A generalized 2D pencil beam scaling algorithm for proton dose calculation in heterogeneous slab geometries

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Purpose: Pencil beam algorithms are commonly used for proton therapy dose calculations. Szymanowski and Oelfke [“Two-dimensional pencil beam scaling: An improved proton dose algorithm for heterogeneous media,” *Phys. Med. Biol.* **47**, 3313–3330 (2002)] developed a two-dimensional (2D) scaling algorithm which accurately models the radial pencil beam width as a function of depth in heterogeneous slab geometries using a scaled expression for the radial kernel width in water as a function of depth and kinetic energy. However, an assumption made in the derivation of the technique limits its range of validity to cases where the input expression for the radial kernel width in water is derived from a local scattering power model. The goal of this work is to derive a generalized form of 2D pencil beam scaling that is independent of the scattering power model and appropriate for use with any expression for the radial kernel width in water as a function of depth.

Methods: Using Fermi-Eyges transport theory, the authors derive an expression for the radial pencil beam width in heterogeneous slab geometries which is independent of the proton scattering power and related quantities. The authors then perform test calculations in homogeneous and heterogeneous slab phantoms using both the original 2D scaling model and the new model with expressions for the radial kernel width in water computed from both local and nonlocal scattering power models, as well as a nonlocal parameterization of Molière scattering theory. In addition to kernel width calculations, dose calculations are also performed for a narrow Gaussian proton beam.

Results: Pencil beam width calculations indicate that both 2D scaling formalisms perform well when the radial kernel width in water is derived from a local scattering power model. Computing the radial kernel width from a nonlocal scattering model results in the local 2D scaling formula under-predicting the pencil beam width by as much as 1.4 mm (21%) at the depth of the Bragg peak for a 220 MeV proton beam in homogeneous water. This translates into a 32% dose discrepancy for a 5 mm Gaussian proton beam. Similar trends were observed for calculations made in heterogeneous slab phantoms where it was also noted that errors tend to increase with greater beam penetration. The generalized 2D scaling model performs well in all situations, with a maximum dose error of 0.3% at the Bragg peak in a heterogeneous phantom containing 3 cm of hard bone.

Conclusions: The authors have derived a generalized form of 2D pencil beam scaling which is independent of the proton scattering power model and robust to the functional form of the radial kernel width in water used for the calculations. Sample calculations made with this model show excellent agreement with expected values in both homogeneous water and heterogeneous phantoms. © 2013 American Association of Physicists in Medicine. [<http://dx.doi.org/10.1118/1.4804055>]

Key words: 2D scaling, proton dose calculation, heterogeneity corrections, nonlocal scattering power models, local scattering power models

I. INTRODUCTION

Advanced proton therapy techniques, including dynamic beam scanning and intensity modulated proton therapy, require dose calculation algorithms that are both fast and accurate. Accuracy is important because it enables the physician to make informed decisions when evaluating target coverage and/or normal tissue sparing in a given treatment plan. The speed of the calculation is also critical since scanning beam treatments can use thousands of proton beam spots in a single field.¹ Monte Carlo methods are generally preferred for cases with complex (i.e., laterally varying) tissue heterogeneities; however, the time consuming nature of most particle transport calculations makes Monte Carlo techniques unsuitable for routine treatment planning.^{2,3} For this reason, most proton therapy centers continue to rely on analytical methods for dose calculation.

Among the more common analytical techniques for proton dose calculation are variations of the pencil beam algorithm.^{4–10} Conceptually similar to the convolution algorithms used in modern photon therapy treatment planning,^{11,12} pencil beam algorithms model the full proton beam as a two-dimensional (2D) convolution of the proton fluence with a pencil beam dose kernel computed over surfaces normal to the proton beam's initial direction at each depth within the calculation geometry. The pencil beam kernel represents the dose distribution in water created by an elementary pencil beam of unit fluence with kinetic energy corresponding to the beam's full range, and can be calculated using various combinations of physics models and/or measured data.^{13–15}

Because pencil beam algorithms are based on dose kernels computed in water, an important consideration is how to properly account for tissue heterogeneities that occur within the patient. Most pencil beam algorithms use path length scaling with or without proton transport theory to model the effects of material heterogeneities on the treatment beam.^{16,17} Path length scaling (also referred to as 1D scaling) scales the kernel dose distribution according to the water equivalent depth (WED) measured along the kernel's central axis in the heterogeneous geometry. This procedure properly accounts for differences in the proton range (neglecting the effects of range straggling); however, it fails to account for differences in multiple Coulomb scattering (MCS) that occur as the beam traverses materials having different atomic properties.^{16,17}

Accounting for differences in MCS requires the use of proton transport/scattering theories to model the radial width of the pencil beam as a function of depth. Deasy¹⁸ has used Molière scattering theory^{19,20} to compute the radial pencil beam width using the Gaussian approximation suggested by Hanson *et al.*²¹ Alternatively, various authors^{22,23} have used differential approximations to Molière theory with transport equations like those given by Fermi-Eyges theory^{23–25} to compute the pencil beam width as a function of depth. These differential approximations describe the change in the mean squared MCS angle as a function of depth and are commonly referred to as the scattering power.^{23,26,27} Fermi-Eyges theory provides a Gaussian approximation to proton transport and

computes the root-mean-squared (RMS) pencil beam width by integrating the projected change in the mean squared MCS angle as a function of depth. Slab geometry is assumed for the calculation, and any material dependence is taken into account in the scattering power model.

Szymanowski and Oelfke have proposed a method where the material dependence is accounted for by scaling the radial width of the proton dose kernel in water by a material dependent factor derived from Highland's parameterization of Molière theory.¹⁷ Based on Fermi-Eyges theory, they demonstrate that this 2D scaling approach (so called because of the additional radial scaling of the proton dose kernel), allows the RMS pencil beam width in slab geometries to be expressed as a discrete sum of scaled terms computed using an expression for the radial width of the pencil beam kernel in water and its derivatives with respect to depth.

Two-dimensional scaling does not require explicit use of a particular scattering model in the calculation. However, due to an assumption about the functional dependence of the scattering power made in the derivation of the 2D scaling method, the accuracy of the final result is critically dependent on the expression used to describe the radial kernel width in water as a function of depth. In this paper, we examine the effects of this assumption on the existing 2D scaling formalism. We also provide an alternative derivation of 2D scaling that forgoes this assumption and yields a result that is independent of both the scattering power as well as the expression used for the radial pencil beam kernel width in water as a function of depth.

II. 2D SCALING THEORY

While a rigorous derivation of 2D pencil beam scaling is beyond the scope of this paper, some background is necessary for the reader to appreciate this work. Here, we briefly review those parts of the theory that are directly related to the calculation of the RMS pencil beam width both in single and multislabs geometries. For a more detailed treatment of 2D scaling theory, the reader is referred to the original papers by Szymanowski and Oelfke.^{17,28}

II.A. General formalism for single slab geometry

The central tenet of 2D scaling is that the RMS width of a proton pencil beam having initial kinetic energy E_0 , after traversing a physical thickness z of material m can be expressed in terms of the scaled width of a pencil beam kernel in water at the WED, $z_{eq} = z \cdot S_w^m$ (where S_w^m is the ratio of linear stopping powers between material m and water). Mathematically, this can be written as

$$\sigma_m^2(z, E_0) \approx \sigma_w^2(z_{eq}, E_0) \cdot (F_w^m)^2, \quad (1)$$

where $\sigma_m^2(z, E_0)$ is the squared width of the pencil beam in material m at depth z , $\sigma_w^2(z_{eq}, E_0)$ is the squared width of the kernel in water at the WED z_{eq} , and F_w^m is a material dependent constant. For homogenous materials, it has been shown¹⁷

that F_w^m can be computed directly using

$$(F_w^m)^2 = (S_w^m)^3 \frac{L_R^w}{L_R^m} \left[1 + 2C_2 \log \left(S_w^m \frac{L_R^w}{L_R^m} \right) \right], \quad (2)$$

where L_R^m is the radiation length of material m and depends on the physical density and chemical composition of the material, and C_2 is an empirical fit parameter found by Lynch and Dahl²⁹ to be 0.088. The superscript w in this case indicates the material is water.

II.B. Extension to multislab geometry

Equations (1) and (2) provide a straightforward means of scaling the radial kernel width as a function of depth in a homogeneous, single slab. To extend the technique to the more realistic case of mixed slab geometries, Szymanowski and Oelfke employ Fermi-Eyges theory to compute the radial pencil beam width as a function of depth directly.

For the discussion that follows, consider a heterogeneous phantom consisting of k slabs of different materials arranged perpendicular to the z axis of a right-handed Cartesian coordinate system. Let coordinates z_{i-1} and z_i mark the entrance and exit planes respectively of the i th slab, and consider $z_0 = 0$ as the entrance plane of the phantom geometry. In their derivation, Szymanowski and Oelfke begin with the Fermi-Eyges description of the RMS width of a Gaussian pencil beam kernel at depth z in a homogenous material m given by

$$\sigma_m^2(z, E_0) = \int_0^z dx' \frac{d\overline{\theta_m^2}}{dz}(x', E_0)(z - x')^2, \quad (3)$$

where $\overline{\theta_m^2}$ is the mean squared MCS angle and $\frac{d\overline{\theta_m^2}}{dz}$ is the corresponding scattering power. Note that Eq. (3) assumes the mean MCS scattering angle at the surface of the phantom is zero. For a multislab geometry, Eq. (3) is rewritten so that the radial width of the kernel exiting the k th slab at depth z_k is expressed as the summation of contributions from all slabs preceding it,

$$\sigma^2(z_k, E_0) = \sum_{i=1}^k \int_{z_{i-1}}^{z_i} dx' \frac{d\overline{\theta_{m_i}^2}}{dz}(x', E_0)(z_k - x')^2. \quad (4)$$

Next, the substitution $u = x' - z_{i-1}$ is used to rewrite the integral in Eq. (4) as

$$\sigma^2(z_k, E_0) = \sum_{i=1}^k \int_0^{\Delta z_i} du \frac{d\overline{\theta_{m_i}^2}}{dz}(u, E_{i-1})(z_k - z_{i-1} - u)^2, \quad (5)$$

where $\Delta z_i = z_i - z_{i-1}$. Using the scaling relation in Eq. (1), Szymanowski and Oelfke go on to show that the squared contribution to the radial pencil beam width due to the i th slab of material m_i , can be written in terms of the scaled width of a pencil beam kernel in water having kinetic energy E_{i-1} after traversing a slab with equivalent thickness $\Delta z_i \cdot S_w^{m_i}$.

The final result of this derivation is given by

$$\begin{aligned} \sigma^2(z_k, E_0) = & \sum_{i=1}^k (F_w^{m_i})^2 \left[\sigma_w^2(\Delta z_i S_w^{m_i}, E_{i-1}) \right. \\ & + \sigma_w^{2'}(\Delta z_i S_w^{m_i}, E_{i-1})(z_k - z_{i-1}) S_w^{m_i} \\ & \left. + \frac{1}{2} \sigma_w^{2''}(\Delta z_i S_w^{m_i}, E_{i-1})(z_k - z_{i-1})^2 (S_w^{m_i})^2 \right], \end{aligned} \quad (6)$$

where $\sigma_w^2(z, E)$ is the Gaussian pencil beam kernel width in water as a function of depth and kinetic energy, and the primed variables are the first and second derivatives of this quantity with respect to z .

II.C. Limitations of the existing model

At first, Eq. (6) seems quite tractable; the expression does not depend explicitly on the scattering power so that computing the radial pencil beam width as a function of depth in mixed slab geometries requires only an expression for the radial kernel width in water as a function of depth and kinetic energy, along with linear stopping power ratios and radial scaling factors for the different slab materials. Further inspection however reveals that Eq. (6) is not completely independent of the scattering power but rather is implicitly linked to the functional form of this quantity through the substitution used to obtain Eq. (5). Specifically, Eq. (5) assumes the scattering power obeys the relationship

$$\frac{d\overline{\theta_{m_i}^2}}{dz}(z_{i-1} + x, E_0) = \frac{d\overline{\theta_{m_i}^2}}{dz}(x, E_{i-1}), \quad (7)$$

which implies the scattering power at depth depends only on local variables, namely the mean proton kinetic energy at depth z_{i-1} and atomic properties of material m_i . This restriction on the scattering power subsequently affects the expression for the radial kernel width in water through the Fermi-Eyges transport equation [cf. Eq. (3)].

As discussed by Gottschalk²⁶ and Kanematsu,²³ scattering power models generally fall into one of two categories: local models which ignore the effects of single scattering and for which Eq. (7) holds true, and nonlocal models, which account for single scattering effects and consider both the type and thickness of overlying material when computing the mean squared MCS angle at depth. While the functional form of local scattering power models tends to be less complex, integrating these models does not necessarily return the correct mean squared MCS angle. A number of nonlocal models have been developed; however, as we show later in this work, use of nonlocal models to derive $\sigma_w^2(z, E)$ is incompatible with the 2D scaling formalism outlined above. In Sec. II.D, we derive a generalized form of the 2D scaling model that is appropriate for use with local or nonlocal scattering power models, as well as other parameterizations of the radial kernel width in water as a function of depth that may be useful in modeling a particular treatment beam.

II.D. An alternative derivation of 2D scaling in multislabs geometries

As our starting point, consider Eq. (4), which describes the radial width of a Gaussian pencil beam kernel exiting the k th slab at depth z_k , expressed as a summation over contributions from each individual slab. For this derivation as in Ref. 17, we make use of the following identities:

$$\sigma_m^2(z, E) = 2 \int_0^z dz' (z - z') \overline{\theta_m^2}(z', E), \quad (8)$$

$$\sigma_m^{2'}(z, E) \equiv \frac{\partial}{\partial z} \sigma_m^2(z, E) = 2 \int_0^z dz' \overline{\theta_m^2}(z', E), \quad (9)$$

$$\sigma_m^{2''}(z, E) \equiv \frac{\partial^2}{\partial z^2} \sigma_m^2(z, E) = 2 \overline{\theta_m^2}(z, E), \quad (10)$$

which were obtained via manipulation of the Fermi-Eyges transport equation. Rather than using a substitution to shift the coordinates of each slab to the origin, we rewrite Eq. (4) as the difference of two integrals, yielding

$$\sigma^2(z_k, E_0) = \sum_{i=1}^k \left[\int_0^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_k - z')^2 - \int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_k - z')^2 \right]. \quad (11)$$

Integrating by parts yields

$$\sigma^2(z_k, E_0) = \sum_{i=1}^k \left[\overline{\theta_{m_i}^2}(z', E_0) (z_k - z')^2 \Big|_0^{z_i} - \overline{\theta_{m_i}^2}(z', E_0) (z_k - z')^2 \Big|_0^{z_{i-1}} + 2 \int_{z_{i-1}}^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_k - z') \right]. \quad (12)$$

Evaluating the first two terms in this expression and combining the results with Eq. (10) yields

$$\sigma^2(z_k, E_0) = \sum_{i=1}^k \left[\frac{1}{2} \sigma_{m_i}^{2''}(z_i, E_0) (z_k - z_i)^2 - \frac{1}{2} \sigma_{m_i}^{2''}(z_{i-1}, E_0) (z_k - z_{i-1})^2 + 2 \int_{z_{i-1}}^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_k - z') \right]. \quad (13)$$

In the remaining integral in Eq. (13), we add and subtract z_i from the term in parenthesis, which allows this integral to be rewritten as

$$\begin{aligned} & 2 \int_{z_{i-1}}^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_k - z') \\ &= 2(z_k - z_i) \left[\int_0^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) - \int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) \right] \\ &+ 2 \left[\int_0^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_i - z') \right. \\ &\quad \left. - \int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_i - z') \right]. \end{aligned} \quad (14)$$

The first three integral terms on the right hand side (RHS) of this expression can be rewritten using Eqs. (8) and (9),

$$\begin{aligned} & 2 \int_{z_{i-1}}^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_k - z') \\ &= (z_k - z_i) \sigma_{m_i}^{2'}(z_i, E_0) \\ &\quad - (z_k - z_i) \sigma_{m_i}^{2'}(z_{i-1}, E_0) + \sigma_{m_i}^2(z_i, E_0) \\ &\quad - 2 \int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_i - z'). \end{aligned} \quad (15)$$

To evaluate the remaining integral in Eq. (15), we add and subtract z_{i-1} from the integrand before splitting the integral into two, yielding

$$\begin{aligned} & -2 \int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_i - z') \\ &= -2 \left[\int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_{i-1} - z') \right. \\ &\quad \left. + \int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_i - z_{i-1}) \right]. \end{aligned} \quad (16)$$

Again using Eqs. (8) and (9), the above expression can be rewritten as

$$\begin{aligned} & -2 \int_0^{z_{i-1}} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_i - z') \\ &= -\sigma_{m_i}^2(z_{i-1}, E_0) - (z_i - z_{i-1}) \sigma_{m_i}^{2'}(z_{i-1}, E_0). \end{aligned} \quad (17)$$

Substituting the RHS of Eq. (17) back into Eq. (15) and collecting terms results in

$$\begin{aligned} & 2 \int_{z_{i-1}}^{z_i} dz' \overline{\theta_{m_i}^2}(z', E_0) (z_k - z') \\ &= (z_k - z_i) \sigma_{m_i}^{2'}(z_i, E_0) \\ &\quad - (z_k - z_{i-1}) \sigma_{m_i}^{2'}(z_{i-1}, E_0) + \sigma_{m_i}^2(z_i, E_0) \\ &\quad - \sigma_{m_i}^2(z_{i-1}, E_0). \end{aligned} \quad (18)$$

Substituting this expression back into Eq. (13) and applying the scaling relation in Eq. (1) yields the final result,

$$\begin{aligned} \sigma^2(z_k, E_0) &= \sum_{i=1}^k (F_w^{m_i})^2 \left[\sigma_w^2(z_i^{\text{eq}}, E_0) - \sigma_w^2(z_{i-1}^{\text{eq}}, E_0) \right. \\ &\quad + \sigma_w^{2'}(z_i^{\text{eq}}, E_0) (z_k - z_i) S_w^{m_i} \\ &\quad - \sigma_w^{2'}(z_{i-1}^{\text{eq}}, E_0) (z_k - z_{i-1}) S_w^{m_i} \\ &\quad + \frac{1}{2} \sigma_w^{2''}(z_i^{\text{eq}}, E_0) (z_k - z_i)^2 (S_w^{m_i})^2 \\ &\quad \left. - \frac{1}{2} \sigma_w^{2''}(z_{i-1}^{\text{eq}}, E_0) (z_k - z_{i-1})^2 (S_w^{m_i})^2 \right], \end{aligned} \quad (19)$$

where z_i^{eq} and z_{i-1}^{eq} are water equivalent depths corresponding to geometric depths z_i and z_{i-1} respectively.

Equation (19) describes the RMS width of a Gaussian proton pencil beam after traversing a heterogeneous slab geometry expressed in terms of the scaled contributions from each water equivalent slab. Compared to Szymanowski and

Oelfke's original formulation [cf. Eq. (6)], this new formula is expressed in terms of the radial kernel width in water evaluated at the location of the entrance and exit planes of each slab rather than at a depth corresponding to the water equivalent slab thickness. Additionally, Eq. (19) depends on the initial kinetic energy of the proton pencil beam as opposed to the residual energy at each depth. No assumptions about the scattering power model were used in the derivation of Eq. (19). As such, this model places no restrictions on the parameterization used for $\sigma_w^2(z, E_0)$. Finally, in the case of only one slab, Eq. (19) reduces to $(F_w^m)^2 \sigma_w^2(z_k S_w^m, E_0)$, which is the expected result from Eq. (1).

III. MATERIAL AND METHODS

To evaluate the performance of the different 2D scaling formalisms, we first calculated the RMS Gaussian pencil beam width as a function of depth in water by: (1) integrating a local scattering power model using the Fermi-Eyges transport equation, (2) integrating a nonlocal scattering power model using the Fermi-Eyges transport equation, and (3) using the parameterization suggested by Hong *et al.*⁵ for the nonlocal generalized Highland formula. This provided three different curves for $\sigma_w^2(z, E_0)$ which were used to compute the RMS pencil beam width as a function of depth in various geometries using both the local and generalized 2D scaling models. Calculated pencil beam widths were also used with proton depth dose curves obtained from Monte Carlo simulations to compute dose distributions for a Gaussian proton beam with RMS fluence width of 5 mm.

Calculations were performed for proton pencil beams with initial kinetic energies of 100, 160, and 220 MeV, incident on a semi-infinite water phantom composed of 1 mm thick slabs. For the 160 MeV case, calculations were also performed in two heterogeneous slab phantoms. The first contained 3 cm of hard bone starting at a depth of 10 cm while the second contained 3 cm of dry air at the same location. For the heterogeneous calculations, only $\sigma_w^2(z, E_0)$ curves stemming from integration of the local and nonlocal scattering power models were used since the parameterized Highland formula only accounts for tissue heterogeneities through the use of 1D scaling (i.e., without properly accounting for proton transport), which has shown to be inaccurate¹⁷ and would therefore not provide a reliable baseline for comparison. In Secs. III.A–III.D, the methods used to obtain the three expressions for $\sigma_w^2(z, E)$ along with details pertaining to the 2D scaling RMS width and dose calculations are described.

III.A. Local Fermi-Rossi ICRU model: $\sigma_{w-IC}^2(z, E)$

The first scattering power model employed in this work was described by Fermi and Rossi in the early 1940s.³⁰ This model was developed using statistical methods and assumes that broadening of the pencil beam is due only to MCS, i.e., neglecting the effects of single scattering. In their initial derivation, Fermi and Rossi model the nucleus as an unscreened point charge and assume the scattering probability vanishes at very small angles. Rossi however, later modi-

fied the model to better include the effects of nuclear screening by atomic electrons.³¹ The result of this modification is the scattering power formula given for electrons in ICRU Report 35 (Ref. 27) and adapted for radiotherapy protons by Gottschalk.²⁶ The latter formula (referred to as T_{IC} in Ref. 26) is used in this work and is given by

$$\frac{d\theta_{IC}^2}{dz}(z, E_0) = \left(\frac{E_s}{pv}\right)^2 \frac{1}{X_s}, \quad (20)$$

where E_s is a constant equal to 15 MeV, pv is the product of the proton momentum and velocity, and X_s is a scattering length, defined as²⁶

$$\frac{1}{\rho X_s} = \alpha N r_e^2 \frac{Z^2}{A} \{2 \log(33219(AZ)^{-1/3}) - 1\}. \quad (21)$$

In Eq. (21), α is the fine structure constant, N is Avogadro's number, r_e is the classical electron radius, Z is the atomic number, and A the atomic mass of the material being traversed. For water, $X_s = 46.88$ cm. Substituting Eq. (20) into the Fermi-Eyges transport equation allows for the calculation of σ_{w-IC}^2 .

III.B. Nonlocal Molière-Hanson model: $\sigma_{w-MH}^2(z, E)$

As mentioned above, local scattering power models fail to consider the effects of single scattering on the broadening of the proton angular distribution as a function of depth. Accounting for such effects requires a correction that depends on both the type and amount of material overlying the calculation point. Gottschalk²⁶ developed such a correction factor by fitting a linear polynomial in $\log(1 - (pv/p_0 v_0)^2)$ to the ratio of the ideal scattering power (taken to be the numerical derivative of the mean squared MCS angle computed using Molière/Bethe/Hanson theory with respect to depth) to $\frac{d\theta_{IC}^2}{dz}$ calculated as a function of overlying material thickness normalized to the incident proton range for various energies and materials. The result of this fit is given by Eq. (22). Note that p_0 and v_0 in Eq. (22) represent the initial proton momentum and velocity, respectively.

$$\begin{aligned} f_{MH}(pv, p_0 v_0) = & 0.5244 \\ & + 0.1975 \log(1 - (pv/p_0 v_0)^2) \\ & + 0.2320 \log(pv) \\ & - 0.0098 \log(pv) \log(1 - (pv/p_0 v_0)^2). \end{aligned} \quad (22)$$

Applying this correction factor to Eq. (20) gives

$$\frac{d\theta_{MH}^2}{dz}(z, E_0) = f_{MH}(pv, p_0 v_0) \times \left(\frac{E_s}{pv}\right)^2 \frac{1}{X_s}, \quad (23)$$

which can be integrated using the Fermi-Eyges transport equation to obtain σ_{w-MH}^2 . Note that Eq. (23) corresponds to T_{dM} in Ref. 26 and has been described in the context of dose calculation in the master's thesis by Chapman.³²

III.C. Parameterization of the Nonlocal generalized Highland formula: $\sigma_{w-H}^2(z, E)$

The Highland approximation³³ is a fit to the Bethe form of Molière theory that describes the characteristic scattering angle in terms of the radiation length of the target material, L_R^m . Gottschalk extended the theory to thick targets by considering Highland's expression to be the result of scattering interactions taking place in a single infinitesimal slab, then integrating the contributions from multiple slabs in quadrature.³⁴ He also removed a logarithmic term from the integral, treating it instead as a correction factor that depends on the total thickness of the target, normalized to the radiation length. The result is the generalized Highland approximation given by Eq. (24):

$$\theta_H(z, E_0) = \left[14.1 + \frac{14.1}{9} \log_{10} \left(\frac{z}{L_R^m} \right) \right] \times \left[\int_0^z \left(\frac{1}{pv} \right)^2 dx' \right]^{1/2} \frac{L_R^m}{L_R^w}. \quad (24)$$

As pointed out by Gottschalk,²⁶ Eq. (24) is not the result of integrating a scattering power model, rather it stems directly from a fit to Molière theory. To obtain the radial kernel width, the integrand in Eq. (24) is projected onto measuring planes normal to the beam's central axis, giving

$$\sigma_{w-H}^2(z, E_0) = \left[14.1 + \frac{14.1}{9} \log_{10} \left(\frac{z}{L_R^w} \right) \right]^2 \times \int_0^z \left(\frac{z - x'}{pv} \right)^2 \frac{\rho}{L_R^w} dx'. \quad (25)$$

Equation (25) allows for calculation of the radial pencil beam width at the residual energy corresponding to each depth as prescribed by Szymanowski and Oelfke's 2D scaling method; however, for simplicity we have chosen to use the parameterization suggested by Hong *et al.*⁵ which makes use of the fact that the function σ_{w-H}^2 , when normalized to its maximum value and plotted as a function of the fractional range, is a universal curve independent of energy.^{5,34,35} After creating a lookup table of $\sigma_{w-H}^2(z, E_0)/\sigma_{w-H}^2(R, E_0)$ for one proton energy, the absolute kernel width at a given depth can be recovered knowing only the residual range (corresponding to the residual energy at a given starting depth) along with the kernel width at that residual range. Following Hong *et al.*,⁵ we compute the latter using a quadratic fit to a plot of σ_{w-H}^2 vs $R(E)$, the result of which is given by

$$\sigma_{w-H}^2(R) = 0.02275R + 0.12085 \times 10^{-4} R^2. \quad (26)$$

III.D. Calculation details

Calculation of the radial kernel width in water, $\sigma_w^2(z, E)$, depends on the product pv as a function of depth. This quantity was evaluated using the weak Øverås relation²⁶ given by

$$(pv)^2 = (p_0 v_0)^2 \left(1 - \frac{z}{R_0} \right), \quad (27)$$

FIG. 1. Integrated depth dose curves for 100, 160, and 220 MeV proton pencil beams computed with MCNPX. Depth dose curves shown were used as input for pencil beam dose calculations.

where p_0 , v_0 , and R_0 correspond to the initial momentum, velocity, and range of the incident beam. The initial proton range was taken to be the CSDA range obtained from PSTAR,³⁶ while the product of the initial momentum and velocity was computed from the initial kinetic energy using the kinematic relation

$$p_0 v_0 = \frac{\tau + 2}{\tau + 1} E_0, \quad (28)$$

where $\tau = E_0/mc^2$ is the reduced kinetic energy.

To compute the dose for a Gaussian proton pencil beam appropriate for use in a scanning beam treatment, the following model was used

$$D(r, z) = \text{PDD}(z_{\text{eq}}) \cdot \frac{1}{2\pi\sigma_{\text{total}}^2(z)} \exp\left(-\frac{r^2}{2\sigma_{\text{total}}^2(z)}\right), \quad (29)$$

where PDD is the proton depth dose, and $\sigma_{\text{total}}^2 = \sigma_{\text{kernel}}^2 + \sigma_{\text{fluence}}^2$ describes the radial width of the beam and accounts for both the initial width of the Gaussian fluence distribution as well as the broadening of the beam due to scattering. For our calculations, PDD was taken to be the radially integrated depth dose for a proton pencil beam kernel obtained from Monte Carlo simulations performed in MCNPX.^{37,38} The Monte Carlo simulations assumed a monodirectional point source of protons with energy spread equal to 1% of the nominal kinetic energy incident on a semi-infinite water phantom. Depth dose profiles for the 100, 160, and 220 MeV beams used in this work are shown in Fig. 1.

All kernel width and dose calculations were performed using MATLAB version 7.9 (R2009b) (The Mathworks, Natick, MA). Table I lists material properties and computed factors necessary for the calculations. Note that the 2D scaling factors for bone and air listed in the table are different from those given in Ref. 17 because we used the scattering length instead of the radiation length in Eq. (2) to obtain better agreement with the Fermi-Rossi ICRU and Molière-Hanson models.

Trapezoidal integration was used to obtain the radial pencil beam width in water from the Fermi-Eyges transport equation with the two different scattering power models,

TABLE I. Material properties used for calculations.

Material	Density (g/cc)	S_w^m	L_R^m (cm)	X_S (cm)	F_w^m
Water	1.00	1.00	36.08	46.88	1.000
Hard bone	1.85	1.72	16.46	18.19	0.716
Air	1.205×10^{-3}	1.07×10^{-3}	30.4×10^3	38.8×10^3	995.0

as well as to perform the integral found in the generalized Highland formula. The step size for this integration was set to the minimum radiological slab width for each phantom, except when computing the residual energy kernel widths for the local 2D scaling calculations. For these calculations, the step size was decreased by a factor of 50 to avoid stability issues arising from larger variations in the nonlocal scattering models that occur near the phantom surface. This factor was determined empirically by decreasing the step size until no further changes in the computed result were observed. All derivatives required for both 2D scaling models were computed numerically.

IV. RESULTS

IV.A. Homogeneous water phantom calculations

Figure 2 shows RMS kernel widths as a function of depth computed for 100, 160, and 220 MeV proton pencil beams in a homogeneous water slab phantom. The top row shows nominal kernel widths computed using the local Fermi-Rossi ICRU and nonlocal Molière-Hanson scattering powers with Eq. (3), as well as the nonlocal generalized highland formula computed with Eq. (6). Subsequent rows show the accuracy of both 2D scaling formalisms in reproducing each nominal curve when said curve is used as input for the calculations. In all cases, the dashed vertical line indicates the depth corresponding to 97% of the CSDA range in water for the incident energy proton beam. This depth is considered the limit of validity for the multiple scattering theories used in this work and generally corresponds to the depth of the Bragg peak.²⁶

Figure 3 shows normalized lateral dose profiles taken at 97% of the CSDA range for 5 mm Gaussian proton beams computed for the different proton energies using Eq. (29) and the kernel widths shown in Fig. 2. The layout for the subplots is the same as in Fig. 2, with the top row illustrating dose profiles computed using the nominal kernel widths and subsequent rows showing dose profiles resulting from kernel widths computed using the local and general 2D scaling algorithms. For the nominal dose profiles, Fermi-Rossi ICRU maximum doses were used for normalization while for 2D scaling calculations maximum doses from the nominal profiles were used.

From these two figures, we see differences in the nominal kernel widths computed using the three models are small for low and intermediate proton energies (on the order of 0.1 mm). This translates into dose differences in the narrow proton beam example of only a few percent at the Bragg peak for the 160 MeV case. For the highest energy case examined, there is a larger difference observed with the generalized Highland formula at the depth of the Bragg peak, resulting in a 7% dose difference.

Looking now at kernel widths computed using the different 2D scaling algorithms, both 2D scaling methods perform well when the local Fermi-Rossi ICRU model is used to derive the kernel width in water for the calculations, independent of the proton energy. However, for cases where the kernel width in water is derived from the nonlocal Molière-Hanson scattering power or nonlocal generalized Highland model, the local 2D scaling method under-predicts the pencil beam width, resulting in dose differences at the Bragg peak of 3% and 3% respectively for the 100 MeV case, 13% and 14%, respectively, for the 160 MeV case, and 28% and 32%, respectively, for the 220 MeV case, illustrating an increase in the discrepancy with increasing proton energy. The generalized 2D scaling model continues to perform well in all cases, with dose differences on the order of the floating point precision.

IV.B. Heterogeneous slab phantom calculations

Results of RMS kernel width calculations performed for 160 MeV proton pencil beams in the two heterogeneous phantoms are shown in Fig. 4. As before, nominal kernel widths are shown in the top row, while subsequent rows illustrate the performance of the local and generalized 2D scaling algorithms at reproducing the nominal curves. Figures 5 and 6 show normalized dose distributions for a 5 mm Gaussian proton beam in the heterogeneous phantoms using Eq. (29) and the kernel widths shown in Fig. 4. Also shown are normalized dose difference maps computed between the nominal dose distribution and the dose calculated using the local and generalized 2D scaling methods.

Looking first at Fig. 4, a similar pattern is observed in the heterogeneous phantoms as was seen in the homogeneous case. Nominal kernel widths in the heterogeneous phantoms differ only slightly, regardless of whether the presenting heterogeneity is hard bone or air. The local 2D scaling model continues to perform well when the kernel width in water is generated with the local Fermi-Rossi ICRU model; however, local 2D scaling under-predicts the radial kernel width as a function of depth when the nonlocal Molière-Hanson model is used. The observed discrepancy is greater with the air heterogeneity, likely because increased penetration of the beam results in a compounding of errors for a greater number of slabs. This effect is similar to what is seen with the higher energy beams in water. The generalized 2D scaling method continues to perform well in both scenarios.

Looking now at Figs. 5 and 6, differences in radial kernel widths resulting from use of the local 2D scaling model produce maximum dose discrepancies of 9% at the Bragg peak in the bone heterogeneity phantom and 17% at the Bragg peak in the air heterogeneity phantom for the 5 mm Gaussian proton beam. Dose discrepancies observed with the generalized 2D scaling model were small in all cases, on the order of 0.3%.

V. DISCUSSION

In this work we have derived an alternative, generalized 2D scaling algorithm. No assumptions about the locality of

FIG. 2. RMS pencil beam kernel widths as a function of depth computed for 100, 160, and 220 MeV proton beams in a homogeneous water slab phantom. The top row shows nominal kernel widths computed using the local Fermi-Rossi ICRU and nonlocal Molière-Hanson scattering powers as well as the nonlocal generalized Highland formula. Subsequent rows show the accuracy of both 2D scaling algorithms in reproducing the nominal curves when each is used as input for the calculations. The dashed vertical lines indicate the depth corresponding to 97% of the CSDA range in water for the incident energy proton beams.

the scattering model were assumed in the derivation of the new model, and hence it is independent of the scattering process used to derive the primary input parameter, namely the radial pencil beam kernel width as a function of depth in water. As such, this model should be considered for use

with any expression for the radial kernel width in water as a function of depth that matches the user's treatment beam, regardless of its origin. Specifically in this work, we have demonstrated the model's effectiveness when the radial kernel width in water is derived from a local scattering power, a

FIG. 3. Normalized lateral dose profiles taken at the depth of the Bragg peak for 5 mm Gaussian scanning proton beams computed using Eq. (29) and the kernel widths shown in Fig. 2. The layout for the plots is the same as in Fig. 2 with the top row illustrating dose profiles computed using the nominal kernel widths and subsequent rows showing dose profiles resulting from kernel widths computed using the local and general 2D scaling algorithms. Nominal profiles are normalized to the Fermi-Rossi ICRU maximum doses while normalization of the different 2D scaling calculations is to the maximum of the nominal dose profile in each case.

nonlocal scattering power, and a parameterization of the generalized Highland theory.

As interest in proton therapy continues to grow, so too will the need for fast and accurate dose calculation algorithms.

Advances in computing technology may eventually make Monte Carlo techniques practical for routine treatment planning,³⁹ however, at present, pencil beam algorithms remain the primary dose calculation tool for the majority of

FIG. 4. RMS pencil beam kernel widths as a function of depth computed for a 160 MeV proton beam in a water slab phantom containing 3 cm of hard bone (left) or 3 cm of air (right). The top row shows nominal kernel widths computed using the local Fermi-Rossi ICRU and nonlocal Molière-Hanson scattering powers. Subsequent rows show the accuracy of both 2D scaling algorithms in reproducing the nominal curves when each model is used to generate the radial kernel width in water as a function of depth. The dashed vertical lines indicate the locations of the bone or air heterogeneities.

proton therapy centers, as they provide an acceptable balance between speed and accuracy for most clinical situations.

Pencil beam algorithms commonly combine 1D scaling of the kernel dose distribution with proton transport/scattering models to account for tissue heterogeneities in the patient. This is likely adequate for treatment sites such as the prostate and brain where only minor density variations occur near the target.⁸ However, for disease sites with more severe tissue heterogeneities, including the lung, base of skull and sinus cavities or in cases where high density prosthetics or dental fillings are present, greater calculation accuracy may be required to ensure optimal target coverage and/or sparing of normal tis-

sues. In such cases, Monte Carlo methods would be indicated, despite the associated increase in calculation time.

The primary limitation of pencil beam algorithms is their inability to accurately account for laterally varying tissue heterogeneities. This limitation arises from the slab approximation used in proton transport models to describe the broadening of the proton pencil beam kernel with depth. Early attempts to correct for tissue heterogeneities used a 1D scaling of the broad beam dose distribution performed along rays running parallel to the beam's central axis. As described by Schaffner *et al.*,¹⁶ this ray-casting technique results in sharp, nonphysical discontinuities in the dose distribution at the

FIG. 5. (Top row) Normalized nominal dose distributions computed for a 160 MeV, 5 mm Gaussian scanning proton beam in a water phantom containing 3 cm of hard bone. Doses were computed using Eq. (29) with the nominal kernel widths shown in the top left panel of Fig. 4. Below each nominal dose distribution are normalized difference maps computed between the nominal dose and the dose calculated using the local and generalized 2D scaling algorithms. Notice the difference in color scale for the local 2D scaling calculation when the nonlocal Molière-Hanson scattering power model is used to derive the radial kernel width in water.

FIG. 6. (Top row) Normalized nominal dose distributions computed for a 160 MeV, 5 mm Gaussian scanning proton beam in a water phantom containing 3 cm of dry air. Doses were computed using Eq. (29) with the nominal radial kernel widths shown in the top right panel of Fig. 4. Below each nominal dose distribution are normalized difference maps computed between the nominal dose and the dose calculated using the local and generalized 2D scaling algorithms. Notice the difference in color scale for the local 2D scaling calculation when the nonlocal Molière-Hanson scattering power model is used to derive the radial kernel width in water.

edges of laterally varying heterogeneities and can lead to significant dose errors. Modern pencil beam algorithms compute the dose by convolving the proton fluence with individually scaled dose kernels. This tends to blur out the discontinuities seen at the edges of lateral heterogeneities, though as discussed by Schaffner, the blurring is not perfect and dose errors on the order of 10% can persist, particularly when the offending heterogeneity occurs near the Bragg peak.

The 2D scaling formalism suggested by Szymanowski and Oelfke describes a pencil beam algorithm based on Fermi-Eyges transport theory and the generalized Highland scattering model. As such, 2D scaling implicitly assumes slab geometry for the calculations. In their original paper,¹⁷ Szymanowski and Oelfke demonstrate a benefit with 2D scaling in computing dose with laterally varying heterogeneities; however, this improvement was relative to 1D scaling alone (i.e., without the proper use of transport/scattering theory) and thus reflects the possible improvements with a convolution method when the individual kernel widths are accurately represented.

While the physics behind the 2D scaling model is fundamentally the same as that used in more traditional transport/scattering approaches to pencil beam dose calculation, the real difference lies in the fact that 2D scaling uses a parameterization of the dose kernel width as a function of depth in water which, when scaled appropriately, produces the corresponding kernel width as a function of depth in an arbitrary heterogeneous slab geometry. The utility of 2D scaling will depend somewhat on the user's application; however, we see potential benefits when modeling a treatment beam using a fit to empirical or simulated data since the method allows for great flexibility when deciding on the functional form of the primary input parameter, namely the radial kernel width in water as a function of depth (as long as the solution is continuous and at least twice differentiable). The 2D scaling method also has direct application in extending the parameterized Highland model described by Hong *et al.*,⁵ which by default relies on 1D scaling alone to determine the radial kernel width as a function of depth in heterogeneous geometries.

While undoubtedly 2D scaling can be useful for certain applications, the implementation described by Szymanowski and Oelfke is limited to cases where the parameterized kernel width in water describes a local scattering process. As demonstrated in this work, the sensitivity of the local 2D scaling model is such that failure to strictly adhere to this requirement results in discrepancies in the radial kernel width that in turn can lead to large dose errors for narrow proton beam calculations. These discrepancies tend to increase with increasing phantom penetration (due either to an increase in the initial proton kinetic energy or to the presence of lower density materials in the beam) and will vary in magnitude depending on the numerical step size used in the calculation. The former effect can be attributed to the summation in Eq. (6) compounding the kernel width errors in each individual slab calculation, while the latter stems from the fact that the nonlocal corrections used in both the Molière-Hanson and generalized Highland models affect the model strongly near the phantom surface. Because the local 2D scaling model requires integrating

this region of the scattering power model at each residual energy (to find the radial kernel width in water as a function of depth at that residual energy), the calculation result depends strongly on the numerical step-size used for the integration.

VI. CONCLUSIONS

Analytic pencil beam algorithms continue to be a mainstay for clinical proton therapy treatment planning. Szymanowski and Oelfke have presented a proton pencil beam algorithm that uses a two-dimensional scaling of the kernel dose distribution to account for tissue heterogeneities in the patient. Due to a limiting assumption made in their derivation, this method only applies in cases where the radial kernel width in water as a function of depth (the primary input for the calculation) is derived from a local scattering process. Failure to adhere to this requirement results in miscalculation of the radial kernel width in slab geometries and can lead to significant dose errors with narrow proton beams. We have presented an alternative formulation of the 2D scaling method appropriate for use with radial kernel widths in water derived from both local and nonlocal scattering models, as well as other sources including fits to measured or simulated data. Calculations performed with radial kernel widths derived from local and nonlocal scattering power models indicate that our algorithm performs well in both homogeneous and heterogeneous slab geometries.

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