Published in final edited form as:

Ultrason Imaging. 2013 April ; 35(2): 90–108. doi:10.1177/0161734613479246.

Variance and Covariance of Accumulated Displacement Estimates

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Abstract

Tracking large deformations in tissue using ultrasound can enable the reconstruction of nonlinear elastic parameters, but poses a challenge to displacement estimation algorithms. Such large deformations have to be broken up into steps, each of which contributes an estimation error to the final accumulated displacement map. The work reported here measured the error variance for single-step and accumulated displacement estimates using one-dimensional numerical simulations of ultrasound echo signals, subjected to tissue strain and electronic noise. The covariance between accumulation steps was also computed. These simulations show that errors due to electronic noise are negatively correlated between steps, and therefore accumulate slowly, while errors due to tissue deformation are positively correlated and accumulate quickly. For reasonably low electronic noise levels, the error variance in the accumulated displacement estimates is remarkably constant as a function of step size, but increases with the length of the tracking kernel.

Keywords

elastography; displacement estimation; accumulation; multicompression; nonlinear

1 Introduction

Ultrasound elastography uses ultrasound echo signals to track tissue displacement, usually with the aim of imaging tissue stiffness. In many tissues, however, stiffness changes with deformation, and this nonlinear behavior can potentially be used to differentiate tissue types.¹ While obtaining some indicator of stiffness, such as the tissue strain, can be done with very small displacements, estimating the elastic nonlinearity requires much larger deformations. Measurements of these deformations can be combined with a nonlinear elastic model of tissue to solve an inverse problem and reconstruct maps of a nonlinearity parameter,^{2,3} which could be clinically useful.

Tracking the required large deformations, however, poses a problem for displacement estimation. As a tissue undergoes strain, its echo signal will change, or “decorrelate,” so that tracking a portion of signal between image frames becomes more uncertain. One solution to this problem is to break large deformations into a series of smaller steps that are then added together. This strategy decreases the strain-induced error in each individual estimate, but each step also contributes a separate error induced by the electronic noise of the imaging system. The way that these two types of errors accumulate will depend on the number and size of the tracking steps, whether the errors are correlated between steps, and various parameters of the estimation algorithm. The aim of the present work is to characterize these dependencies using a relatively simple simulated model.

The idea of breaking up a large-strain displacement estimation problem into smaller steps arose early in the strain imaging literature. A number of authors have explored it, often under the name of “multicompression.”^{4–8}

O'Donnell et al.,⁸ for example, proposed the accumulation of many small displacement estimates in order to minimize error due to strain-induced decorrelation. They showed simulations of strain estimation error over a single tracking step as a function of applied strain, and used accumulated displacements to create strain images of phantoms. They did not simulate the accumulation process itself, however, and did not try to optimize the size of the tracking step. They also made the common assumption that the displacement estimation errors in one step are uncorrelated with previous and subsequent steps. The present study will show that assumption to be false; the correlation of errors between steps is in fact very important.

Later studies by Varghese, Ophir, and Céspedes^{4,5} analyzed the effects of multicompression more carefully, using the concept of the strain filter⁹ and including the technique of temporal stretching. They built a theoretical model of the strain image SNR based on the Cramér-Rao lower bound for time-delay estimation, as derived by Walker and Trahey,¹⁰ as well as the modified Ziv-Zakai lower bound and the Barankin lower bound described by Weinstein and Weiss.¹¹ The authors concluded that one could obtain better strain SNR by combining multicompression and temporal stretching, and predicted that an optimal strain step increment, in the sense of maximizing SNR, existed at 0.5% strain. This work also had some limitations, though. The authors included only limited simulation support for these results, and like O'Donnell made the assumption that displacement errors were uncorrelated between steps.

Du et al.⁶ performed another study on optimizing the multicompression approach that used theoretical estimates and data from phantoms and *in vivo* breast scans to find an optimal strain increment, in the sense of minimizing displacement variance. Their theoretical considerations, for one particular set of parameters, found 0.35% to be the optimal increment, while heuristic measures of variance on their *in vivo* data led them to recommend a 0.15% increment for tissue. Notably, this study pointed out the importance of step covariance in multicompression techniques. They did not explain the source of the covariance or characterize it in detail, however.

The studies discussed so far have used multicompression to improve strain imaging, but other researchers have used sequences of displacement estimates to demonstrate the elastic nonlinearity of tissues. One method is to form small-strain images at different levels of prior deformation, or to form images of large accumulated strain, and monitor changes in strain contrast. Varghese, Ophir and Krouskop¹² presented a theoretical model and simulated images to show how contrast and CNR of strain images would change if a medium's Young's modulus varied with applied deformation. Pavan et al.,¹³ in demonstrating the construction of a nonlinear elastic ultrasound phantom, created incremental strain images at different levels of pre-deformation as well as accumulated strain images using multicompression. Again, differing nonlinear properties made the strain contrast between the phantom's background and inclusion change as the phantom was deformed.

Sequences of displacements, incremental or accumulated, have also been used to directly reconstruct an elastic modulus or other material parameter. Emelianov et al.¹⁴ acquired an image sequence of an *ex vivo* kidney subjected to a total of 16% average strain, dividing the sequence into steps in order to analyze the organ's nonlinear elastic properties. They then reconstructed Young's modulus images at different levels of pre-deformation and monitored the change. In a similar setup, a later paper by Nitta and Shiina¹⁵ used accumulated

displacements to estimate a nonlinearity parameter from *ex vivo* chicken gizzard and pig kidney. Oberai et al.² accumulated a large-deformation data set over time and reconstructed a different nonlinearity parameter from images of *in vivo* breast lesions. Being focused on the nonlinearity of materials, rather than on motion tracking algorithms, none of these studies explicitly analyzed their displacement accumulation processes.

The present work focuses attention on the displacement accumulation problem, investigated for a single displacement estimation site using one-dimensional simulations subjected to strain and tracked over time up to a total strain of 20%. The variance of accumulated displacement estimates was computed for different strain step sizes, electronic noise levels, and tracking kernel lengths. The results demonstrate that covariance between estimation steps is an important effect, contrary to the assumptions of many previous authors. A one-dimensional simulation model, though simple, was sufficient to show important features of the accumulation process. Though our ultimate objective is to improve the input to reconstructions of elastic nonlinearity of tissues (in 3D), we expect the results will also be useful for the formation of conventional strain images, and will make a contribution toward further understanding of the statistics of motion tracking.

2 Methods

2.1 Simulation

Radiofrequency (RF) ultrasound signals were simulated in small, one-dimensional pieces by convolving a windowed sinusoidal pulse with a random scattering medium. In these simulations the wavelength corresponding to the pulse's center frequency was taken as the fundamental unit of distance, and is denoted by λ . The results may be translated into concrete units of echo time or range by applying appropriate scaling factors, but concrete units are not necessary in the context of these simulations.

The scattering medium was modeled as a collection of point scatterers randomly placed along a line, whose amplitudes were normal random variables. A simulated scattering section contained 400 scatterers spread over a distance of 40 wavelengths, or 10 scatterers per wavelength. Each section represented a single tracking site, independent of all other sections and small enough that its deformation could be modeled simply. These simulations did not produce full echo images and displacement maps, but rather ensembles of individual speckle patches and displacement estimates.

The ultrasonic imaging process was modeled as a simple convolution with a generic pulse-echo point-spread function (PSF). The depth dependence of a real imaging system's PSF was neglected for simplicity. First, scatterers were generated as arrays of positions and amplitudes. To enable standard digital convolution, the set of scatterers was then transformed into a sampled scattering function by binning them into a discrete set of locations, with 100 spatial samples per wavelength. If two or more scatterers were placed within the same location bin, their amplitudes were summed. The resulting numerical scattering function was then convolved with the one-dimensional point-spread function, consisting of a sine wave multiplied by a Hann window. A Hann window was chosen because it is similar in shape to the often-simulated Gaussian window but, unlike the Gaussian, has compact support without needing to truncate the function. The length of the Hann window, from beginning to end, was 4 wavelengths, resulting in a full width half maximum bandwidth of about 50%.

To create a strained version of this signal, the positions of the underlying scatterers were subjected to a uniform compressive strain. The set of scatterers was strained in 180 incremental steps up to a total of 20% strain. The strain increments had a uniform size of

about 0.12%, relative to that step's configuration of scatterers. In other words, at each step the length of the scattering section was reduced by about 0.12% of its current length, not of its initial length. This process ensured that every step had the same incremental strain, regardless of its position in the sequence. As a consequence, the total strain of 20% is equal to the incremental strain, 0.12%, compounded over 180 steps, rather than multiplied by 180 steps.

Deformation of the scatterer positions occurred about the center point of the scattering length, which later was also used as the center point of the data kernel used for tracking. The displacement of this center point was considered the true displacement of the simulated tissue. Since these simulations assumed a shift-invariant PSF, a simple translation of the scattering medium would not be a source of error and was therefore arbitrary. In this case the center-point displacement was set at zero. After each application of strain, the scatterers were again binned into a scattering function and again convolved with the same point-spread function.

Uncorrelated noise was then added to each signal, to simulate the effects of electronic noise. Each signal with a different level of strain received an independent noise realization. The noise was modeled as Gaussian, with the same power spectrum as the signal. It was produced by convolving a vector of Gaussian random variables with the same point-spread function, followed by appropriate scaling to achieve the desired SNR. A range of SNRs was simulated.

This entire simulation process was repeated for 800 independently generated scatterer distributions and sets of noise realizations. These scatterer and noise realizations were saved so that any simulated case might be reproduced at will. To reduce the amount of data stored, the same scatterers and noise realizations were used for each kernel size, changing only the proper motion tracking parameter; and for each SNR, changing only the scaling factor between signal and noise. Since all estimation of statistical values was performed for a particular kernel size and SNR, and not among differing kernel sizes or SNRs, there was no need to produce independent realizations of data for every distinct estimation parameter.

2.2 Displacement estimation

Finally, displacement was estimated between each pair in the set of 181 signals (the original signal plus 180 strained versions). The number of elementary strain increments that had occurred between each pair of signals determined the strain step size for the estimate. Displacement estimation was performed by selecting a segment, or kernel, of data in whichever signal in the pair had lower overall strain, called the "pre-deformation" or "reference" signal, and computing the value of a matching function with kernels of data from the second, "post-deformation" signal. The center of the kernel in the post-deformation signal was varied, and the location of the peak of the matching function was then taken to be the displacement estimate. The estimation error is simply the difference between the estimate and the known value of zero displacement.

During this displacement estimation process, no companding or temporal stretching^{16,17} was performed to compensate for tissue strain. While these methods have been shown to improve tracking performance, they add complexity and computational expense. This work, instead, has analyzed the basic block-matching tracking method.

Several aspects of the displacement estimation method require more detail to fully understand the results. First, the matching function search region was constricted to a range of two-thirds of a wavelength on either side of the true displacement. This limitation sped up the simulation and also purposely eliminated tracking errors larger than a wavelength. In the

ultrasonic motion tracking literature these large errors are known as “peak-hop” errors, since they arise from a false peak in the correlation function. In rejecting these types of errors, the simulation was restricted to the case where there is enough prior information to know, within about a wavelength, what the true displacement is. This prior information may be gained, for example, through the use of “guided search” strategies in creating a displacement map.^{18–21} In other words, it was assumed that some existing strategy could select the correct peak in the matching function, so only errors in the precise localization of that peak were measured.

Second, although zero overall displacement has been simulated for these sets of scatterers, the results are not limited to that case. It has been argued previously that an overall displacement is arbitrary at this level of approximation, because the assumed PSF is shift-invariant. Therefore simulations of zero-displacement scatterers would be equivalent to sets of scatterers undergoing translations, as long as the location of the data kernel was also translated at each step to ensure that it was centered over the same piece of scattering medium. Such a tracking algorithm would essentially follow a single piece of tissue along its whole trajectory of motion. This method of tracking and accumulation operates in “material” or “Lagrangian” coordinates, that is, coordinates that denote a particular piece of a medium, and follow the piece as it moves.²²

Material coordinates contrast with coordinates that denote a particular point in space, called “spatial” or “Eulerian” coordinates. A motion tracking algorithm in spatial coordinates would estimate incremental displacements at a certain set of spatial locations—a fixed grid of image coordinates, for instance—while the medium moves around them. A motion tracking algorithm in material coordinates would move its tracking locations to match tissue motion, so that each incremental displacement estimate operates on the same piece of tissue. In material coordinates, the accumulated displacement of a piece of tissue is simply the sum of all its incremental displacements. In spatial coordinates, computing an accumulated displacement map requires a series of coordinate transformations.¹³ Because it is more convenient for accumulation, the material coordinates system has been simulated here.

Finally, the sampling of the signal should be addressed. Because the simulated echo signals were sampled, the computed maximum of the matching function could only occur at a sample location. Measuring a finer displacement required a method of sub-sample estimation, the subject of much research in its own right.^{23,24} In general, the accuracy of a displacement estimate depends on the sub-sample displacement estimation method, and on where the true displacement falls between adjacent sample locations. The present study was not interested in the effects of sub-sample estimation, however. To reduce its influence these simulations used the high sampling frequency of 100 samples per wavelength, with an additional sub-sample estimation using quadratic fitting around the matching function peak.²³ The normal sampling rate for a commercial ultrasound system, in contrast, might be 40 MHz, which for a signal of 10 MHz center frequency would record only 4 samples per wavelength. With a high simulated sampling rate, sub-sample displacement effects were considered to be negligible. Future work will assess the effects of coarser, more realistic sampling.

Displacement estimates were carried out using four kernel sizes—4, 8, 12, and 16 wavelengths—and three matching functions—normalized cross-correlation (NCC), sum of squared differences (SSD), and non-normalized cross-correlation (CC). Though the results show subtle differences between these functions, for brevity only the NCC results will be shown.

2.3 Accumulation

Since displacement was estimated between each possible pair of signals, accumulating to 20% strain only required selecting the proper estimates and summing them. To illustrate, denote the displacement estimation error between signals i and j as $\tilde{D}_{i,j}$. Then the accumulated error using a strain-step size of 1 (expressed as the number of elementary strain steps, instead of a strain percentage) is

$$\tilde{D}_{accum}(1) = \sum_{k=0}^{N-1} \tilde{D}_{k,k+1} \quad (2.1)$$

where N is the total number of elementary strain steps, and the simulated signals are labeled from 0 to N . Similarly, the accumulated displacement error using a strain-step size of m , and assuming that N is divisible by m , is

$$\tilde{D}_{accum}(m) = \sum_{k=0}^{N/m-1} \tilde{D}_{mk,m(k+1)} \quad (2.2)$$

$$= \tilde{D}_{0,m} + \tilde{D}_{m,2m} + \cdots + \tilde{D}_{N-m,N} \quad (2.3)$$

The post-deformation signal for one term in the sum is always the pre-deformation signal for the next term. If N is not divisible by m , then let the first step in the accumulation sequence have a size smaller than m . Let a be the remainder when dividing N by m . This gives

$$\tilde{D}_{accum}(m) = \tilde{D}_{0,a} + \sum_{k=0}^{\lfloor N/m \rfloor - 1} \tilde{D}_{a+mk,a+m(k+1)} \quad (2.4)$$

$$= \tilde{D}_{0,a} + \tilde{D}_{a,a+m} + \tilde{D}_{a+m,a+2m} + \cdots + \tilde{D}_{N-m,N} \quad (2.5)$$

where $\lfloor \cdot \rfloor$ stands for rounding down to the nearest integer.

Of course, the accumulated error is just the sum of many single-step errors. As a sum of random variables, its overall variance is equal to the sum of error covariances between each pair of individual steps. If the indices above are abbreviated so that $\tilde{D}_i$ simply indicates “the error in the i th step of an accumulation sequence,” then the accumulated variance can be expressed as

$$\text{Var}(\tilde{D}_{accum}) = \sum_{i,j} \text{Cov}(\tilde{D}_i, \tilde{D}_j) \quad (2.6)$$

The variances of single-step and accumulated estimates, as well as the covariance between individual estimation steps, were all computed over the ensemble of 800 simulated data sets for each kernel size and SNR.

3 Results

3.1 Single-step and accumulated variance

The accumulated displacement error for a given strain step size depends on the error in each step. Figure 1 shows plots of single-step variance as a function of tissue strain for three different electronic noise levels, quantified by their SNR. More precisely, they are plots of $\text{Var}(\tilde{D}_{0,k})$, the variance of the estimation error between the 0th and k th signals, with the single-step strain between each signal pair on the x -axis. Variance is plotted rather than some other measure of error, like standard deviation or strain SNR, because it is simpler to analyze the accumulation of error in terms of variance.

Figure 1 shows that single-step error variance is initially relatively constant as a function of strain, being sensitive instead to the level of electronic noise. As strain increases, the corresponding strain-induced error comes to dominate, resulting in a curve that appears roughly quadratic with strain. Larger kernels better suppress the effects of electronic noise, but are more sensitive to strain-induced errors, so their error variance is lower at small strains but increases more rapidly as strain increases.

The effects of accumulating these individual steps are shown in Figure 2, which plots the error variance of the accumulated displacement estimate described in section 2.3. In the plots of noise-free results, accumulated variance increases with kernel size and with step size, though the step size effect is surprisingly weak except for the largest kernel size. Adding noise also has a surprisingly weak effect, except for the smallest kernel sizes and highest noise level.

3.2 Covariance of consecutive displacement estimates

Because an accumulated displacement error is just the sum of all the single-step errors in a sequence, knowing the covariance between the errors of each step will aid in understanding the accumulated error. Since there are two sources of estimation error, electronic noise and tissue strain, their effects will be examined separately before considering them in combination.

3.2.1 Covariance due to electronic noise—Figure 3 shows the covariance matrix for a simulation that only included the effects of electronic noise. The previously described simulation procedure was followed, except that no strain was induced in the scatterers. Each x or y position in the figure represents an estimation step, and the brightness of the image at the point (x, y) is proportional to the covariance between the errors of those two steps. Only the first 30 steps are shown for clarity.

The bright line of positive covariance down the main diagonal of the matrix is simply the variance of each estimation step. The off-diagonal elements are close to zero, except for the superdiagonal and subdiagonal, immediately above and below the main diagonal. These side diagonals have *negative* covariance, with a magnitude approximately half that of the main diagonal.

3.2.2 Covariance due to tissue strain—The covariance structure for the strain-induced error has a very different appearance. Figure 4 shows a representative covariance matrix for an accumulation simulation in which no electronic noise has been added, so that the only source of error is tissue strain. Each estimation step in this figure is the size of two elementary strain steps, or about 0.25% strain. The 90 steps shown in the matrix cover the entire 20% simulated deformation.

The most important feature of this matrix is that it is non-zero throughout the matrix, not only along a few diagonals. The magnitude tends to decrease in the corners of the matrix, meaning that estimates that have a greater separation in the accumulation sequence—steps 10 and 90, for example—have lower covariance than steps that are closer together.

Even though no electronic noise has been added, the pattern of superdiagonal and subdiagonal with depressed covariance also appears. Relative to the covariance values in the rest of the matrix, the amount that these side diagonals are depressed tends to be about half the amount that the center diagonal is elevated, just as in the electronic noise case. Nevertheless, this main diagonal pattern is much smaller in magnitude than the similar effect of electronic noise, as will be seen when the two sources of error are combined.

3.2.3 Electronic noise and strain combined—A more realistic displacement estimation experiment includes both electronic noise and strain decorrelation. The simulated electronic noise was generated independently from the simulated echo signal, so the errors induced by electronic noise and by strain should also be independent. In that case their covariances simply add together.

Figures 5 and 6 are two examples of combining these sources of error, for different electronic noise levels and strain step sizes, and are consistent with the expectation of independence. The first rows of the figures show covariance matrices for electronic noise alone and strain alone. The second rows compare the results from simply adding the previous two matrices together, and from performing a full simulation with both noise and strain. Though these latter matrices are not identical, they are similar enough to justify treating electronic noise and strain errors as approximately independent.

3.3 Comparison of accumulation models

The covariance matrices in the previous section show that the estimates in an accumulation sequence are far from independent. To demonstrate the consequences of this finding, the variance of this study's accumulated estimates can be compared with the predictions of the uncorrelated model that is common in the literature. By assuming the error in each step is uncorrelated, this model predicts that the accumulated variance will simply be the product of the single-step variance and the number of steps. Figure 7 plots this study's accumulated variance, here called the "correlated model," against the predictions of the uncorrelated model, demonstrating their divergent results.

The effect of electronic noise is larger in the uncorrelated model, where error variance increases greatly as SNR falls. For the correlated model, in contrast, the variance curves for different noise levels are often so close together as to be nearly indistinguishable; the smallest kernel size and SNR is the only exception. The uncorrelated model also predicts that the contribution of strain-induced error to the variance (most clearly visible in the zero-noise lines in the plots) will be negligible for near-zero step sizes and increase roughly linearly with step size. In the correlated model, however, the variance begins at a finite value for near-zero step size and increases only slightly for larger steps.

4 Discussion

The accumulated variance plot of Figure 2 shows that optimizing an accumulated estimate is a different problem from optimizing a single-step estimate. As a simple example, suppose one wanted to select the optimal kernel size for a strain step size of 1%, SNR of 13 dB, and an accumulated strain of 20%. Figure 1 would suggest using a 12λ kernel to obtain the lowest single-step variance. Because of the covariance of consecutive estimates, however,

Figure 2 reveals that a 4λ kernel would be a better choice, giving an accumulated variance only one-third as high as the 12λ kernel would produce.

This difference is explained by the structure of the covariance matrices in Figures 3 and 4. This section begins with an elaboration of the uncorrelated accumulation model and proceeds to an interpretation of the covariance matrices.

4.1 The uncorrelated model

Some simple calculations and approximations can express the general behavior of the uncorrelated model. Later, with an understanding of the real covariance between estimation steps, the model will be modified to explain the observed accumulated variance curves.

Section 3.2.3 has shown that errors due to electronic noise and errors due to strain are approximately independent, so the variance for a single step is the sum of a strain-induced term and a noise-induced term. Furthermore, suppose that the strain-induced variance depends quadratically on the strain step size, as might be suggested by the single-step variance plots in Figure 1. The single-step variance can then be expressed as a function of the strain step size, denoted by

$$\sigma_{step}^2(s_0) \approx A s_0^2 + \sigma_{noise}^2 \quad (4.1)$$

where A is a coefficient, constant in s_0 , and σ_{noise}^2 is the error variance due to electronic noise, which does not depend on strain but does depend on the SNR and the kernel size.

If the uncorrelated model were correct, then the total accumulated variance for n estimates would be simply n times the single-step variance.

$$\begin{aligned} \sigma_{tot,uncorr}^2 &\approx n \cdot \sigma_{step}^2(s_0) \\ &\approx n \cdot A s_0^2 + n \cdot \sigma_{noise}^2 \end{aligned} \quad (4.2)$$

To express this in terms of total strain s_{tot} rather than number of steps n , substitute $n = s_{tot}/s_0$:

$$\begin{aligned} \sigma_{tot,uncorr}^2 &\approx \frac{s_{tot}}{s_0} \cdot A s_0^2 + \frac{s_{tot}}{s_0} \cdot \sigma_{noise}^2 \\ &\approx s_{tot} \left(A s_0 + \frac{\sigma_{noise}^2}{s_0} \right) \end{aligned} \quad (4.3)$$

For a given experiment s_{tot} can be treated as a fixed quantity, so the accumulated variance depends only on the term inside the parentheses. This term becomes large for both small s_0 , where the large number of estimation steps create a large build-up of error from electronic noise, and large s_0 , where strain-induced errors become dominant. Minimizing the

expression would yield an optimal step size of $s_0 = \sqrt{\sigma_{noise}^2/A}$.

The formula in (4.3) very much resembles the plots of uncorrelated model results in Figure 7. In those plots, with zero electronic noise the model variance is nearly a straight line passing through the origin, and at higher noise levels the $1/s_0$ dependence becomes apparent. Unfortunately for this model, the simulation results are very different. The step covariance matrices provide an explanation.

4.2 Covariance due to electronic noise

The covariance matrix for the uncorrelated model would consist of only a non-zero main diagonal, with zeros in all other positions. The matrix for error due to electronic noise shown in Figure 3, in contrast, featured a bright main diagonal, anti-correlated superdiagonal and subdiagonal, and zero covariance elsewhere. The reason that anti-correlation appears is that a pair of consecutive displacement estimation processes share an echo signal, and are therefore not independent. Consider a sequence of three echo signals, with zero deformation:

$$\begin{aligned} r_1(x) &= f(x) + n_1(x) \\ r_2(x) &= f(x) + n_2(x) \\ r_3(x) &= f(x) + n_3(x) \end{aligned}$$

where f is a noise-free echo signal and the n_i are independent noise realizations. Using the notation of section 2.3, let the displacement estimate $\hat{D}_{i,j}$ be a functional of two of these echo signals, $\hat{D}_{i,j} = \hat{D}[r_i, r_j]$. Then

$$\begin{aligned} \hat{D}_{1,2} &= \hat{D}[r_1, r_2] = \hat{D}[f + n_1, f + n_2] \\ \hat{D}_{2,3} &= \hat{D}[r_2, r_3] = \hat{D}[f + n_2, f + n_3] \end{aligned}$$

The component f does not change between signals, so it is not a source of error. Notice, however, that the noise realization $n_2(x)$ appears in both $\hat{D}_{1,2}$ and $\hat{D}_{2,3}$; it appears in $\hat{D}_{1,2}$ as the post-deformation signal and in $\hat{D}_{2,3}$ as the pre-deformation signal. Because the two estimates share a signal component, but in opposite positions, they will be anti-correlated.

For example, suppose the effect of n_2 was to make signal r_2 appear shifted slightly in the $+x$ direction relative to the other signals. Then $\hat{D}_{1,2}$ would err in that same direction. For the next estimate the signal r_2 again appears shifted in the $+x$ direction relative to r_3 , but because r_2 is now the reference signal the error in $\hat{D}_{2,3}$ is in the $-x$ direction. Whatever the influence of n_2 on $\hat{D}_{1,2}$, it tends to have the opposite effect on $\hat{D}_{2,3}$, creating a negative covariance. The same argument holds for any pair of consecutive estimates. A pair of estimates separated by more than one estimation step, however, would not share any noise realizations, and therefore could be expected to have zero covariance. This pattern of negative and zero covariance is precisely what appears in Figure 3.

The consequence of this negative component in the covariance matrix is that electronic noise has far less impact on the accumulated variance than the uncorrelated model would predict. In summing the covariance matrix, all three center diagonals contribute. Let the values of the superdiagonal and subdiagonal be $\rho_{noise} \cdot \sigma_{noise}^2$, where ρ_{noise} is a correlation coefficient and σ_{noise}^2 is the single-step error variance due to electronic noise. Then

$$\begin{aligned} \sigma_{tot,noise}^2 &= \sum_{i,j} \text{Cov}(\hat{D}_i, \hat{D}_j) \\ &\approx n \cdot \sigma_{noise}^2 + 2(n-1)\rho_{noise} \cdot \sigma_{noise}^2 \quad (4.4) \\ &\approx \sigma_{noise}^2 [1 + (n-1)(1 + 2\rho_{noise})] \\ &\approx \sigma_{noise}^2 \left[-2\rho_{noise} + \frac{s_{tot}}{s_0} (1 + 2\rho_{noise}) \right] \end{aligned}$$

where the last step substitutes $n = s_{tot}/s_0$. The variance takes the form an inverse- s_0 curve plus a constant, which is consistent with the electronic noise component of the curves in Figure 2.

If $\rho_{noise} = 0$, the formula recovers the prediction of the electronic noise portion of the uncorrelated model. In the simulations, in contrast, the value of ρ_{noise} was usually slightly less negative than -0.5 . At the limit of $\rho_{noise} = -0.5$, the second term in (4.4) vanishes, so that the accumulated variance for n steps is the same as for a single step. For ρ_{noise} slightly less negative than -0.5 , the variance increases with the number of steps, but at a much slower rate than in the uncorrelated model. This effect explains why the correlated-model and uncorrelated-model plots of accumulated variance in Figure 7 differ so much in their behavior under electronic noise. The appendix contains more details about the values of ρ_{noise} for different estimation parameters and noise levels.

4.3 Covariance due to strain

The strain-induced error covariance matrices shown in Figure 4 had three notable features. The first and most important feature was that all off-diagonal elements were nonzero. The matrices were not entirely uniform, however. The matrices also possessed an elevated center diagonal and depressed superdiagonal and subdiagonal, similar to the covariance matrices from electronic noise simulations, and covariance decreased between estimates that were far apart in the accumulation sequence. Each of these features can be explained by the error mechanisms for echo signals from strained media.

When a portion of tissue undergoes a deformation, the echo signal distorts in such a way as to roughly track the tissue deformation. This distortion of the signal is what makes displacement estimation possible. But it also causes a mismatch between pre- and post-deformation signals, introducing error into displacement estimation. The distortion is not as apparent when comparing small sections of signal, so the mismatch is not as bad for smaller data kernels. This type of strain-induced error was explained in more detail in an early paper by Céspedes et al.²⁵

Error that arises from this kind of mismatch is random in that it depends on shape of the echo signal, which is determined by the random positions of small scatterers in the tissue. But for a given section of echo signal tracked through time, each tissue strain step will induce a similar signal distortion, and the resulting tracking errors will be highly correlated between steps. Consider the first estimation step in a sequence, for example, between 0% and 0.2% strain. The post-deformation signal will be distorted by 0.2%, creating a small mismatch in the two data kernels and inducing an error in the displacement estimate. In the next estimation step, from 0.2% to 0.4% strain, the signal will be distorted by another 0.2%. The mismatch between signals in this second step will be very similar to the mismatch in the previous step, and the estimation error is likely to be nearly the same. These similar mismatches, then, explain the nonzero covariance between estimation errors that appears in Figure 4.

In addition to distorting the echo signals, tissue deformation “decorrelates” them, gradually changing the speckle pattern. An echo signal is formed from the coherent sum of echoes from individual scatterers, interfering with one another. Deformation on a scale smaller than the ultrasound pulse changes this coherent sum in a way that depends on the random locations of the scatterers. By producing new configurations of these scatterers, tissue deformation effectively introduces new, uncorrelated information into the signal. More details on this effect in two dimensions, including other types of deformations besides uniform strain, can be found in a paper by Meunier and Bertrand.²⁶

This new information in the echo signal could explain the remaining features of the strain-induced covariance matrix. First, the addition of some small amount of uncorrelated information at each strain step could mimic the effect of electronic noise, which would explain the pattern of a brighter center diagonal with depressed superdiagonal and

subdiagonal that appears in the strain-only matrices. Second, as the accumulation sequence progresses, increasingly more uncorrelated information is incorporated into the echo signals. As the signals change, so would the estimation errors, explaining the decrease in the covariance between estimates at the beginning and estimates at the end of the sequence.

Of these three features of the strain-only covariance matrices, the latter two will not matter as much for predicting the ultimate accumulated variance. The elevated and depressed covariances in the three center diagonals tend to cancel each other out, and they have small magnitudes to begin with. The decrease in covariance at the corners will have a greater effect on the sum over the matrix, but is still not as important as the overall positive covariance. As an approximation, then, let every element in the covariance matrix have the same value, $\alpha \cdot \sigma_{strain}^2$, where α is a correlation coefficient, close to unity, and σ_{strain}^2 is the single-step error variance due to strain. The number of elements in the matrix is the square of the number of strain steps, n , so its sum, the accumulated variance, is

$$\sigma_{accum}^2 \approx n^2 \alpha \sigma_{strain}^2 \quad (4.5)$$

Combining this approximation with a previous assumption, that the single-step error variance is roughly quadratic with strain, produces the curious result that the accumulated variance should be independent of strain step size. Substituting the strain-induced portion of (4.1) into (4.5), recalling that s_{tot} is the total strain and s_0 is the size of the strain step,

$$\begin{aligned} \sigma_{tot, strain}^2 &\approx n^2 \alpha \sigma_{strain}^2 \\ &\approx n^2 \alpha A s_0^2 \\ &\approx \left(\frac{s_{tot}}{s_0}\right)^2 \alpha A s_0^2 \\ &\approx \alpha A s_{tot}^2 \end{aligned} \quad (4.6)$$

Adding this result to $\sigma_{tot, noise}^2$ above and consolidating many of the constants produces an overall accumulated error variance of the form

$$\sigma_{tot}^2 \approx a + \frac{b}{s_0} \quad (4.7)$$

where a and b are constants.

As in the equivalent formula for the uncorrelated model, (4.3), buildup of error due to electronic noise increases the variance for low s_0 , though here that buildup is greatly dampened. Unlike the uncorrelated formula, however, the variance settles into a constant value for high s_0 . Although this result is surprising, it is consistent with the results in Figure 2. Past a certain s_0 , when the electronic noise effects have faded, those accumulated variance curves are remarkably at as a function of step size. The major exception is the 16λ kernel. The upward trajectory of that curve indicates that for such a large kernel, the single-step variance increases at a rate faster than quadratic. One could also draw that conclusion from the single-step plots of Figure 1.

The prediction of an accumulated variance that is constant as a function of step size only holds as far as the single-step variance behavior remains quadratic. We expect that at a high enough strain, though, single-step variances for *all* kernel sizes will begin to increase at a faster-than-quadratic rate. Nevertheless, the step size at which the quadratic approximation breaks down and accumulated variance begins to increase (perhaps 2% strain for the 16λ

tracking kernel, more than 4% strain for the shorter kernels) seems to be larger than the usual operating point for multicompression strain imaging strategies.

Here the limitations of these simulations become important. Unlike these simulated scattering functions, tissue is three-dimensional and far from uniform, even on the scale of an ultrasound pulse. Tissue deformation results in more complex motion and more signal decorrelation than was modeled here. With more signal decorrelation, we would expect single-step variance to increase more rapidly with strain, causing accumulated variance to begin to increase at a smaller step size.

A more complex displacement estimation algorithm, such as one incorporating companding, could also alter these results. Since companding, in particular, compensates for how the ultrasound signal distorts in accord with tissue motion, it would likely change the covariance structure of strain-induced errors. Testing the effects of companding or other advanced algorithms on displacement estimate accumulation would be an interesting avenue of future work.

We would not expect more realistic deformation or more advanced algorithms to change the core finding, however, that the steps in an accumulation sequence have important covariances. Rather they would add new effects in addition to those already described. The relative simplicity of the present work isolates the most important aspects of displacement estimation, granting insight into the problem and serving to guide future studies. Extensions of this work to 2D and 3D deformation, including shear strain, are underway.

5 Conclusion

These simulations revealed some general statistical properties of accumulated displacement estimates. Most importantly, the errors in an accumulation sequence are not uncorrelated, but have an important covariance structure that affects the final accumulated displacement estimate. Displacement estimate errors caused by electronic noise are anti-correlated with the estimation steps immediately before and after, while errors caused by tissue strain are positively correlated with all other estimation steps in the sequence. As a result of this covariance structure, accumulated displacement estimates behave differently from single-step estimates.

While the one-dimensional model presented here cannot provide precise guidance to tracking algorithms for tissue, some qualitative recommendations are possible. Compared to a single-step displacement estimate, an accumulated estimate is robust to the presence of electronic noise but is sensitive to strain-induced error. Therefore it may be more effective to use smaller kernel sizes for accumulation than would be used for single-step estimates. The present results suggest that kernel sizes only somewhat larger than a pulse length may be safely used unless there is very high electronic noise. This work also indicates that accumulated variance should depend only weakly on strain step size, so the choice of step size in accumulation may not need to be tightly optimized.

These statistical properties and recommendations should help to guide the creation of accumulated displacement maps, both for the reconstruction of elasticity parameters and for strain images that use a multicompression method.

Acknowledgments

This work was supported by NIH grants R01CA140271 and T32CA009206.

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Appendix: The anti-covariance caused by electronic noise

Sections 3.2.1 and 4.2 showed that with only electronic noise and no tissue strain, consecutive displacement estimates are anti-correlated, with a correlation coefficient close to -0.5 . This appendix shows estimated values of ρ_{noise} , as well as how those estimates change for various combinations of simulation and estimation parameters.

Correlation coefficients were computed by taking the covariance over all pairs of displacement errors that shared an echo signal and dividing it by the error variance. Again using the notation of section 2.3, let $\tilde{D}_{ij}$ be the displacement estimation error between signals i and j . Then ρ_{noise} can be expressed as

$$\rho_{noise} = \frac{\text{Cov}(\tilde{D}_{ij}, \tilde{D}_{jk})}{\text{Var}(\tilde{D}_{ij})} \quad (5.1)$$

The results of this computation for different SNR and kernel lengths are plotted in Figure 8, which helps to further explain the accumulated variance results in Figure 2. For high and moderate SNR, the correlation coefficient is very close to -0.5 for all kernel lengths, so error due to electronic noise largely cancels out and little variance accumulates. At low SNR, the anti-correlation is still strong for the larger kernels tested, but decays considerably for the smaller kernels. Thus in Figure 2 the larger kernels show little accumulated effect from any level of electronic noise, while the smaller ones, especially the 4λ length, build up considerable error from electronic noise. As a rule, it seems that variations in parameters that lower the noise-induced error variance, such as high SNR or larger kernels, will also tend to bring the correlation coefficient of consecutive estimates closer to -0.5 .

Figure 1.

Single-step displacement estimate error variance as a function of strain. Larger kernels have a greater dependence on strain and smaller kernels have a greater dependence on electronic SNR.

Figure 2. Accumulated displacement estimate error variance as a function of strain step size. The accumulated variance is remarkably insensitive to step size and electronic noise level.

Figure 3. Displacement estimate error covariance matrix for zero strain and an electronic SNR of 20 dB. The kernel size was 4λ . Results for other SNR and kernel sizes have similar patterns but different magnitudes.

Figure 4. Displacement estimate error covariance matrix for zero electronic noise, but including strain. The kernel length was 12λ and the step size was 2 elementary steps, or 0.25%.

Figure 5.

Displacement estimate error covariance matrices for (a) strain only, (b) noise only, (c) the actual covariance matrix for strain and noise simulated together, and (d) the individual matrices summed together. The kernel length was 12λ and the electronic SNR was 20 dB. The strain step was 5 elementary strain steps, or about 0.6% strain. These parameters were chosen so that the dynamic ranges of the noise and strain covariance matrix images would be similar.

Figure 6.

The same types of example matrices as in Figure 5, but using a kernel length of 16λ , a strain step of 1% and an SNR of 10 dB.

Figure 7.

Comparison between correlated and uncorrelated models. Accumulated displacement estimate error variance was calculated in (a) and (c), the correlated model, from accumulated simulation results. The uncorrelated model in (b) and (d) assumes no step covariance and simply multiplies the single-step variance by the number of steps. The marker types represent different electronic noise levels, and each row of the figure reports a different kernel size. Note that the no-noise lines can be difficult to see because they are often overlaid with other lines.

Figure 8. Correlation coefficients for consecutive displacement estimates where electronic noise is the only source of error.