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Including fluid shear viscosity in a structural acoustic finite element model using a scalar fluid representation

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Abstract

An approximate boundary condition is developed in this paper to model fluid shear viscosity at boundaries of coupled fluid-structure system. The effect of shear viscosity is approximated by a correction term to the inviscid boundary condition, written in terms of second order in-plane derivatives of pressure. Both thin and thick viscous boundary layer approximations are formulated; the latter subsumes the former. These approximations are used to develop a variational formulation, upon which a viscous finite element method (FEM) model is based, requiring only minor modifications to the boundary integral contributions of an existing inviscid FEM model. Since this FEM formulation has only one degree of freedom for pressure, it holds a great computational advantage over the conventional viscous FEM formulation which requires discretization of the full set of linearized Navier-Stokes equations. The results from thick viscous boundary layer approximation are found to be in good agreement with the prediction from a Navier-Stokes model. When applicable, thin viscous boundary layer approximation also gives accurate results with computational simplicity compared to the thick boundary layer formulation. Direct comparison of simulation results using the boundary layer approximations and a full, linearized Navier-Stokes model are made and used to evaluate the accuracy of the approximate technique. Guidelines are given for the parameter ranges over which the accurate application of the thick and thin boundary approximations can be used for a fluid-structure interaction problem.

Keywords

fluid viscous boundary layer; fluid-structure interaction; structural acoustic; finite element method

1. Introduction

There are many problems in acoustics and structural acoustics where fluid shear viscosity plays an important role. For example, in micro-electro-mechanical systems and micro-scale biological structures, an inviscid approximation to the flow results in large errors in the predicted response. Using a linearized Navier-Stokes (N-S) solution, however, increases the number of unknowns by at least a factor of three compared to an inviscid approximation.

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Hence, there is a large computational burden associated with incorporating the correct physics. Significant effort has been devoted to develop accurate numerical methods to model fluid-structure interaction including viscosity (4; 5; 3; 13). It is possible to discretize and solve the three dimensional (3-D) linearized N-S equations with fluid viscosity using a personal computer or small cluster of parallel nodes (3; 4). Models with appropriate choice of fluid, structure, and coupling elements are able to predict accurate results, but the computational cost is high as a typical 3-D viscous fluid model utilizes fluid velocity and pressure as independent variables. In this paper, we show how to seamlessly include a boundary perturbation term into an inviscid finite element formulation (with one scalar nodal unknown) to accurately approximate the effects of shear viscosity for a wide range of boundary layer (BL) thicknesses in a structural acoustic system.

In order to avoid the computational burden associated with solving the 3-D N-S equations, simplifications of the viscous fluid field equations have been used to develop different approximate schemes. For interior structural acoustics problems when the viscous BL is comparable to the smallest dimension of the interior fluid domain, Beltman (1) used the thin-film fluid approximations to develop an efficient and accurate finite element method. For the case where the viscous BL is thin compared to the dimensions of the problem (or the domain is unbounded), Holmes and Cole (6) derived a modified boundary condition including viscosity. The modified boundary condition was obtained by matching the inner (inside the BL) and outer (outside the BL) solutions of pressure and velocity. The resulting equation was written in terms of pressure and its normal derivatives only. The Holmes and Cole boundary condition is not readily applicable in a numerical scheme because it involved higher order derivatives in the out-of-plane direction. A hybrid numerical and analytical solution for thermo-viscous fluid was presented by Bossart *et al.* (2). The viscous boundary condition was derived from the NS equation with the assumption that the viscous and thermal BL thicknesses are much smaller than the dimension of the problem. Inside the BL, the assumptions are the same as in Beltman's approach (1). The viscous correction term is incorporated into a variational scheme and, like the Holmes and Cole condition, involves higher order out-of-plane derivatives. In Bossart *et al.* (2), the higher order derivative is eliminated by introducing the wavenumber in the normal direction. The wavenumber is unknown but bounded and an iterative scheme to find the wavenumber was proposed (however, results for only a single iteration were presented). Lim and Steele (8) used an alternate approach to model fluid viscosity in the cochlear duct through a semi-analytical WKB (Wentzel-Kramers-Brillouin) method (see also Steele and Taber (11)). The viscous BL at fluid-structure interface was included and a single mode of propagation included. The analytical approximation offers some advantages as no assumptions were necessary on the relative magnitude between in-plane and out-of-plane gradients since the in-plane gradients can be explicitly represented by in-plane wave numbers for each WKB-mode (or solution to the eikonal equation).

In this paper, we develop a new shear viscous boundary correction to an inviscid formulation. We are primarily interested in liquid-structure interaction so that the thermal effect of the fluid is not considered. However, if the fluid is air, its thermal effect is coupled to the fluid pressure and density. In this case, the thermal effect becomes important and should be taken into account in the model (e.g., Beltman (1) and Bossart *et al.* (2)). We will show how to implement the modified boundary condition in an existing inviscid finite element scheme with minor changes to retain its low computational cost. As an example, this new formulation is used to analyze a fluid-structure interaction system comprised of a rectangular fluid-filled duct with a flexible structure occupying part of the boundary.

2. Method

Consider a viscous fluid contained within a rectangular enclosure shown in Fig. 1. The fluid geometry is given by $\Omega = \{0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq L_z\}$. We assume that the viscous BL thickness δ satisfies $\delta \ll L_x$ and L_y . The interior surface is Ω and non-rigid boundaries are contained in Γ . For clarity of presentation, we restrict the flexible structures to the constant- z surfaces ($z = 0$ and $z = L_z$).

2.1. Shear Viscous Boundary Layer Correction Theory

The mathematical model for shear viscous and compressible acoustic fluid is based on continuity equation, compressibility equation, and linearized N-S equations are given next:

$$\frac{\partial \rho_f}{\partial t} + \rho_f \nabla \cdot \mathbf{v} = 0, \quad (1)$$

$$\beta = \frac{1}{\rho_f} \frac{\partial \rho_f}{\partial P} \Big|_{\text{adiabatic conditions}}, \quad (2)$$

$$\rho_f \frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \mu \nabla^2 \mathbf{v} + \frac{1}{3} \mu \nabla (\nabla \cdot \mathbf{v}), \quad (3)$$

where ρ_f is the fluid density, $\beta (= 1/\rho_f c^2)$ is the compressibility of the fluid, and $\mathbf{v}$ is the fluid velocity vector. Time harmonic solutions of the form $e^{j\omega t}$ are sought. The velocity field is decomposed into the gradient of a scalar field Φ and the curl of a vector field $\tilde{\Psi}$ as

$$\mathbf{v} = \nabla \Phi + \nabla \times \tilde{\Psi}, \quad (4)$$

where Φ is related to pressure P by $\Phi = -(a/j\omega\rho_f)P$, and $a = 1 + (j\omega 4\mu/3\rho_f c^2)$. Eliminating Φ and $\mathbf{v}$ in Eqs. 1-3 gives the following equations,

$$\nabla^2 P + \frac{\omega^2}{ac^2} P = 0, \quad \text{in } \Omega, \quad (5)$$

$$\tilde{\Psi} = \frac{\mu}{j\omega\rho_f} \nabla^2 \tilde{\Psi}, \quad \text{in } \Omega. \quad (6)$$

For clarity of illustration, in the following we show the derivation of BL correction to the constant- z surfaces (the BL correction to the other surfaces follows directly, as will see later). Within the BL, one can scale the normal direction of the z variable by the BL thickness δ (6), and expand the viscosity vector field $\tilde{\Psi}$ in terms of δ as well. From the asymptotic analysis, the normal derivative, $\partial/\partial z$, dominates the other derivatives in the governing equation for $\tilde{\Psi}$ (Eq. 6), and the leading order approximation term of $\tilde{\Psi}$ (denoted as Ψ) in Eq. 6 is

$$\frac{\partial^2 \Psi}{\partial z^2} - \frac{j\omega\rho_f}{\mu} \Psi = 0, \quad \text{in } \Omega. \quad (7)$$

Following Steele and Taber (11), a solution for the vector potential Ψ in Eq. 7 is,

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} A_1(x, y) e^{\gamma z} + B_1(x, y) e^{\gamma(L_z - z)} \\ A_2(x, y) e^{\gamma z} + B_2(x, y) e^{\gamma(L_z - z)} \\ 0 \end{pmatrix}, \quad (8)$$

where $\gamma = \sqrt{j\omega\rho_f/\mu}$. The real part of γ is chosen to be negative so that $e^{\gamma z}$ decays with increasing z . The BL thickness δ is defined as $|1/\Re(\gamma)|$, given by $\delta = \sqrt{2\mu/\rho_f\omega}$. Ψ plays a significant role inside the BLs. Assuming negligible in-plane structural motion on Γ , one has the no-slip boundary condition at $z = 0$ and $z = L_z$,

$$v_x = \frac{\partial\Phi}{\partial x} - \frac{\partial\Psi_2}{\partial z} = 0, \quad v_y = \frac{\partial\Phi}{\partial y} + \frac{\partial\Psi_1}{\partial z} = 0, \quad \text{on } z=0, L_z. \quad (9)$$

Extensions to non-zero in-plane motions are possible (see Appendix A), but they are not included here. Except when explicitly mentioned, $v = 0$ is prescribed on the other surfaces at $x = 0$, $x = L_x$, $y = 0$, and $y = L_y$.

The z -dependence of Ψ is now determined. Next we will use the boundary conditions in Eq. 9 to obtain the coefficients A_1 , A_2 , B_1 , and B_2 (in Eq. 8) for two cases.

2.1.1. Case 1: thin boundary layer approximation ($\delta \ll L_z$)—If $\delta \ll L_z$, then we assume that BL perturbation at one surface is not coupled to the other surface. For instance, the solution at $z = 0$ is taken to be

$$\Psi_1 = A_1(x, y) e^{\gamma z}, \quad \Psi_2 = A_2(x, y) e^{\gamma z}, \quad \Psi_3 = 0. \quad (10)$$

The unknown coefficients A_1 and A_2 are solved from the no-slip boundary condition at $z = 0$ (Eq. 9),

$$A_1(x, y) = -\frac{1}{\gamma} \frac{\partial\Phi}{\partial y} \Big|_{z=0}, \quad A_2(x, y) = \frac{1}{\gamma} \frac{\partial\Phi}{\partial x} \Big|_{z=0}. \quad (11)$$

The continuity of the normal velocity at $z = 0$ gives,

$$v_z = \frac{\partial\Phi}{\partial z} + \frac{\partial\Psi_2}{\partial x} - \frac{\partial\Psi_1}{\partial y} = j\omega W(x, y), \quad \text{on } z=0, \quad (12)$$

where $W(x, y)$ is the normal displacement of the structure, defined to be positive in the positive z -direction. Substitution of Ψ_1 and Ψ_2 with Φ (by using Eqs. 10 and 11) into Eq. 12, and replacement of Φ by P to have

$$\frac{\partial P}{\partial z} + \frac{1}{\gamma} \nabla_{xy}^2 P = \frac{1}{\alpha} \rho_f \omega^2 W(x, y), \quad \text{on } z=0, \quad (13)$$

where ∇_{xy}^2 is the Laplace operator in the xy -plane. The second term on the left-hand-side of Eq. 13 is the correction term at $z = 0$ due to the viscosity.

Similarly, if we consider the viscous BL at $z = L_z$ alone, the solution of Ψ has the form,

$$\Psi_1 = B_1(x, y) e^{\gamma(L_z - z)}, \quad \Psi_2 = B_2(x, y) e^{\gamma(L_z - z)}, \quad \Psi_3 = 0. \quad (14)$$

Applying the no-slip boundary condition at $z = L_z$ yields,

$$B_1(x, y) = \frac{1}{\gamma} \frac{\partial \Phi}{\partial y} \Big|_{z=L_z}, \quad B_2(x, y) = -\frac{1}{\gamma} \frac{\partial \Phi}{\partial x} \Big|_{z=L_z}. \quad (15)$$

The corresponding pressure boundary condition at $z = L_z$ is

$$\frac{\partial P}{\partial z} - \frac{1}{\gamma} \nabla_{xy}^2 P = -\frac{1}{\alpha} \rho_f \omega^2 W(x, y), \quad \text{on } z=L_z. \quad (16)$$

Equations 13 and 16 are the viscous corrections for thin BLs on $z = 0$ and $z = L_z$ respectively. These equations can be used either individually (on either $z = 0$ or $z = L_z$) or simultaneously (on both $z = 0$ and $z = L_z$) to account viscosity on the constant- z surfaces.

Equations 13, 16 are then generalized to any portion of Ω

$$-\frac{\partial P}{\partial n} + \frac{1}{\gamma} \nabla_s^2 P = \frac{1}{\alpha} \rho_f \omega^2 W, \quad \text{on } \partial\Omega. \quad (17)$$

where W is the normal velocity on the structure, n is the outward normal of a surface, and ∇_s^2 is the in-plane (tangent) Laplace operator on the surface where n is defined. Equation 17 is a higher order approximation for the normal pressure derivative on the boundary (see also Appendix B). It is applicable on any surface Ω (including the surfaces normal to the x and y directions) in this system because $\delta \ll L_x, L_y, L_z$ is assumed in this derivation.

2.1.2. Case 2: thick boundary layer approximation ($\delta \sim L_z$)—Next we consider the case where $\delta \ll L_x, L_y$ (the thin BL case handled in 2.1.1), but $\delta \sim L_z$ so that the viscous BL at $z = 0$ couples to the BL at $z = L_z$. Assuming the $z = 0$ and $z = L_z$ surfaces are parallel (this method of thick boundary layer approximation is not valid if the two surfaces are not parallel) and following the same procedure as before, one solves the coefficients A_1, A_2, B_1 and B_2 (in Eq. 8) from the no-slip boundary condition,

$$\begin{aligned} A_1 &= -\frac{\frac{\partial \Phi}{\partial y} \Big|_{z=0} - \frac{\partial \Phi}{\partial y} \Big|_{z=L_z} e^{\gamma L_z}}{\gamma(1-e^{2\gamma L_z})}, & B_1 &= -\frac{\frac{\partial \Phi}{\partial y} \Big|_{z=0} e^{\gamma L_z} - \frac{\partial \Phi}{\partial y} \Big|_{z=L_z}}{\gamma(1-e^{2\gamma L_z})}, \\ A_2 &= \frac{\frac{\partial \Phi}{\partial y} \Big|_{z=0} - \frac{\partial \Phi}{\partial y} \Big|_{z=L_z} e^{\gamma L_z}}{\gamma(1-e^{2\gamma L_z})}, & B_2 &= \frac{\frac{\partial \Phi}{\partial y} \Big|_{z=0} e^{\gamma L_z} - \frac{\partial \Phi}{\partial y} \Big|_{z=L_z}}{\gamma(1-e^{2\gamma L_z})}, \end{aligned} \quad (18)$$

The terms with the coefficient $e^{\gamma L_z}$ are not negligible in this case. The components of Ψ are written as,

$$\begin{aligned} \Psi_1 &= \frac{2e^{\gamma L_z}}{\gamma(1-e^{2\gamma L_z})} \left[-\frac{\partial \Phi}{\partial y} \Big|_{z=0} \cosh \gamma(L_z - z) + \frac{\partial \Phi}{\partial y} \Big|_{z=L_z} \cosh \gamma z \right], \\ \Psi_2 &= \frac{2e^{\gamma L_z}}{\gamma(1-e^{2\gamma L_z})} \left[\frac{\partial \Phi}{\partial x} \Big|_{z=0} \cosh \gamma(L_z - z) - \frac{\partial \Phi}{\partial x} \Big|_{z=L_z} \cosh \gamma z \right]. \end{aligned} \quad (19)$$

Substitution Ψ into the normal velocity at $z = 0$ and $z = L_z$ gives the viscous boundary conditions for thick BLs,

$$\frac{\partial P}{\partial z} \Big|_{z=0} + \frac{\beta_1}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=0} - \frac{\beta_2}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=L_z} = \frac{1}{\alpha} \rho_f \omega^2 W \Big|_{z=0}, \quad (20)$$

$$\left. \frac{\partial P}{\partial z} \right|_{z=L_z} - \frac{\beta_1}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=L_z} + \frac{\beta_2}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=0} = -\frac{1}{\alpha} \rho_f \omega^2 W \Big|_{z=L_z}, \quad (21)$$

where

$$\beta_1 = \frac{1 + e^{2\gamma L_z}}{1 - e^{2\gamma L_z}}, \quad \beta_2 = \frac{2e^{\gamma L_z}}{1 - e^{2\gamma L_z}}. \quad (22)$$

Equations 20 and 21 differ from Eqs. 13 and 16 by having an additional term that couples the local pressure to the pressure on the opposite boundary. When the BL thickness is much smaller than the duct height, the real part of γL_z is a large negative number. In this case, $|\beta_1|$ is close to 1 and $|\beta_2|$ is approximately 0; Eqs. 20 and 21 reduce to Eqs. 13 and 16 for the thin (non-coupling) viscous BL correction. Hence, the thick BL case subsumes the thin BL approximation, but takes extra connectivity in the finite element method (FEM) scheme as will be seen.

2.2. Strong form

Here we state the strong form of the coupled fluid-structure problem.

In the fluid domain Ω , the governing equation is (Eq. 5)

$$\nabla^2 P + \frac{\omega^2}{\alpha c^2} P = 0, \quad \text{in } \Omega,$$

subjected to (Eq. 17)

$$-\frac{\partial P}{\partial n} + \frac{1}{\gamma} \nabla_s^2 P = \frac{1}{\alpha} \rho_f \omega^2 W, \quad \text{on } \partial\Omega$$

for the thin viscous BL case. If the viscous BL is thick on the z surfaces, P/z is replaced by (Eqs. 20 and 21)

$$\begin{aligned} \left. \frac{\partial P}{\partial z} \right|_{z=0} + \frac{\beta_1}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=0} - \frac{\beta_2}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=L_z} &= \frac{1}{\alpha} \rho_f \omega^2 W \Big|_{z=0}, \\ \left. \frac{\partial P}{\partial z} \right|_{z=L_z} - \frac{\beta_1}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=L_z} + \frac{\beta_2}{\gamma} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \Big|_{z=0} &= -\frac{1}{\alpha} \rho_f \omega^2 W \Big|_{z=L_z}. \end{aligned}$$

The structures on the $z = 0$ and $z = L_z$ surfaces interact with the fluid in the duct. The governing equation of the structure can be expressed as

$$\mathcal{L}(W) = -P, \quad \text{on } \Gamma, \quad (23)$$

where $\mathcal{L}$ is a linear operator represents the structural dynamics. In this paper, examples are given for a membrane as the structural model with simply supported boundary condition,

$$\tilde{T}_x \frac{\partial^2 W}{\partial x^2} + \tilde{T}_y \frac{\partial^2 W}{\partial y^2} + \rho_p \omega^2 W = \alpha P, \quad \text{on } \Gamma, \quad (24)$$

subjected to

$$W=0, \quad \text{on } \partial\Gamma, \quad (25)$$

where $\tilde{T}_x = T_x(1 + j\eta)$ and $\tilde{T}_y = T_y(1 + j\eta)$ are the membrane tensions in the x and y directions, respectively, with a damping coefficient η . ρ_p is the membrane density per unit area.

2.3. Variational formulation

The variational formula for the governing equation of the fluid (Eq. 5) is

$$\int_{\Omega} \left(\nabla \bar{P} \cdot \nabla P - \frac{\omega^2}{\alpha c^2} \bar{P} P \right) d\Omega - \int_{\partial\Omega} \bar{P} \nabla P \cdot \mathbf{n} dS = 0, \quad (26)$$

where $\bar{P}$ is a weighting function and P is a trial solution (7), $\mathbf{n}$ is the outward normal on the boundary, and S is the surface of Ω .

2.3.1. 3-D formulation for thin boundary layer (case 1)—The surface integral in Eq. 26 (third term) can be rewritten using the thin BL correction (Eq. 17),

$$\int_{\partial\Omega} \bar{P} \nabla P \cdot \mathbf{n} dS = \frac{1}{\gamma} \int_{\partial\Omega} \bar{P} \nabla_s^2 P dS - \frac{1}{\alpha} \rho_f \omega^2 \int_{\Gamma} W \bar{P} dxdy. \quad (27)$$

The first integral on the RHS of Eq. 27 is a sum of six surface integrations on Ω . Taking the $z = 0$ surface as an example, by using the relation $\bar{P} \nabla^2 P = \nabla \cdot (\bar{P} \nabla P) - \nabla \bar{P} \cdot \nabla P$, the surface integral can be rewritten as

$$\underbrace{\frac{1}{\gamma} \int_{z=0} \bar{P} \nabla_{xy}^2 P dxdy}_{O(\delta)} = \frac{1}{\gamma} \int_{z=0} \nabla_{xy} \cdot (\bar{P} \nabla_{xy} P) dxdy - \frac{1}{\gamma} \int_{z=0} \nabla_{xy} \bar{P} \cdot \nabla_{xy} P dxdy, \quad (28)$$

where ∇_{xy} is a in-plane gradient operator in the x and y directions. Equation 28 is of order $O(\delta)$ since $1/\gamma$ has the same order as δ . By using the divergence theorem in 2-D, the first integral on the RHS of Eq. 28 becomes

$$\frac{1}{\gamma} \int_{z=0} \nabla_{xy} \cdot (\bar{P} \nabla_{xy} P) dxdy = \frac{1}{\gamma} \int_0^{L_x} \left(\bar{P} \frac{\partial P}{\partial y} \Big|_{z=0} \right) \Big|_{y=0}^{y=L_y} dx + \frac{1}{\gamma} \int_0^{L_y} \left(\bar{P} \frac{\partial P}{\partial x} \Big|_{z=0} \right) \Big|_{x=0}^{x=L_x} dy. \quad (29)$$

Applying the thin BL correction (Eq. 17) again to P/y and P/x on the RHS of Eq. 28 gives

$$\frac{1}{\gamma} \int_{z=0} \nabla_{xy} \cdot (\bar{P} \nabla_{xy} P) dxdy = \underbrace{\frac{1}{\gamma^2} \int_0^{L_x} \left(\bar{P} \nabla_{xz}^2 P|_{z=0, y=0} + \bar{P} \nabla_{xz}^2 P|_{z=0, y=L_y} \right) dx}_{O(\delta^2)} + \underbrace{\frac{1}{\gamma^2} \int_0^{L_y} \left(\bar{P} \nabla_{yz}^2 P|_{z=0, x=0} + \bar{P} \nabla_{yz}^2 P|_{z=0, x=L_x} \right) dy}_{O(\delta^2)}. \quad (30)$$

Equation 30 (the first integral on the RHS of Eq. 28) is of order $O(\delta^2)$, which is negligible compared to the order of Eq. 28 [$O(\delta)$]. Hence, Eq. 28 is simplified as

$$\frac{1}{\gamma} \int_{z=0} \bar{P} \nabla_{xy}^2 P dxdy \approx -\frac{1}{\gamma} \int_{z=0} \nabla_{xy} \bar{P} \cdot \nabla_{xy} P dxdy. \quad (31)$$

By a similar derivation (Eqs. 28-30) to the other five surfaces ($z = L_z$, $x = 0$, L_z , $y = 0$, L_y), the surface integral in Eq. 26 can be written as

$$\int_{\partial\Omega} \bar{P} \nabla P \cdot \mathbf{n} dS \approx -\frac{1}{\gamma} \int_{\partial\Omega} \nabla_s \bar{P} \cdot \nabla_s P dS - \frac{1}{\alpha} \rho_s \omega^2 \int_{\Gamma} W \bar{P} dxdy, \quad (32)$$

where ∇_s is the in-plane gradient operator on the surfaces. The first term on the RHS of Eq. 32 is a sum of six surface integrals on Ω . Substitution Eq. 32 into Eq. 26 yields,

$$\int_{\Omega} \nabla \bar{P} \cdot \nabla P d\Omega - \int_{\Omega} \frac{\omega^2}{\alpha c^2} \bar{P} P d\Omega + \frac{1}{\gamma} \int_{\partial\Omega} \nabla_s \bar{P} \cdot \nabla_s P dS + \frac{1}{\alpha} \rho_f \omega^2 \int_{\Gamma} W \bar{P} dxdy = 0, \quad (33)$$

which is the 3-D variational equation with thin viscous BL corrections. Both α and γ are functions of the viscosity μ . When $\mu = 0$, then $\alpha = 1$ and $1/\gamma = 0$, the inviscid form is recovered.

Remark: Equation 13 is consistent with Holmes and Cole's viscous boundary condition (6) although a different approach is used here. Using our sign convention for γ and including compressibility, the dimensional form of Holmes and Cole's equation at $z = 0$ is,

$$\frac{\partial P}{\partial z} - \frac{1}{\gamma} \left(\frac{\partial^2 P}{\partial z^2} + \frac{\omega^2}{\alpha c^2} P \right) = \frac{1}{\alpha} \rho_f \omega^2 W. \quad (34)$$

The difference between the above equation in Ref. (6) and Eq. 13 is that we keep the correction written in terms of the in-plane derivative as $\nabla_s^2 P / x^2 + \nabla_s^2 P / y^2$. This is the key step enabling the development of a variational formulation that can be used with an existing inviscid structural acoustic code with only minor modification. Otherwise, if we used Eq. 34, the boundary term at $z = 0$ becomes,

$$\int_{z=0} \bar{P} \frac{\partial P}{\partial z} dxdy = -\frac{1}{\gamma} \int_{z=0} \left[\bar{P} \frac{\partial^2 P}{\partial z^2} + \frac{\omega^2}{\alpha c^2} \bar{P} P \right] dxdy - \frac{1}{\alpha} \rho_f \omega^2 \int_{\Gamma} \bar{P} W dxdy. \quad (35)$$

The first term on the RHS of the equation cannot be evaluated by integration by parts because the higher-order boundary derivatives are not in the in-plane direction.

2.3.2. 3-D formulation for thick boundary layer (case 2)—Using the thick BL correction (Eqs. 20, 21) on the $z = 0$ and $z = L_z$ surfaces and thin BL correction (Eq. 17) on the other surfaces, and following the derivation in Eqs. 27-33 (dropping order $1/\gamma^2$ terms), one obtains the final variational equation as

$$\begin{aligned} & \int_{\Omega} \nabla \bar{P} \cdot \nabla P d\Omega \\ & - \int_{\Omega} \frac{\omega^2}{\alpha c^2} \bar{P} P d\Omega \\ & + \frac{1}{\alpha} \rho_f \omega^2 \int_{\Gamma} W \bar{P} dxdy \\ & + \frac{\beta_1}{\gamma} \int_{z=0, L_z} \nabla_s \bar{P} \cdot \nabla_s P dxdy \\ & + \frac{1}{\gamma} \int_{x=0, L_x} \nabla_s \bar{P} \cdot \nabla_s P dydz \\ & + \frac{1}{\gamma} \int_{y=0, L_y} \nabla_s \bar{P} \cdot \nabla_s P dx dz \\ & - \frac{\beta_2}{\gamma} \int \left(\nabla_s \bar{P} \Big|_{z=L_z} \cdot \nabla_s P \Big|_{z=0} + \nabla_s \bar{P} \Big|_{z=0} \cdot \nabla_s P \Big|_{z=L_z} \right) dxdy = 0. \end{aligned} \quad (36)$$

In this case, the viscous BL is sufficiently thick that the two BLs interact in the z direction. The last integral in Eq. 36 requires an extra connectivity that couples the elements on $z = 0$ to those on $z = L_z$ (Fig. 2).

2.3.3. Formulation for the structure—The variational form of Eq. 24 follows in a standard way as in Eq. 26. Since simply supported boundary conditions are used, the weighting function $\bar{W}$ vanishes on the boundary of the structure. Therefore, the variational form of Eq. 24 is

$$\int_{\Gamma} \left(\tilde{T}_x \frac{\partial \bar{W}}{\partial x} \frac{\partial W}{\partial x} + \tilde{T}_y \frac{\partial \bar{W}}{\partial y} \frac{\partial W}{\partial y} \right) dx dy - \int_{\Gamma} \rho_p \omega^2 \bar{W} W dx dy + \int_{\Gamma} \alpha \bar{W} P dx dy = 0, \quad (37)$$

which is applicable on both $z = 0$ and $z = L_z$.

Equations 33 (or 36) and 37 close the set of equations for the fluid-structure coupled viscous BL correction problem.

3. Results

In this section, we present simulation results using the viscous BL approximation methods developed in this paper and compared to predictions obtained from a 3-D FEM model based on linearized N-S equations (3). The geometry and the dimensions of the example problem were given in Ref. (3) (where experimental results are also presented) and are reproduced here in Fig. 3. In the present study, the duct height is varied among $15 \mu\text{m}$, $50 \mu\text{m}$, or $110 \mu\text{m}$. Symmetry boundary conditions are used at the $y = 0$ plane so that only half the geometry is modeled. The rectangular input membrane on $z = 0$ is driven by a constant external pressure field. The tapered structure on $z = 0$ is an orthogonally tensioned membrane. The input and tapered membranes are the only flexible surfaces on Ω , and thus Eq. 37 is only applied on $z = 0$. The material properties of the system are given in Table 1. The fluid in the model was silicone oil. The thickness of the tapered membrane was $1.7 \mu\text{m}$, upon which the areal density was calculated [also see Ref. (3) for more information]. In the results presented here, the simulation frequency is 12 kHz, which gives a BL thickness of $12.07 \mu\text{m}$ for the fluid properties listed in Table 1. The BL correction is only used on the $z = 0$ and $z = L_z$ surfaces as $\delta/L_x, \delta/L_y < 1/500$ (as will be discussed in the next section). The mesh sizes for the problem are 602 elements in x -direction, 20 elements in z -direction, and 24 elements in y -direction for 3-D models.

3.1. Prediction from a 3-D model

In Fig. 4, the central-line structural displacement normalized to the input pressure is shown with different duct heights ($L_z = 110 \mu\text{m}$, $50 \mu\text{m}$, and $15 \mu\text{m}$). Predictions are made from the N-S model, the thick BL model (Eq. 36), and the thin BL model (Eq. 33) applied to both the $z = 0$ and $z = L_z$ surfaces. As expected, when $L_z \gg \delta$, the thin BL and thick BL approximations closely match the N-S model. However, when $L_z \sim \delta$, the thin BL model over predicts the structural displacement (see the dashed line for the $15 \mu\text{m}$ case). The thick BL model matches the N-S solution for all duct heights. Although the BL correction on the x and y surfaces is neglected, we still get good results because $L_x/\delta, L_y/\delta > 500$ in the model. In this case, including the viscous BL correction on the x and y surfaces does not improve the accuracy of the results (results not shown). In Fig. 5, the normalized (with respect to the input pressure) longitudinal fluid displacement u_x on the central-plane ($y = 0$) is shown for $L_z = 110 \mu\text{m}$. Predictions are made from the N-S model, the thick BL model, and the thin BL model on the $z = 0$ and $z = L_z$ surfaces. The fluid displacement is not directly computed in the structural acoustic models (pressure is the only variable). The method used for

estimating u_x in the fluid and producing the results in Fig. 5 is given in Appendix C. Good agreement is achieved among three models for this $110\ \mu\text{m}$ duct height. When the duct height is $50\ \mu\text{m}$, the thick BL model prediction of u_x still matches the N-S model, but the prediction from the thin BL model has noticeable differences compared to the thick BL model (results not shown) because small differences in pressure are magnified in post-processing.

Next (in Figs. 6 and 7), we explore the effect of including the thin BL correction on only one surface at $z = 0$ (Eq. 13), and compare the results to predictions from (1) a fully inviscid model and (2) to the thick BL model applied to both the $z = 0$ and $z = L_z$ surfaces (Eqs. 20 and 21). In Fig. 6, the central-line structural displacement from the inviscid model is strongly oscillatory and shows large errors compared to the thick BL solution, which is nearly identical to the N-S solution (as shown in Fig. 4). The model with one thin BL approximation at $z = 0$ clearly improves the prediction compared to the inviscid model. However, the BL approximation at $z = L_z$ is required in order to obtain accurate structural response at $z = 0$ even when $\delta/L_z \sim 1/10$ for the $L_z = 110\ \mu\text{m}$ case [see Fig. 6(a)]. When $L_z = 15\ \mu\text{m}$ [Fig. 6(b)], the difference between inviscid and BL correction (either one or both layers) models are more dramatic.

In Fig. 7, the normalized longitudinal fluid velocity v_x on the central-plane ($y = 0$) is plotted. The velocity predicted by the inviscid model (thin solid lines) does not vanish at two surfaces because the model is not subjected to the non-slip boundary condition. Using one thin BL approximation at $z = 0$ (thick solid lines) enforces zero velocity at $z = 0$ while thick BL approximation (dashed lines) affects the velocity at both $z = 0$ and $z = L_z$ surfaces. The normalized thick BL solution seamlessly transits from the thin BL case to the Poiseuille-like case [as can be seen in Fig. 7(b)].

From Fig. 4 we can see that the structural prediction from the thick BL approximation model matches well with the N-S model for various duct heights. Thus, we use the prediction from the thick BL model as a reference solution (denoted as $\hat{W}$). Then we quantify the structural displacement error for four cases: (i) the thin BL model on both $z = 0$ and $z = L_z$, one BL correction either on (ii) $z = 0$ or on (iii) $z = L_z$ and (iv) the inviscid model (all denoted as W) using root mean square error (RMSE) by

$$\text{RMSE} = \left[\frac{\sum_{i=1}^N (W_i - \hat{W}_i)^2}{\sum_{i=1}^N \hat{W}_i^2} \right]^{\frac{1}{2}} \times 100\%, \quad (38)$$

where N is the total number of points on the center line of the structure from the FEM discretization. Table 2 lists the RMSEs from different models with various L_z/δ ratios. The geometry is the same the model shown in Fig. 3. The simulation frequency is still 12 kHz. Predictions with $\text{RMSE} < 3\%$ are considered to have sufficient accuracy and are highlighted in the table.

3.2. Including shear viscosity in a cross-mode model

A cross-mode FEM model (9; 10) uses a modal solution for the fluid pressure in the width direction (y) of a duct to reduce the dimensionality of a 3-D problem to 2-D. If the geometry admits such a formulation (as the current model example does, see Fig. 3), the cross-mode FEM model is generally faster than a 3-D FEM model, but the fluid viscosity cannot be included in the direction where the modal solution is applied (e.g., the y direction in Fig. 3). The derivation of the cross-mode model with FEM implementation, including approximations of fluid viscosity as in 2.3, is given in Appendix D. In Fig. 8, the prediction

from a 3-D model (including BL approximations on the z surfaces) and a cross-mode model with 7 even cross-sectional pressure modes is compared. The duct height is $110\text{ }\mu\text{m}$ and thin BL correction is implemented. Figure 8 shows good agreement between the two models.

3.3. Computational cost

3.3.1. 3-D model—With 603 nodes in the x direction, 25 nodes in the y direction, and 21 nodes in the z direction, the N-S model can be simulated on two 25-GB Intel cores, and takes 3.05 hours to run. However, for the same mesh size, the viscous BL approximation model developed here can be simulated on a 4-GB Intel Core™2 machine (which cannot run the N-S model) with only 3.12 minutes. If accurate structural displacement is the only output quantity of interest, using 15 nodes in the y direction yields sufficient accuracy (results not shown). In this case, the N-S model can be simulated on two 4-GB Intel cores with 21.71 minutes, while the acoustic model takes 42.7 s to be simulated on the same 4-GB Intel Core™2 machine. The coupled elements for the thick BL model increase the bandwidth of the stiffness matrix in FEM, and thus may potentially increase the computational cost. In this example, however, the node numbering allows the pressure nodes at the top of the channel to be near the nodes on the bottom; hence the bandwidth is not significantly increased, and the difference in computational cost between the thin and thick BL approximations is negligible.

3.3.2. Cross-mode model—A cross-mode model with 7 cross-sectional modes gives accurate prediction compared to the 3-D model (Fig. 8) and reduces the problem size to less than $1/3$ (compared to the mesh with 25 nodes in the y direction). Since the computational cost is in the order of $O(b^2n)$ (b is the bandwidth and n is the size of the problem), using a cross-mode model further speeds the computation. Indeed, the 7-modes cross-mode model only takes 4.3 s to run on the same 4-GB machine. The only limitation for the cross-mode model is that the viscosity cannot be implemented in the y direction (not a problem in this example because $L_y/\delta > 500$), where multiple mode shapes are utilized and integrated. Hence, when the viscous BL in y has non-negligible effect, a 3-D model is required.

4. Discussion and conclusion

In the example problem used here, the structure is located on the $z = 0$ surface. Sufficiently accurate solutions ($\text{RMSE} < 3\%$) require the use of thick BL approximation (with increased complicity in the FEM formulation) if $L_z \sim \delta$ (see Fig. 4, for the $15\text{ }\mu\text{m}$ case only the thick BL approximation yields high accuracy). When δ/L_z is much less than 1, constant pressure approximation (lubrication theory) in Beltman's model (1) becomes simpler and accurate. Using thin BL approximation on both the $z = 0$ and $z = L_z$ surfaces is sufficient if L_z is larger than $\sim 5\delta$ (see Table 2 and the $50\text{ }\mu\text{m}$ case in Fig. 4). Using one BL on the $z = 0$ surface also provides sufficient accuracy if L_z is larger than $\sim 100\delta$ (Table 2). Here we can see that the influence of the viscous BL extends beyond the region of the normally-defined BL thickness $[\delta = |1/\Re(\gamma)|]$, see Figs. 6(a) and 7(a)]. Although the BL correction on $z = L_z$ is important when L_z is less than $\sim 100\delta$, including the BL correction on the flexible surface (in this case $z = 0$) is more important than including the correction on the rigid surfaces (see Table 2). In the example, the distance from the x and y surfaces to the structure is much larger than 500δ ; hence, the error for not including BLs on these two surfaces is less than $6.56 \times 10^{-3}\%$ (see Table 2). For this reason, the BL correction on the x and y surfaces are negligible in the example. The effect of different viscous BL correction (thin or thick) on the accuracy of the prediction of fluid pressure is similar to that on the structure (results not shown).

In 2.1.1, the thin viscous BL condition is derived for the $z = 0$ and $z = L_z$ surfaces (Eqs. 13 and 16), and the result is generalized to all the boundaries on Ω (Eq. 17). The generalization is based on the assumption that the BL on one surface does not couple to the BLs on the other surfaces. This assumption holds well away from the corners or the intersection of two surfaces. In general, the thin viscous BL condition is applicable to any geometry (not necessarily a rectangular box used in this example) as long as the position of interest is away from surfaces of small radius of curvature or sharp boundaries, where one BL is coupled to another. We conjecture that the extension of this approach to curved surface using the thin BL (case 1) approach is straightforward. Thus, whenever applicable, the overall approximation for the whole system is a linear superposition of the BL approximations on each surface. In this case, viscosity can be flexibly implemented either on a single surface (e.g., $z = 0$ in Eq. 13) or on multiple surfaces. Sufficient accuracy can be achieved by using thin BL approximation if the minimum dimension of a system is larger than $\sim 5\delta$ (Table 2). For the thick BL condition, the linear superposition of the BL contributions no longer holds because the BLs from 6 surfaces are coupled. This work illustrates the implementation of thick BL approximation on one pair of parallel surfaces ($z = 0$ and $z = L_z$ in this paper). If BLs are thick in all dimensions, the current method does not apply, and an alternative approximation method (e.g., Stinson (12) or Beltman (1)) should be used.

As mentioned in 2.1, in this work we only consider the out-of-plane motion of the structure. Including in-plane motion is also possible. In Appendix A the derivation and implementation of the shear viscous BL correction for in-plane structure motion is given (by taking the thin BL case on the $z = 0$ surface as an example). This correction is found to be one order (δ) less than the out-of-plane correction. Although in-plane structure displacement will introduce shear stress on the fluid due to the non-slip boundary condition, the contribution of the viscous effect on the system is typically of a lower order, as derived in Appendix A.

To summarize, a viscous BL approximation is formulated in this work to include fluid viscosity in fluid-structure coupling problems. The method can be easily implemented in FEM to convert an inviscid acoustic model solved for pressure to a viscous one without changing the primary variable. Thus, the total degrees of freedom is unchanged and the size of the problem is well-controlled. The modified boundary condition has additional terms written in second order in-plane derivatives of the pressure evaluated at the boundaries. In the FEM model, the additional terms act as added damping in the stiffness matrix. Although in this paper examples are for interior flows, the thin BL approximation is applicable to exterior flows as well.

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Appendix A. Inclusion of in-plane structure motion in the shear viscous boundary layer correction

Here we illustrate how to include and implement in-plane structure motion in the model by using the thin BL case (derived in 2.1.1) applied on the $z = 0$ surface as an example.

Suppose the structure on the $z = 0$ surface has displacement $U(x, y)$, $V(x, y)$, and $W(x, y)$ in the x , y , and z directions, respectively. Then the non-slip boundary condition on $z = 0$ in Eq. 9 is replaced by

$$v_x = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi_2}{\partial z} = j\omega U(x, y), \quad v_y = \frac{\partial \Phi}{\partial y} + \frac{\partial \Psi_1}{\partial z} = j\omega V(x, y), \quad \text{on } z=0. \quad (\text{Appendix A.1})$$

Taking the solution of Ψ_1 and Ψ_2 as given in Eq. 10, the two coefficients A_1 and A_2 are solved as

$$A_1(x, y) = -\frac{1}{\gamma} \frac{\partial \Phi}{\partial y} \Big|_{z=0} + \frac{j\omega}{\gamma} V, \quad A_2(x, y) = \frac{1}{\gamma} \frac{\partial \Phi}{\partial x} \Big|_{z=0} - \frac{j\omega}{\gamma} U. \quad (\text{Appendix A.2})$$

Following the same derivation in 2.1.1, Eq. 13 is now written as

$$\frac{\partial P}{\partial z} + \frac{1}{\gamma} \nabla_{xy}^2 P = \frac{1}{\alpha} \rho_f \omega^2 \left(W + \frac{1}{\gamma} \frac{\partial U}{\partial x} + \frac{1}{\gamma} \frac{\partial V}{\partial y} \right), \quad \text{on } z=0, \quad (\text{Appendix A.3})$$

which is the thin BL correction on $z=0$ with both out-of-plane and in-plane structure motion.

For the FEM implementation, substituting Eq. Appendix A.3 into the surface integral in Eq. 26 for $z=0$, we have

$$-\int_{z=0} \bar{P} \frac{\partial P}{\partial z} dx dy = \frac{1}{\gamma} \int_{z=0} \bar{P} \nabla_{xy}^2 P dx dy - \frac{1}{\alpha} \rho_f \omega^2 \int_{z=0} \bar{P} \left(W + \frac{1}{\gamma} \frac{\partial U}{\partial x} + \frac{1}{\gamma} \frac{\partial V}{\partial y} \right) dx dy. \quad (\text{Appendix A.4})$$

Equation Appendix A.4 differs from Eq. 27 (for the $z=0$ surface) by having the following two additional terms (forget about the factor $\rho_f \omega^2 / \alpha$ for now):

$$\frac{1}{\gamma} \int_{z=0} \bar{P} \frac{\partial U}{\partial x} dx dy \quad \text{and} \quad \frac{1}{\gamma} \int_{z=0} \bar{P} \frac{\partial V}{\partial y} dx dy. \quad (\text{Appendix A.5})$$

Taking the first term for an example (the second term follows in the same way), we can evaluate it by integration by part:

$$\frac{1}{\gamma} \int_{z=0} \bar{P} \frac{\partial U}{\partial x} dx dy = \frac{1}{\gamma} \int_0^{L_y} \left(\int_0^{L_x} \bar{P} \frac{\partial U}{\partial x} dx \right) dy = \frac{1}{\gamma} \int_0^{L_y} \left(\bar{P} U \Big|_{x=0}^{x=L_x} - \int_0^{L_x} U \frac{\partial \bar{P}}{\partial x} dx \right) dy. \quad (\text{Appendix A.6})$$

Both $U|_{x=0}$ and $U|_{x=L_x}$ vanish since they are either evaluated on the rigid boundary (if the structure does not occupy the whole x dimension) or due to the simply supported boundary condition for the structure. Hence

$$\frac{1}{\gamma} \int_{z=0} \bar{P} \frac{\partial U}{\partial x} dx dy = -\frac{1}{\gamma} \int_{z=0} U \frac{\partial \bar{P}}{\partial x} dx dy, \quad (\text{Appendix A.7})$$

which is ready to be incorporated into the FEM scheme.

Note that Eq. Appendix A.7 is of order δ (see 2.3.1 for more detail) compared to the W term in Eq. Appendix A.4. Hence, the contribution of the shear viscous effect from the in-plane displacement is a second order correction.

Appendix B. Asymptotic analysis

Consider a generic surface with normal coordinate n and in-plane (tangent) coordinates s_1 and s_2 . Inside the BL, we can scale the normal coordinate with the BL thickness $\delta(6)$ as n/δ .

Let L be a characteristic scale of the surface. $\delta \ll L$ is satisfied. Then we can introduce the following non-dimensional variables,

$$s_1^* = \frac{s_1}{L}, \quad s_2^* = \frac{s_2}{L}, \quad n^* = \frac{n}{\delta}, \quad \epsilon = \frac{\delta}{L} \ll 1, \quad P^* = \rho_f \omega^2 L^2 P. \quad (\text{Appendix B.1})$$

On the rigid walls $\Omega \Gamma$, Eq. 17 becomes

$$\frac{\partial P^*}{\partial n^*} + \frac{1}{\sqrt{2j}} \epsilon^2 \left(\frac{\partial^2 P^*}{\partial s_1^{*2}} + \frac{\partial^2 P^*}{\partial s_2^{*2}} \right) = 0. \quad (\text{Appendix B.2})$$

Expand P^* with the small parameter ϵ ,

$$P^* = P_0^* + \epsilon P_1^* + \epsilon^2 P_2^* + \dots \quad (\text{Appendix B.3})$$

Substitution of Eq. Appendix B.3 into Eq. Appendix B.2 gives the following orders of equations,

$$\epsilon^0: \frac{\partial P_0^*}{\partial n^*} = 0, \quad (\text{Appendix B.4})$$

$$\epsilon^1: \frac{\partial P_1^*}{\partial n^*} = 0, \quad (\text{Appendix B.5})$$

$$\epsilon^2: \frac{\partial P_2^*}{\partial n^*} + \frac{1}{\sqrt{2j}} \left(\frac{\partial^2 P_0^*}{\partial s_1^{*2}} + \frac{\partial^2 P_0^*}{\partial s_2^{*2}} \right) = 0. \quad (\text{Appendix B.6})$$

Hence, for the first and second order approximation we have $P/n = 0$ on the boundary. Equation 17 represents a higher order viscous BL correction.

Appendix C. Derivation of longitudinal fluid velocity in the acoustic model

In the acoustic model, pressure is the only degree of freedom in the fluid domain, and thus the fluid velocity/ displacement is not explicitly solved. In this part, we derive the longitudinal fluid velocity/displacement in the acoustic model with viscous BL approximation on the z surfaces.

In case 1 (thin viscous BL), when BL correction is applied on $z = 0$ only (Eq. 13), Eq. 4 gives

$$v_x = \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial x} \Big|_{z=0} e^{\gamma z}. \quad (\text{Appendix C.1})$$

The correction term in Eq. Appendix C.1 (second term on the RHS) decays exponentially away from $z = 0$ since the real part of γ is chosen to be negative. When BL correction is applied on $z = L_z$ only (Eq. 16), the corresponding velocity v_x is,

$$v_x = \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial x} \Big|_{z=L_z} e^{\gamma(L_z-z)}, \quad (\text{Appendix C.2})$$

Again, the correction term in Eq. Appendix C.2 decays exponentially away from $z = L_z$.

If corrections at $z = 0$ and $z = L_z$ are applied simultaneously, the fluid velocity v_x can be approximated as,

$$v_x = \frac{\partial \Phi}{\partial x} - \left. \frac{\partial \Phi}{\partial x} \right|_{z=0} e^{\gamma z} - \left. \frac{\partial \Phi}{\partial x} \right|_{z=L_z} e^{\gamma(L_z-z)}. \quad (\text{Appendix C.3})$$

which is a linear superposition of the contribution of viscous correction at each surface. When the BL is thin, $e^{\gamma z}$ (or $e^{\gamma(L_z-z)}$) essentially vanishes at $z = L_z$ (or $z = 0$). Hence, Eq. Appendix C.3 satisfies the non-slip boundary condition in an approximated manner.

In case 2 (thick viscous BL), v_x is

$$v_x = \frac{\partial \Phi}{\partial x} + \beta_2 \left[\left. \frac{\partial \Phi}{\partial x} \right|_{z=0} \sinh \gamma (L_z - z) + \left. \frac{\partial \Phi}{\partial x} \right|_{z=L_z} \sinh \gamma z \right]. \quad (\text{Appendix C.4})$$

Once the velocity v_x is obtained, the displacement u_x is given by $u_x = v_x / j\omega$.

Appendix D. Variational formulation for a cross-mode model

Similar to 2.3, in this part, we derive the the variational formulation for a cross-mode model for both the fluid and the structure, and illustrate the implementation of BL approximation in the model.

The cross-mode model uses a model decomposition in the y direction for the fluid pressure and the structure displacement, i.e.,

$$P(x, y, z) = \sum_m P_m(x, z) \cos(\sigma_m y), \quad W(x, y) = W_x(x) \cos\left(\frac{\pi \zeta}{b}\right), \quad (\text{Appendix D.1})$$

where $b = b(x)$ is the structure width, $\sigma_m = m\pi/L_y$. The coordinate ζ has its origin at the center of the membrane ($\zeta = y - L_y/2$). Without loss of generality, only the first structure mode is considered.

Integration over the y -direction of the acoustic governing equation gives,

$$\frac{\partial^2 P_m}{\partial x^2} + \frac{\partial^2 P_m}{\partial z^2} + \left(\frac{\omega^2}{\alpha c^2} - \sigma_m^2 \right) P_m = 0. \quad (\text{Appendix D.2})$$

Hence, the original 3-D problem (in x , y , and z) is reduced to a 2-D problem (in x and z).

Since BL cannot be implemented in the y -direction, only the correction on $x = 0$, L_x and $z = 0$, L_z surfaces are considered in the derivation. For thin BLs, the boundary conditions are

$$-\frac{\partial P_m}{\partial n} + \frac{1}{\gamma} \left(\frac{\partial^2 P_m}{\partial z^2} - \sigma_m^2 P_m \right) = 0, \quad \text{on } x=0, L_x, \quad (\text{Appendix D.3})$$

$$-\frac{\partial P_m}{\partial n} + \frac{1}{\gamma} \left(\frac{\partial^2 P_m}{\partial x^2} - \sigma_m^2 P_m \right) = \frac{\rho_f \omega^2 e_m A_m}{\alpha L_y} W_x, \quad \text{on } z=0, L_z. \quad (\text{Appendix D.4})$$

For thick BLs on $z = 0$, L_z , Eq. Appendix D.4 is replaced by

$$\frac{\partial P_m}{\partial z}\bigg|_{z=0} + \frac{\beta_1}{\gamma} \left(\frac{\partial^2}{\partial x^2} - \sigma_m^2 \right) P_m\bigg|_{z=0} - \frac{\beta_2}{\gamma} \left(\frac{\partial^2}{\partial x^2} - \sigma_m^2 \right) P_m\bigg|_{z=L_z} = \frac{\rho_f \omega^2 e_m A_m}{\alpha L_y} W_x\bigg|_{z=0}, \quad (\text{Appendix D.5})$$

$$\frac{\partial P_m}{\partial z}\bigg|_{z=L_z} - \frac{\beta_1}{\gamma} \left(\frac{\partial^2}{\partial x^2} - \sigma_m^2 \right) P_m\bigg|_{z=L_z} + \frac{\beta_2}{\gamma} \left(\frac{\partial^2}{\partial x^2} - \sigma_m^2 \right) P_m\bigg|_{z=0} = -\frac{\rho_f \omega^2 e_m A_m}{\alpha L_y} W_x\bigg|_{z=L_z}. \quad (\text{Appendix D.6})$$

In the above equations, A_m and e_m are defined as,

$$A_m(x) = \int_{\Gamma_b(x)} \cos(\sigma_m y) \cos\left(\frac{\pi \zeta}{b}\right) d\zeta, \quad e_m = \begin{cases} 1 & \text{for } m=0 \\ 2 & \text{for } m \neq 0 \end{cases}, \quad (\text{Appendix D.7})$$

where $\Gamma_b(x)$ represents the domain where the structure exists.

The variational form for Eq. Appendix D.2 is,

$$\int \left(\frac{\partial \bar{P}_m}{\partial x} \frac{\partial P_m}{\partial x} + \frac{\partial \bar{P}_m}{\partial x} \frac{\partial P_m}{\partial z} \right) dx dz + \left(\sigma_m^2 - \frac{\omega^2}{\alpha c^2} \right) \int \bar{P}_m P_m dx dz - \int_0^{L_x} \bar{P}_m \frac{\partial P_m}{\partial z}\bigg|_{z=0}^{z=L_z} dx - \int_0^{L_x} \bar{P}_m \frac{\partial P_m}{\partial x}\bigg|_{x=0}^{x=L_x} dz = 0, \quad (\text{Appendix D.8})$$

where $\bar{P}_m$ is the weighting function for each cross-section mode. Similar to the 3-D model, the boundary terms in Eq. Appendix D.8 can be replaced by the corresponding BL corrections [Eqs. Appendix D.3, Appendix D.4 (or Appendix D.6, Appendix D.5)].

In the structural domain, multiplication of the structural mode shape to its governing equation (Eq. 24) and integration over the y -direction gives

$$\tilde{T}_x \left(\frac{d^2 W_x}{dx^2} D_0 + 2 \frac{dW_x}{dx} D_1 + W_x D_2 \right) + \tilde{T}_y W_x E_3 + \rho_p \omega^2 D_0 W_x = -\alpha \sum_m P_m A_m, \quad (\text{Appendix D.9})$$

where the coefficients are functions of x , given by

$$\begin{aligned} D_0 &= \int_{\Gamma_b(x)} \eta \eta d\zeta, & D_1 &= \int_{\Gamma_b(x)} \frac{\partial \eta}{\partial x} \eta d\zeta, \\ D_2 &= \int_{\Gamma_b(x)} \frac{\partial^2 \eta}{\partial x^2} \eta d\zeta, & E_3 &= \int_{\Gamma_b(x)} \frac{\partial^2 \eta}{\partial y^2} \eta d\zeta. \end{aligned} \quad (\text{Appendix D.10})$$

where $\eta = \cos(\pi \zeta / b)$. The variational form of Eq. Appendix D.9 is,

$$\tilde{T}_x \int \frac{d\bar{W}_x}{dx} \frac{dW_x}{dx} D_0 dx - \int \bar{W}_x W_x \left(\tilde{T}_x D_2 + \tilde{T}_y E_3 + \rho_p \omega^2 D_0 \right) dx + \alpha \sum_m \int \bar{W}_x P_m A_m dx = 0, \quad (\text{Appendix D.11})$$

where $\bar{W}_x$ is the weighting function for the structure.

Combination of Eqs. Appendix D.8 and Appendix D.11 gives the final set of equations for the viscous cross-mode FEM model,

$$\begin{bmatrix} \mathbf{K}_f & \mathbf{K}_c \\ \mathbf{K}_c & \mathbf{K}_m \end{bmatrix} \begin{pmatrix} \mathbf{P} \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_f \\ \mathbf{f}_m \end{pmatrix}. \quad (\text{Appendix D.12})$$

The viscous BL approximation can be implemented in a cross-mode FEM model.

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Figure 1.

(Color online) Schematic of a generic fluid-structure coupled system. Ω is the interior of fluid filled duct. The boundary of the duct is denoted by $\partial\Omega$, and $\Gamma \subset \partial\Omega$ is the flexible structure on the $z = 0$ and $z = L_z$ surfaces.

Figure 2.
(Color online) Schematic of the 3-D coupling boundary layer correction. The elements on $z = 0$ are coupled to the elements on $z = L_z$ through a 2-D configuration.

Figure 3.
The dimension of the duct and the structure (not to scale).

Figure 4.

Comparison of the normalized structure displacements at the central line with respect to the input pressure among the N-S model (thin solid lines), the thin BL model (dashed lines), and the thick BL model (thick solid lines) on the $z = 0$ and $z = L_z$ surfaces with different duct heights ($L_z = 110\mu\text{m}$, $50\mu\text{m}$, and $15\mu\text{m}$). The three curves overlap in the $L_z = 110\mu\text{m}$ case.

Figure 5.

(Color online) Comparison of normalized (with respect to the input pressure) fluid longitudinal displacement u_x on the central-plane ($y = 0$) among the N-S model (thin solid lines), the thin BL model (dashed lines), and the thick BL model (thick solid lines, overlapped with dashed lines) on the $z = 0$ and $z = L_z$ surfaces. The numbers on each group of curves indicate the selected longitudinal locations in the duct. The duct height is $110\mu\text{m}$.

Figure 6.

(Color online) Central-line structural displacement normalized to the input pressure from the 3-D acoustic model. Thin solid lines: inviscid model. Thick solid lines: one thin BL correction on $z = 0$. Thick dashed lines: thick BL correction. Thin dashed lines: N-S solution. (a). $L_z = 110 \mu\text{m}$, $L_z/\delta = 9.12$. (b). $L_z = 15 \mu\text{m}$, $L_z/\delta = 1.24$.

Figure 7.

Variation of normalized longitudinal fluid velocity v_x (at $x = 30 \text{ mm}$) in the vertical direction in 3-D acoustic models. Thin solid lines: an inviscid model. Thick solid lines: one BL correction at $z = 0$. Thick dashed lines: thick BL correction. (a) $L_z = 110 \mu\text{m}$ ($L_z/\delta = 9.12$). The velocity is normalized to the velocity at $z = 5.5 \mu\text{m}$. (b) $L_z = 15 \mu\text{m}$ ($L_z/\delta = 1.24$). The velocity is normalized to the velocity at $z = 0.75 \mu\text{m}$.

Figure 8.

Comparison between a 3-D model and a cross-mode model with 7 even cross-sectional modes. The duct height is $110\ \mu\text{m}$. (a) Normalized central-line structural displacement with respect to the input pressure. (b) Normalized longitudinal fluid velocity v_x (with respect to the velocity at $z = 5.5\ \mu\text{m}$) in the central-plane ($y = 0$) at $x = 25\ \text{mm}$.

Table 1

Material properties for the example problem

Symbols	Values	Units	Physical Meaning
ρ_f	911	kg/m ³	[fluid density]
c	1000	m/s	[fluid wave speed]
μ	5×10^{-3}	kg/(s·m)	[fluid viscosity]
ρ_p	3.6×10^{-3}	kg/m ²	[plate area density]
T_x	30	N/m	[tension in x]
T_y	240	N/m	[tension in y]
η	0.01		[structural damping factor]

Table 2

Root mean square errors of structural displacement for different L_z/δ ratios and models.

L_z/δ	1	5	10	50	100	500
Model						
inviscid	8788.27%	568.31%	520.25%	64.04%	48.53%	19.21%
thin BL on $z = L_z$	301.08%	103.96%	83.28%	49.17%	45.78%	19.21%
thin BL on $z = 0$	300.80%	99.25%	68.72%	6.15%	1.40%	$6.56 \times 10^{-3}\%$
two thin BLs	203.68%	0.15%	$1.09 \times 10^{-2}\%$	~ 0	~ 0	~ 0