)
\end{aligned}$$

Our goal is to evaluate $\widehat{F}(z) = -\left(2\pi^2|z|^2\right)^{-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_j(z) \tilde{b}_k(z) P(z)$ in terms of the data we collected, which is in terms of the discrete Dirichlet-to-Neumann map, $\mathbf{L}_\gamma$. Thus, we need to expand $P(z)$ in a Fourier series, which may be simplified given certain symmetries of the boundary of the domain.

3.1.1. Elliptical domain—As in Murphy *et al* (2007), we will define the boundary of an ellipse by $r(\theta) = ab[a^2 \sin^2(\theta) + b^2 \cos^2(\theta)]^{-1/2}$, where a is the semi-major(minor) axis length along the x-axis and b is the semi-minor(major) axis length along the y-axis. Before proceeding to expanding the terms of (18) we note that $r(\theta)$ and $s(\theta)$ are even functions. We use this fact to eliminate some of the coefficients in the Fourier series expansions. The constant terms in the expansions are all zero due to the existence and uniqueness of solutions. Using these simplifications we get series representations of $s(\theta)r^j(\theta) \cos(j\theta)$, $s(\theta)r^j(\theta) \sin(j\theta)$, $r^k(\theta) \cos(k\theta)$ and $r^k(\theta) \sin(k\theta)$ which can be found in Appendix A.

Now we will expand each of the terms in (18). The idea is to replace each function in the inner products with its Fourier series expansion. The full work will be shown in Appendix B. Combining (15), (16) and the expansions of (18), we get an approximation for $\widehat{F}(z)$ in terms of our measured data.

$$\begin{aligned}
\widehat{F}(z) = & -\frac{L\Delta\theta}{4\pi^2|z|^2A} \left\{ \sum_{j=1}^{\frac{L}{2}-1} \sum_{k=1}^{\frac{L}{2}-1} \tilde{a}_j(z) \tilde{b}_k(z) \left[\widehat{\mathbf{L}}_\gamma^{j,k} + \widehat{\mathbf{L}}_\gamma^{j+\frac{L}{2},k+\frac{L}{2}} + i \left(\widehat{\mathbf{L}}_\gamma^{j,k+\frac{L}{2}} - \widehat{\mathbf{L}}_\gamma^{j+\frac{L}{2},k} \right) \right] \right. \\
& + \sqrt{2} \sum_{k=1}^{\frac{L}{2}-1} \tilde{a}_{\frac{L}{2}}(z) \tilde{b}_k(z) \left(\widehat{\mathbf{L}}_\gamma^{\frac{L}{2},k} + i \widehat{\mathbf{L}}_\gamma^{\frac{L}{2},k+\frac{L}{2}} \right) + \sqrt{2} \sum_{j=1}^{\frac{L}{2}-1} \tilde{a}_j(z) \tilde{b}_{\frac{L}{2}}(z) \left(\widehat{\mathbf{L}}_\gamma^{j,\frac{L}{2}} - i \widehat{\mathbf{L}}_\gamma^{j+\frac{L}{2},\frac{L}{2}} \right) \\
& \left. + 2\tilde{a}_{\frac{L}{2}}(z) \tilde{b}_{\frac{L}{2}}(z) \widehat{\mathbf{L}}_\gamma^{\frac{L}{2},\frac{L}{2}} \right\}. \quad (19)
\end{aligned}$$

where

$$\widehat{\mathbf{L}}_\gamma \equiv \begin{bmatrix} \mathbf{A}\mathbf{L}_\gamma^{1:\frac{L}{2},1:\frac{L}{2}} \mathbf{C} & \mathbf{A}\mathbf{L}_\gamma^{1:\frac{L}{2},\frac{L}{2}+1:L-1} \mathbf{D} \\ \mathbf{B}\mathbf{L}_\gamma^{\frac{L}{2}+1:L-1,1:\frac{L}{2}} \mathbf{C} & \mathbf{B}\mathbf{L}_\gamma^{\frac{L}{2}+1:L-1,\frac{L}{2}+1:L-1} \mathbf{D} \end{bmatrix} = [\mathbf{A} \quad \mathbf{B}] \mathbf{L}_\gamma \begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix}$$

and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have elements $A_{j,k}$, $B_{j,k}$, $C_{j,k}$ and $D_{j,k}$ respectively. Again note that the summations over j and k are truncated based on the number of sin/cosine patterns applied.

Now that $\widehat{F}(z)$ can be written in terms of the measured data, the inverse Fourier transform will yield an approximation to the conductivity. Before taking the transform, Calderón (1980) also states that

$$\widehat{\gamma}(z)\widehat{\eta}\left(\frac{z}{\sigma}\right)=\widehat{F}(z)\widehat{\eta}\left(\frac{z}{\sigma}\right)+R(z)\widehat{\eta}\left(\frac{z}{\sigma}\right)$$

will yield a good approximation when the inverse Fourier transform is taken, where $\widehat{\eta}$ is a sufficiently chosen mollifying function. Bikowski and Mueller (2008) provide such a

mollifier to be $\widehat{\eta}\left(\frac{z}{\sigma}\right)=te^{-\pi|z|^2/t}$, where $\sigma=1/\sqrt{t}$. This is the same mollifier used in this paper, with t used as a regularization parameter.

The conductivity is found in this paper by taking the Fourier inverse of $\widehat{F}(z)\widehat{\eta}\left(\frac{z}{\sigma}\right)$. Since $\widehat{F}(z)$ is written in terms of the polar coordinates of $\mathbf{z}$, the Fourier inverse needs to be taken in the polar coordinate system.

$$\begin{aligned}\gamma(\mathbf{x}) &\approx \mathcal{F}^{-1}\left(\widehat{\gamma}(z)\widehat{\eta}\left(\frac{z}{\sigma}\right)\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{F}(z)\widehat{\eta}\left(\frac{z}{\sigma}\right)e^{-i2\pi\mathbf{x}\cdot\mathbf{z}}dz_1dz_2 \\ &=\int_0^{\infty}\int_0^{2\pi}|z|\widehat{F}(|z|,\phi)\widehat{\eta}\left(\frac{z}{\sigma}\right)e^{-i2\pi\mathbf{x}\cdot\mathbf{z}}d\phi d|z| \\ &\approx \int_0^{T_z}\int_0^{2\pi}|z|\widehat{F}(|z|,\phi)\widehat{\eta}\left(\frac{z}{\sigma}\right)e^{-i2\pi\mathbf{x}\cdot\mathbf{z}}d\phi d|z|\end{aligned}\quad (20)$$

In truncating the integration over the radial variable, a second regularization parameter, T_z , is introduced.

4. Numerical reconstruction method

Images were reconstructed on a 50 by 50 rectangular mesh in the xy-plane. The conductivity is assumed to be element-wise constant, so the Fourier inverse transform of $\widehat{F}(z)\widehat{\eta}(z)$ is approximated in each element using Simpson's method for a function of two variables. The Fourier inversion regularization parameter used was $T_z = 0.65$. The mollifier regularization parameter used was $t = 0.1$. These parameters were chosen to minimize oscillations while still being able to see the target in the centered copper pipe experiment below. The resulting image was then interpolated onto a 200 by 200 grid. This reconstruction procedure follows that prescribed in Bikowski and Mueller (2008).

To reconstruct difference images, first the static images are reconstructed for the tank filled with saline solution for a homogeneous background. Then the static image is reconstructed for an inhomogeneous configuration with target pipes placed in the saline solution. Finally, we subtract the homogeneous static image from the inhomogeneous image to yield the difference image. The benefit of performing this difference imaging is that artefacts caused by the general experimental configuration will be the same across all static images and thus removed in the difference image.

5. Methods

Data was collected using Rensselaer Polytechnic Institute's ACT 4 system operating at 33 kHz. ACT 4 applies a voltage and measures the resulting currents across 32 electrodes. Essentially, ACT 4 makes a two-terminal measurement since voltages are measured on current-carrying electrodes. For the results in this paper, the maximum voltage amplitude was set at 0.5 V. For image reconstruction, we use the measured voltages and currents to generate the voltages that would have been measured had we applied the currents, (4), with amplitude 1 mA.

The targets used were a copper pipe and a PVC pipe, representing a conductor and insulator, respectively. The pipes had diameter 3.3 cm. Since these pipes represent large perturbations from the background conductivity, we also used agar targets with physiologically representative conductivities that are closer to the background conductivity. These agar targets were shaped to represent a two-dimensional view of a heart and lungs. Graphite was added to the agar to produce a complex admittivity, though we only present conductivity results in this paper (Kao *et al* 2008). The heart has conductivity of approximately 0.5 S/m and the lungs have a conductivity of around 0.1 S/m. The elliptical tank was filled with a saline solution to a depth of 2.51 cm. For the pipe experiments, the saline had a conductivity of 0.2 S/m and for the heart and lung experiment, it had a conductivity of 0.225 S/m. The tank has a semi-major axis of 16.5 cm and a semi-minor axis of 11.9 cm. The walls of the tank are lined with 32 electrodes measuring 2.5 cm in width and 4.6 cm in height, extending the full height of the tank. The electrodes, made of silver, are evenly spaced around the tank.

To compare the reconstructions, we introduce the measure of flattening of the reconstructed targets. Define flattening, f , to be

$$f = \frac{a - b}{a}$$

where a is the longer dimension (semi-major axis) and b is the shorter dimension (semi-minor axis). By this definition, the closer f is to zero, the more circular the object is. Our targets are circular, so we expect their reconstructions to have flattening numbers to be zero or close to it. As an example, we find that the flattening of the elliptical tank is $f = 0.279$. To pick out our reconstructed targets, we plot black contours at 75% of the maximum reconstructed value (to identify the copper pipe) and 75% of the minimum reconstructed conductivity (to identify the PVC pipe). Our measurements of a and b were taken from these contours for each target. Note that a similar analysis may be done with the shape deformation parameter defined by Adler *et al* (2009).

6. Experimental results

In this section difference images of targets in an elliptical tank are presented. All values in the reconstructions are in Siemens/meter. We show four different experiments in the tank. These can be seen in figures 1, 3, 5 and 7, and are described in Table 1. Electrode 1 is located at the bottom of the pictures, or on the negative y-axis of the reconstructions, and the rest of the electrodes are numbered counter-clockwise around the tank. For each configuration difference images were reconstructed using Calderón's method. The first image for each configuration shows the reconstruction if the domain had been assumed to be a circle of radius 14.2 cm. This radius was determined by averaging the semi-major and semi-minor axes of the elliptical tank. The second image shows reconstruction assuming the tank's true dimensions. Figures 2, 4, 6, 8 show these difference images. Since difference images are made by subtracting the homogeneous data from the inhomogeneous data, we expect conductors to show up as positive and insulators as negative with a zero background.

In the reconstructions for configuration A (Figure 1), the vertical distortion of the target is visually obvious when the domain is incorrectly assumed to be a circle. The flattening of the target in figure 2(a) is 0.286, while the flattening is 0.083 in figure 2(b). Thus, modelling the domain correctly greatly reduces the distortion of target shape. This vertical distortion can be explained by the fact that in the actual tank, the target is closer to the electrodes through which the minor axis (in this case the y-axis) passes. The data reflects that the target is

closest to those same electrodes regardless of the domain assumed. In the case of assuming a circular domain, this would vertically stretch the target.

For configurations B and C, we find similar results. For configuration B, when we assume a circular domain (figure 4(a)), the flattening of the reconstructed copper pipe is 0.25 and the flattening of the reconstructed PVC pipe is 0.2. When we model the actual elliptical domain (figure 4(b)), the flattening of the copper pipe is 0.214 and that of the PVC pipe is 0.143. For configuration C, in figure 6(a), the flattening of both the copper and PVC pipes is 0.2. In figure 6(b), the flattening of the copper pipe is 0.077 and that of the PVC pipe is 0.2. It is important to note that although the reconstructed targets are not perfectly circular when we correctly model the elliptical domain, they are more circular than when we incorrectly assume a circular domain. The addition of a second target contributes to this flattening in the correctly modelled domain. We can see that the flattening effect of having two targets is increased as the targets get closer to each other. Also of importance is that the flattening of the targets in the assumed circular domain is comparable to the flattening of the elliptical tank. We remarked earlier that the pipes used represented larger perturbations from the background constant and that Calderón's method relies on an assumption of a small perturbation from a constant conductivity. These reconstructions show that despite breaking this assumption, Calderón's method is still able to reconstruct these targets in their relative positions but the magnitudes of the conductivities may be incorrect. For configuration D, with the agar targets, we do not use a flattening analysis since the agar targets are not perfectly circular. It is visually clear that assuming a circular domain, as in figure 8(a), the targets are vertically distorted and their orientation is skewed.

7. Conclusion

It can be seen that correct modelling of the domain in Calderón's method improves the accuracy of reconstructions. A connection is shown between the distortion of the assumed and correct domains and the distortion of the reconstructed targets. Although this algorithm has been demonstrated in an elliptical domain, it may be applied to more general domains, but with the understanding that care must be made when transforming the Dirichlet-to-Neumann matrix since some simplifications were made based on the symmetries of the ellipse. Further studies will need to be done to investigate reconstructing the size of the targets more accurately. Future studies can also include investigating the effects of the domain model on the distortion in more detail.

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Appendix A: Fourier series

Here are the expansions of the functions found in equation (18).

$$\begin{aligned}
s(\theta) r^j(\theta) \cos(j\theta) &= \sum_{m=1}^{\infty} A_{j,m} \cos(m\theta) \\
s(\theta) r^j(\theta) \sin(j\theta) &= \sum_{m=1}^{\infty} B_{j,m} \sin(m\theta) \\
r^k(\theta) \cos(k\theta) &= \sum_{n=1}^{\infty} C_{k,n} \cos(n\theta) \\
r^k(\theta) \sin(k\theta) &= \sum_{n=1}^{\infty} D_{k,n} \sin(n\theta)
\end{aligned}$$

where

$$\begin{aligned}
A_{j,m} &= \frac{1}{\pi} \int_{-\pi}^{\pi} s(\theta) r^j(\theta) \cos(j\theta) \cos(m\theta) d\theta \\
B_{j,m} &= \frac{1}{\pi} \int_{-\pi}^{\pi} s(\theta) r^j(\theta) \sin(j\theta) \sin(m\theta) d\theta \\
C_{k,n} &= \frac{1}{\pi} \int_{-\pi}^{\pi} r^k(\theta) \cos(k\theta) \cos(n\theta) d\theta \\
D_{k,n} &= \frac{1}{\pi} \int_{-\pi}^{\pi} r^k(\theta) \sin(k\theta) \sin(n\theta) d\theta
\end{aligned}$$

Appendix B: Inner product expansions

$$\begin{aligned}
&\left(s(\cdot) r^j(\cdot) \cos(j\cdot), \left(\frac{R_\gamma}{A}\right)^{-1} \left(r^k(\cdot) \cos(k\cdot) \right) \right)_L \\
&= \left(\sum_{m=1}^{\infty} A_{j,m} \cos(m\theta), \left(\frac{R_\gamma}{A}\right)^{-1} \left(\sum_{n=1}^{\infty} C_{k,n} \cos(n\theta) \right) \right)_L \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{j,m} C_{k,n} \left(\cos(m\theta), \left(\frac{R_\gamma}{A}\right)^{-1} \cos(n\theta) \right)_L.
\end{aligned}$$

Due to the number of cosine current patterns applied, we only consider $1 \leq m, n \leq \frac{L}{2}$ and use the definition of the normalized current pattern.

$$\begin{aligned}
&\left(s(\cdot) r^j(\cdot) \cos(j\cdot), \left(\frac{R_\gamma}{A}\right)^{-1} \left(r^k(\cdot) \cos(k\cdot) \right) \right)_L \\
&\approx \sum_{m=1}^{\frac{L}{2}} \sum_{n=1}^{\frac{L}{2}} A_{j,m} C_{k,n} \frac{\|\mathbf{T}^m\|_2 \|\mathbf{T}^n\|_2}{M^2} \left(\mathbf{t}^m, \left(\frac{R_\gamma}{A}\right)^{-1} \mathbf{t}^n \right)_L. \quad (\text{B.1})
\end{aligned}$$

Recall that $\mathbf{L}_\gamma(m, n) \equiv \left(\mathbf{t}^m, \left(\frac{R_\gamma}{A}\right)^{-1} \mathbf{t}^n \right)_L$, then (B.1) can be viewed as

$$\begin{aligned}
&\left(s(\cdot) r^j(\cdot) \cos(j\cdot), \left(\frac{R_\gamma}{A}\right)^{-1} \left(r^k(\cdot) \cos(k\cdot) \right) \right)_L \\
&\approx \sum_{m=1}^{\frac{L}{2}} \sum_{n=1}^{\frac{L}{2}} A_{j,m} C_{k,n} \frac{\|\mathbf{T}^m\|_2 \|\mathbf{T}^n\|_2}{M^2} \mathbf{L}_\gamma(m, n) \quad (\text{B.2}) \\
&= \sum_{m=1}^{\frac{L}{2}} \sum_{n=1}^{\frac{L}{2}} \frac{\|\mathbf{T}^m\|_2}{M} A_{j,m} \mathbf{L}_\gamma^{m,n} \frac{\|\mathbf{T}^n\|_2}{M} C_{k,n}.
\end{aligned}$$

The other inner products are expanded similarly in terms of the discrete Dirichlet-to-Neumann map, $\mathbf{L}_\gamma$.

$$\left(s(\cdot) r^j(\cdot) \cos(j\cdot), \left(\frac{R_y}{A}\right)^{-1} \left(r^k(\cdot) \cos(k\cdot) \right) \right)_L \approx \sum_{m=1}^{\frac{L}{2}-1} \sum_{n=1}^{\frac{L}{2}-1} \frac{\|\mathbf{T}^m\|_2}{M} A_{j,m} \mathbf{L}_\gamma^{m,n+\frac{L}{2}} \frac{\|\mathbf{T}^{n+\frac{L}{2}}\|_2}{M} D_{k,n}. \quad (\text{B.3})$$

$$\left(s(\cdot) r^j(\cdot) \sin(j\cdot), \left(\frac{R_y}{A}\right)^{-1} \left(r^k(\cdot) \cos(k\cdot) \right) \right)_L \approx \sum_{m=1}^{\frac{L}{2}-1} \sum_{n=1}^{\frac{L}{2}} \frac{\|\mathbf{T}^{m+\frac{L}{2}}\|_2}{M} B_{j,m} \mathbf{L}_\gamma^{m+\frac{L}{2},n} \frac{\|\mathbf{T}^n\|_2}{M} C_{k,n}. \quad (\text{B.4})$$

$$\left(s(\cdot) r^j(\cdot) \sin(j\cdot), \left(\frac{R_y}{A}\right)^{-1} \left(r^k(\cdot) \sin(k\cdot) \right) \right)_L \approx \sum_{m=1}^{\frac{L}{2}-1} \sum_{n=1}^{\frac{L}{2}-1} \frac{\|\mathbf{T}^{m+\frac{L}{2}}\|_2}{M} B_{j,m} \mathbf{L}_\gamma^{m+\frac{L}{2},n+\frac{L}{2}} \frac{\|\mathbf{T}^{n+\frac{L}{2}}\|_2}{M} D_{k,n}. \quad (\text{B.5})$$

Recall that we use trigonometric current patterns defined by (4), which means

$$\|\mathbf{T}^k\|_2 = \begin{cases} M \sqrt{\frac{L}{2}}, & k \neq \frac{L}{2} \\ M \sqrt{L}, & k = \frac{L}{2}. \end{cases}$$

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Figure 1.
Configuration A: a copper pipe in the center of the tank

Figure 2. Reconstructed images for Configuration A: 2(a) assumes a false circular domain while 2(b) assumes the correct elliptical domain. Note that the values are scaled by 10^{-3} in 2(b).

Figure 3.

Configuration B: a copper pipe at electrode 10 and a PVC pipe at electrode 28, both 2.9 cm from the tank wall.

Figure 4. Reconstructed images for Configuration B: 4(a) assumes a false circular domain while 4(b) assumes the correct elliptical domain.

Figure 5.

Configuration C: a copper pipe at electrode 10 and a PVC pipe at electrode 28, both 8.6 cm from the tank wall

Figure 6. Reconstructed images for Configuration C: 6(a) assumes a false circular domain while 6(b) assumes the correct elliptical domain.

Figure 7.

Configuration D: Agar heart and lungs approximating their positions in a human thorax

Figure 8. Reconstructed images for Configuration D: 8(a) assumes a false circular domain while 8(b) assumes the correct elliptical domain.

Table 1

Experimental configuration descriptions

Configuration A	One copper pipe in the center of the tank
Configuration B	One copper pipe 2.9 cm from electrode 10 and one PVC pipe 2.9 cm from electrode 28
Configuration C	One copper pipe 8.6 cm from electrode 10 and one PVC pipe 8.6 cm from electrode 28
Configuration D	Two agar lungs and one agar heart in an anatomical orientation