Force and torque on a cylinder rotating in a narrow gap at low Reynolds number: Scaling and lubrication analyses

J. Yang,^{1,a)} C. W. Wolgemuth,^{1,2,b)} and G. Huber^{1,2,3,c)}

¹*Center for Cell Analysis & Modeling, University of Connecticut Health Center, Farmington, Connecticut 06030, USA*

²*Department of Cell Biology, University of Connecticut Health Center, Farmington, Connecticut 06030, USA*

³*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*

(Received 4 August 2012; accepted 10 April 2013; published online 7 May 2013)

The hydrodynamic forces and torques on a rotating cylinder in a narrow channel are investigated in this paper using lubrication analysis and scaling analysis. To explore the effect of the shape of the gap, three different geometries are considered. The force and torque expressions from lubrication analysis agree well with numerical solutions when the gap between cylinder and wall is small. The solutions from scaling analysis can be applied over a broader range, but only if the scaling coefficients are properly deduced from numerical solution or lubrication analysis. Self-similarity in the solutions is discussed as well. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4803077>]

I. INTRODUCTION

Spheres and cylinders rotating in microfluidic channels can form the basis of micropumps.^{1–4} Likewise, long filaments rotating in narrow fluid gaps are encountered in the hydrodynamics of microorganisms,⁵ and drive the motility of an important phylum of bacteria, the spirochetes.^{6–8} In this case, the filaments are the bacterial flagella, and the gap is the *periplasm*, the narrow space between the bacterial cell wall and the outer membrane. A prime motivation for our study of rotating cylinders is to understand how the shape changes of swimming spirochetes (such as *Borrelia* and *Treponema*) arise from the forces and torques acting between the rotating flagella and the cell body⁵ (see Figure 1). Periplasmic flagella are attached near each end of the cell cylinder, and extend inward along the cell cylinder beneath the outer membrane with some overlap in the middle of the cell. Each periplasmic flagellum is connected to a rotary motor anchored in the cell wall. Rotation of the periplasmic flagella causes the cell body undulation responsible for the swimming of spirochetes. One would like to treat the flagella and cell body as flexible rods, each with their own shape degrees of freedom and elastic moduli,^{7,9} but even this picture does not completely capture the complex periplasmic environment of spirochetal flagella, sandwiched as they are between a relatively rigid peptidoglycan layer and a fluid, bilayer outer membrane. A simpler, tractable model consists of a single flagellum rotating between two concentric cylinders, which is the starting point for our considerations. The rotating filament treated in this paper is an infinite cylinder of circular cross section, and the confining space is modeled as one of three geometries of increasing complexity (i.e., increasing fidelity to the shape of the periplasmic space). We refer to these geometries as Types I, II, and III (see Figure 2). The fluid nature of the outer membrane also allows the possibility of a rotation of the concentric cylinders relative to each other.

a) jyang@haverford.edu. Present address: Department of Physics, Haverford College, Haverford, Pennsylvania 19041, USA.

b) wolg@email.arizona.edu. Present address: Department of Physics, University of Arizona, Tucson, Arizona 85721, USA.

c) huber@kitp.ucsb.edu.

FIG. 1. Schematic of the periplasmic flagella of *Borrelia burgdorferi*. (a) The *B. burgdorferi* cell body has a planar, wave-like shape. A blow-up and cut-away view of the cell shows how the flagella wrap around the cell cylinder. The flagella and cell cylinder are enclosed within the outer membrane. (b) Cross-section through the cell.

The force and torque on the filament are quantities of paramount importance for an elastohydrodynamical description of the system. Although the flagella are elastic, they are extremely rigid on lengthscales comparable to the width of the periplasmic space, and the resistive forces they experience are determined by the interaction with the fluid.¹⁰ An understanding of the forces and torques that act between the cell cylinder and the flagella is necessary for determining the morphology of a spirochete, as well as its locomotion. It is also the case that the hydrodynamic forces and torques at low Reynolds number are often counter-intuitive. Consider, for instance, an infinitely long, rigid cylinder rotating in the vicinity of a single wall, as pictured in Figure 3(a). It is not immediately evident that the horizontal component of the force on the cylinder is zero. (Jeffery¹¹ did not note this fact in his analytical solution of the problem in 1922, and it waited until Jeffrey and Onishi¹² pointed it out when they revisited Jeffery's solution in 1981.) It is perhaps then even more surprising that the force deviates from zero as a second wall (see Figure 3(b)), parallel to the first, is brought in from infinity.¹³ As we shall see in the three geometries of Figure 2, the presence of a new boundary affects the previous interactions in a nontrivial, non-additive way.

To the best of our knowledge, there are no analytical solutions available for the geometries of Figure 2. The analytical solutions of the biharmonic equation in bipolar coordinates obtained by Jeffery¹¹ apply to two cylinders rotating side by side, or one cylinder inside of another. In the case when the radius of one cylinder goes to infinity, the geometry of a cylinder rotating above an infinite plane is obtained. Further analytical work by Jeffrey and Onishi,¹² and Wannier¹⁴ confirmed and extended Jeffery's original results. Appendix A contains a graphical summary of all of these authors' results. Howland¹⁵ solved semi-numerically the case in which the cylinder is centered in the middle plane of the channel, finding convergence in only a limited range of cylinder radii. Recently, Day and Stone² studied a rotating cylinder in a narrow-channel Poiseuille flow using lubrication analysis, as well as numerics. Champmartin *et al.*¹³ also looked at the same channel geometry and via numerical calculations they were able to investigate the entire parameter space. And more recently, Götze and Gompper⁴ studied various configurations of multiple colloidal magnetic microspheres spinning in microchannels in geometries corresponding to our Types I and III.

In this paper, the problem of the confined rotating cylinder is tackled with both scaling and lubrication analyses, and numerical solutions are employed as a point of comparison for the corresponding analytical solutions. The property of self-similarity in the solutions is also discussed.

FIG. 2. Cross-sectional schematics of the confined rotating cylinder between two (a) parallel flat walls (geometry Type I); (b) concentric curved walls (geometry Type II); and (c) concentric cylinders (geometry Type III).

FIG. 3. (a) The horizontal force on a rotating cylinder parallel to a single wall is zero. (b) The introduction of a second wall makes the force non-zero (unless the cylinder lies on the middle plane of the channel).

II. SCALING ANALYSIS

When a solid body moves through a viscous fluid, forces can arise in two different ways: pressure differences across the body, or fluid shear stresses at the surface of the body. A cylinder rotating near a planar wall experiences no net force, because the pressure difference that is generated by the converging and diverging flows on either side of the gap between the cylinder and wall exactly cancels the shear forces due to the velocity gradient in the gap. When the cylinder rotates in the channel between two walls, the velocity gradients in the gaps between the cylinder and walls are only slightly perturbed as compared with the case where there is just a single wall. However, incompressibility of the fluid requires that the flux of fluid through one gap is equal and opposite to the fluid flux through the other gap. A pressure difference is therefore established across the cylinder in order to ensure the fluxes balance. It is this pressure difference that is mostly responsible for the force that the cylinder experiences when rotating between two walls. We previously showed, using a simplified model, that the pressure argument provides a reasonable estimate for the force on the cylinder as a function of position within the channel.⁵ Here we use a more rigorous analysis based on scaling arguments and lubrication theory to derive an accurate description of the forces and torques that act on the cylinder, as well as the velocity and pressure profiles in the channel. We begin with a scaling analysis, and we shall show that the force on the cylinder scales like

$$F \propto \frac{\varepsilon_b - \varepsilon_t}{\varepsilon_b^{5/2} + \varepsilon_t^{5/2}},$$

where ε_b and ε_t are the dimensionless sizes of the bottom and top gaps, respectively.

Since there is no essential difference between geometries Types I and II (see Figures 2(a) and 2(b)) as far as the scaling analysis is concerned, we focus on the details for geometry Type I, as the same procedures can be applied to geometry Type II. For all the geometries, the radius of the rotating cylinder is r , and the angular velocity of the rotating cylinder is ω . The minimum gaps between the rotating cylinder and the top and bottom walls are $\varepsilon_t r$ or $\varepsilon_b r$, respectively (see Figure 4). (Here, and in general, the subscript “t” stands for “top,” and “b” stands for “bottom.”) The scaling analysis is applied to the top and bottom gap separately. They are then combined together as a complete system subject to certain flux constraints, which must be satisfied in order to solve for the force and torque on the rotating cylinder. Since the fluid dynamics in the top and bottom gaps is essentially the same,

FIG. 4. Schematic for scaling analysis.

only the analysis for one gap needs to be presented; the same procedures are then applied to the other gap. To avoid unnecessary complication, the subscripts “*t*” and “*b*” are dropped when the results can be applied to both gaps.

Coordinate systems xy and $x'y'$ are set up in the bottom and top gaps, respectively (see Figure 4). For either the top or bottom gap, the order of magnitude of y in the lubrication region between the rotating cylinder and wall is the same as the minimum gap between them, i.e., $y \sim \varepsilon r$. The order of magnitude of x is not known initially; let us say it is L , i.e., $x \sim L$. Likewise, the order of magnitude of velocity (u, v) is clearly $u \sim \omega r$ and $v = \omega x \sim \omega L$. Then, according to the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

we must have the balance

$$\frac{\omega r}{L} \sim \frac{\omega L}{\varepsilon r},$$

which then implies that $L \sim \varepsilon^{1/2} r$. Therefore, the order of magnitude of the pressure gradient in the lubrication region is

$$\frac{\partial p}{\partial x} \sim \frac{\Delta p}{2L} = \frac{\Delta p}{2\varepsilon^{1/2} r}. \quad (1)$$

The fluid flux J through the gap is

$$J = \int_0^{\varepsilon r} u|_{x=0} dy = \frac{\varepsilon \omega r^2}{2} - \frac{(\varepsilon r)^3}{12\mu} \frac{dp}{dx} \Big|_{x=0} \quad (2)$$

in which the first term on the right side of the equation is the flux caused by the rotation of the cylinder (assuming the velocity distribution in the gap between the rotating cylinder and the stationary wall is linear between 0 and ωr), and the second term is the flux through a channel caused by a pressure gradient dp/dx . For geometries of Types I or II (see Figures 2(a) and 2(b)), the fluid fluxes through the top and bottom gaps should be equal, i.e., $J_b = J_t$. After solving the resultant flux equation, and substituting Eq. (1),

$$\frac{\varepsilon_b \omega r^2}{2} - \frac{(\varepsilon_b r)^3}{24\mu \varepsilon_b^{1/2} r} \frac{\Delta p}{2\varepsilon_b^{1/2} r} = \frac{\varepsilon_t \omega r^2}{2} + \frac{(\varepsilon_t r)^3}{24\mu \varepsilon_t^{1/2} r} \frac{\Delta p}{2\varepsilon_t^{1/2} r}$$

and the order of magnitude of the pressure difference on the two sides of the rotating cylinder is obtained as

$$\Delta p \sim \frac{12\mu\omega(\varepsilon_b - \varepsilon_t)}{\varepsilon_b^{5/2} + \varepsilon_t^{5/2}}.$$

The order of magnitude of the force on the rotating cylinder is then

$$F \sim 2r \Delta p = \frac{24\mu\omega r(\varepsilon_b - \varepsilon_t)}{\varepsilon_b^{5/2} + \varepsilon_t^{5/2}}.$$

Therefore, the force on the cylinder can be written as

$$F = C_F \frac{24\mu\omega r(\varepsilon_b - \varepsilon_t)}{\varepsilon_b^{5/2} + \varepsilon_t^{5/2}},$$

where C_F is an as-yet undetermined constant coefficient, which can be deduced from numerical solution or experimental data.

The order of magnitude of the torque on the rotating cylinder from either the top or bottom gap (T_b or T_t) can be determined as following:

$$\mu \frac{\partial u}{\partial y} 2Lr \sim \frac{2\mu\omega r^2}{\varepsilon^{1/2}}.$$

The total torque on the rotating cylinder is then the sum of the torques from the top and bottom gaps, i.e.,

$$T = T_b + T_t \sim 2\mu\omega r^2 \left(\frac{1}{\varepsilon_b^{1/2}} + \frac{1}{\varepsilon_t^{1/2}} \right).$$

Therefore, the torque on the cylinder can be written as

$$T = 2C_T\mu\omega r^2 \left(\frac{1}{\varepsilon_b^{1/2}} + \frac{1}{\varepsilon_t^{1/2}} \right), \quad (3)$$

where C_T is an undetermined constant coefficient analogous to C_F .

Likewise, for geometry Type III (see Figure 2(c)), the order of magnitude of the force on the rotating cylinder can also be obtained by scaling analysis. Comparing with geometry Type I (or II), the fluid flux difference between the top and bottom gaps ($J_b - J_t$) for geometry Type III is no longer zero but is equal to the net flux in the closed circular channel, which can be approximated as the flux in a straight channel with the same height and length as the closed circular channel under the same pressure difference. The pressure difference built up between the two sides of the rotating cylinder due to the rotation of the cylinder can be related to the flux difference by

$$J_b - J_t = \frac{\ell^3 \Delta p}{3\pi\mu(R_b + \ell)}, \quad (4)$$

where R_b is the radius of the inner circle, and ℓ is the half height of the channel. Therefore, the pressure difference Δp can be solved from Eq. (4), into which Eq. (2) is substituted, and the force can be written as

$$F \sim 2r\Delta p = \frac{24\mu\omega r(\varepsilon_b - \varepsilon_t)}{\varepsilon_b^{5/2} + \varepsilon_t^{5/2} + d},$$

where $d = \frac{8\xi_b}{\pi k^2(k + \xi_b)}$ and where $k = r/\ell$ is the confinement parameter, and $\xi_b = r/R_b$ is the dimensionless curvature of the inner circle. As for the torque on the rotating cylinder for geometry Type III, it has the same form as Eq. (3).

It can be seen from the scaling analysis that the main contribution to the force on the cylinder is the pressure difference built up on the two sides of the cylinder due to the rotation, and that viscous force surrounding the cylinder plays the major role in the torque.

III. LUBRICATION ANALYSIS

Although the results from scaling analysis agree qualitatively with numerical solutions, they miss some important details of the hydrodynamics. Lubrication analysis is therefore employed here to determine the force and torque on the rotating cylinder for the geometries Types I–III, when the center of the rotating cylinder deviates from the neutral plane of the channel. Such an analysis allows us to calculate the flow field near the surface of the cylinder, which will allow an accurate derivation of the forces and torques on the rotating cylinder. Flow characteristics far from the cylinder, such as Moffatt eddies,¹⁶ will not be captured; however, this should not affect strongly our results.

A. Geometry Type I

Similar to the variables used in the scaling analysis, the radius of the rotating cylinder is r , and the smallest gaps between the cylinder and the top and bottom wall are $\varepsilon_t r$ and $\varepsilon_b r$, respectively. To be more general, the walls can be assigned horizontal velocities, as can the rotating cylinder: v_f for the cylinder, v_t for the top wall, and v_b for the bottom wall (see Figure 5). Two sets of coordinate systems, xy and $x'y'$, are defined for the bottom and top gaps, respectively (as in Figure 4). As in the scaling analysis, lubrication theory is applied on the top and bottom gaps separately, and then the two separate parts are brought together to satisfy certain constraints, hence to determine the force and torque on the rotating cylinder. Since the procedures for the top and bottom gaps are analogous,

FIG. 5. The schematic of geometry Type I (i.e., a cylinder rotating around its axis between two parallel infinite plates) for lubrication analysis.

only one of them is presented in detail. (For the same reason, the subscripts “*b*” or “*t*” are dropped if the analysis applies to either gap.)

Stretched coordinates (X, Y) are defined by rescaling (x, y):

$$X = \frac{x}{r\varepsilon^{1/2}}, \quad Y = \frac{y}{r\varepsilon}.$$

The surface of the cylinder can be approximated as a quadratic, $Y = H = 1 + \frac{1}{2}X^2$, and then the normal and tangential vectors along the cylinder surface are

$$\hat{\mathbf{n}} = (-X\varepsilon^{1/2}, \varepsilon(H - 1) - 1), \quad \hat{\mathbf{t}} = (\varepsilon(1 - H) + 1, X\varepsilon^{1/2}),$$

respectively. The velocities and pressure are scaled as follows:

$$u = (\omega r + v)U, \quad v = (\omega r + v)V\varepsilon^{1/2}, \quad p = \mu(\omega r + v)r^{-1}\varepsilon^{-3/2}P,$$

where v is the relative translation velocity between the cylinder and walls. In the scaled variables, the Stokes equations simplified to leading order become

$$\frac{\partial^2 U}{\partial Y^2} - \frac{\partial P}{\partial X} = 0, \tag{5}$$

$$\frac{\partial P}{\partial Y} = 0, \tag{6}$$

$$\frac{\partial V}{\partial Y} + \frac{\partial U}{\partial X} = 0. \tag{7}$$

The corresponding boundary conditions are

$$Y = 0 : \quad U = 0 \quad V = 0, \tag{8}$$

$$Y = H : \quad U = 1 \quad V = \frac{\omega r X}{\omega r + v}. \tag{9}$$

The simplified Stokes equations (called the Reynolds equations) can be solved analytically.

Integrating Eq. (5) twice, and applying the corresponding boundary conditions for U , the expression for the velocity U is

$$U = \frac{1}{2} \frac{\partial P}{\partial X} (Y - H)Y + \frac{Y}{H}. \tag{10}$$

After substituting Eq. (10) into Eq. (7), integrating the resultant equation once, and applying the corresponding boundary conditions for V , we can obtain the expressions for the velocity V and the differential equation for the pressure P :

$$V = \left(\frac{1}{2H^2} + \frac{1}{4} \frac{\partial P}{\partial X} \right) XY^2 + \frac{1}{2} \frac{\partial^2 P}{\partial X^2} \left(\frac{HY}{2} - \frac{Y}{3} \right) Y^2, \tag{11}$$

$$H \frac{\partial^2 P}{\partial X^2} + 3X \frac{\partial P}{\partial X} = \frac{6X}{H^2} \left(\frac{\omega r - v}{\omega r + v} \right). \tag{12}$$

From Eq. (12), the pressure gradient $\partial P/\partial X$ can be solved for

$$\frac{\partial P}{\partial X} = \frac{3X^2}{H^3} \frac{\omega r - v}{\omega r + v} + \frac{D}{H^3}, \quad (13)$$

where D is an integration constant. Therefore, after integrating Eq. (13), the pressure P can be written as

$$P = \frac{1}{8(\omega r + v)} \left\{ 3[\omega r(D + 2) + v(D - 2)] \left(\frac{X}{H} + \sqrt{2} \tan^{-1} \frac{X}{\sqrt{2}} \right) + \frac{2X}{H^2} [\omega r(D - 6) + v(D + 6)] \right\}$$

and

$$\lim_{X \rightarrow \pm\infty} P = \pm \frac{3\pi [\omega r(D + 2) + v(D - 2)]}{8\sqrt{2}(\omega r + v)}.$$

The volume flux J through the gap is

$$J = \int_0^{\varepsilon r} u dy = \frac{\varepsilon r}{12} (\omega r + v)(6 - D).$$

The integration constants D for the top and bottom gaps can be determined by two extra constraints. One condition is that the volume flux through the top and bottom gaps should satisfy the following condition:

$$J_b - J_t = r(2 + \varepsilon_b + \varepsilon_t) \frac{v_b - v_t}{2},$$

where, on the right side of the equation, the first two factors comprise the total channel width, and the last factor is the mean velocity due to the top and bottom walls. The other condition is that the pressure at infinity ($x = +\infty$ for the bottom gap, $x' = -\infty$ for the top gap) should be identical for the two gaps, since the pressure at the same location should be the same, independent of coordinate system. The two constraint conditions result in the following system of equations:

$$\varepsilon_b(\omega r + v_b)(6 - D_b) - \varepsilon_t(\omega r + v_t)(6 - D_t) - 6(v_b - v_t)(2 + \varepsilon_b + \varepsilon_t) = 0,$$

$$[\omega r(D_b + 2) + v_b(D_b - 2)] \varepsilon_b^{-3/2} + [\omega r(D_t + 2) + v_b(D_t - 2)] \varepsilon_t^{-3/2} = 0.$$

The two integration constants D_b and D_t can be solved from the above equations, and take the form

$$D_b = \Phi(b, t), \quad D_t = \Phi(t, b)$$

in which

$$\Phi(i, j) = \frac{2 \left[v_i \left(\varepsilon_j^{5/2} - 3\varepsilon_j \varepsilon_i^{3/2} - 6\varepsilon_i^{3/2} \right) + v_j \left(6 + 3\varepsilon_i + \varepsilon_j \right) \varepsilon_i^{3/2} + \omega r \left(3\varepsilon_i^{5/2} - 4\varepsilon_j \varepsilon_i^{3/2} - \varepsilon_j^{5/2} \right) \right]}{\left(\varepsilon_i^{5/2} + \varepsilon_j^{5/2} \right) (\omega r + v_i)}$$

and where i, j = subscripts “ b ” or “ t ”, $v_t = v_i - v_f$, and $v_b = v_f - v_b$.

The force in the x direction F_x (the force in the y direction is always zero) and the torque T exerted on the cylinder due to one (or the other) gap can be determined from the following equations:

$$F_x = - \left(\int \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} ds \right)_x = \int_{-\sqrt{2}r}^{\sqrt{2}r} \left(p \frac{x}{r} + \mu \frac{\partial u}{\partial y} \right) dx, \quad (14)$$

$$T = -r \int \hat{\mathbf{t}} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} ds = r \int_{-\infty}^{+\infty} \mu \frac{\partial u}{\partial y} dx, \quad (15)$$

where σ is the stress on the cylinder surface. Therefore, the total force $\mathcal{F}$ and torque $\mathcal{T}$ on the cylinder is the summation of the forces and torques stemming from both the bottom and top gaps, i.e.,

$$\mathcal{F} = F_{xb} - F_{xt} = \frac{\mu}{\sqrt{2}(\varepsilon_b^{5/2} + \varepsilon_t^{5/2})} (\Upsilon v_f - \Xi(b, t) v_t - \Xi(t, b) v_b - \Psi \omega r), \quad (16)$$

where

$$\begin{aligned} \Upsilon &= AB - f(b, t) m(b) - f(t, b) m(t), \\ \Xi(i, j) &= A n(i, j) - g(j, i) m(j) + h(i, j) m(i), \\ \Psi &= w(b) m(b) + w(t) m(t) + A, \\ f(i, j) &= 4 \left\{ 2\varepsilon_j^{5/2} - [(\varepsilon_j - \varepsilon_i + 9)(\varepsilon_i + 3) - 18] \varepsilon_i^{1/2} \right\}, \\ g(i, j) &= 7\varepsilon_i^{5/2} - 3 [(\varepsilon_i + 3)(\varepsilon_j + 3) - 3] \varepsilon_i^{1/2} + 8\varepsilon_j^{5/2}, \\ h(i, j) &= q(i) n(i, j), \quad w(i) = q(i) C, \end{aligned} \quad (17)$$

$$\begin{aligned} m(i) &= \cot^{-1}(\sqrt{\varepsilon_i})/\sqrt{\varepsilon_i}, \quad n(i, j) = 6 + 3\varepsilon_i + \varepsilon_j, \quad q(i) = \varepsilon_i^{1/2}(3 + \varepsilon_i), \\ A &= 3(\varepsilon_b^{1/2} + \varepsilon_t^{1/2}), \quad B = 4(3 + \varepsilon_b + \varepsilon_t), \quad C = 4(\varepsilon_b - \varepsilon_t), \end{aligned}$$

$$\begin{aligned} \mathcal{T} = T_b + T_t &= \frac{\pi \sqrt{2} \mu r}{\varepsilon_b^{5/2} + \varepsilon_t^{5/2}} \left\{ \frac{C}{8} [v_f B - v_t n(b, t) - v_b n(t, b)] \right. \\ &\quad \left. - 2\omega r \left(2 + \sqrt{\frac{\varepsilon_b}{\varepsilon_t}} + \sqrt{\frac{\varepsilon_t}{\varepsilon_b}} \right) (\varepsilon_b^2 - \varepsilon_b \varepsilon_t + \varepsilon_t^2) \right\}. \end{aligned}$$

These solutions for the force and torque (Eqs. (16) and (17)) will be compared with the corresponding numerical solutions based on finite-element methods. It is interesting that the resistive force on a cylinder moving at constant velocity scales differently in the two-wall case than it does in the single-wall case.

B. Geometry Types II and III

When the walls of the channel are curved (i.e., geometries Types II and III (see Figures 2(b) and 2(c))), the complication of finite curvature can affect the force and torque on the rotating cylinder. The procedures for deriving the force and torque on the rotating cylinder are similar to those for the zero-curvature walls in Sec. III A. With the same coordinate systems as in the flat-wall case, the surface of the curved wall can be approximated as

$$G = -\frac{\xi}{2} X^2,$$

where $\xi = r/R$ is the dimensionless curvature, and R is the radius of the curved wall. $\xi = 0$ for a flat wall, $\xi > 0$ for a convex wall, and $\xi < 0$ for a concave wall. (The same procedure can be generalized to cases with either two convex or two concave walls—the only difference is the sign of the ξ .) Therefore, comparing with the case of flat walls, the governing equations for curved-wall cases are the same as Eqs. (5)–(7), but the corresponding boundary conditions, Eqs. (8) and (9), are modified to

$$Y = G : \quad U = 0 \quad V = 0, \quad (18)$$

$$Y = H : \quad U = 1 \quad V = X. \quad (19)$$

(Only stationary walls, and cylinders rotating without translation ($v = 0$) are considered in this paper for geometries Types II and III.) After solving the governing equations with the appropriate boundary conditions, we can obtain the pressure distribution along the gap and its limiting values:

$$P = \frac{1}{4(1+\xi)} \left\{ \frac{4X[(\Gamma+6)\xi + (\Gamma-6)]}{[2 + (1+\xi)X^2]^2} + \frac{3X[(\Gamma-2)\xi + (\Gamma+2)]}{2 + (1+\xi)X^2} + \frac{3[(\Gamma-2)\xi + (\Gamma+2)] \tan^{-1}(\sqrt{(1+\xi)/2}X)}{[2(1+\xi)]^{1/2}} \right\} \quad (20)$$

and

$$\lim_{X \rightarrow \pm\infty} P = \pm \frac{3\pi[(\Gamma+2) + \xi(\Gamma-2)]}{8\sqrt{2}(1+\xi)^{3/2}},$$

where Γ is an integration constant. Moreover, the two integration constants from the top and bottom curved gaps can be determined from the pressure and flux constraints, i.e.,

$$\frac{(\Gamma_b - 2)\xi_b + (\Gamma_b + 2)}{[\varepsilon_b(1 + \xi_b)]^{3/2}} + \frac{(\Gamma_t - 2)\xi_t + (\Gamma_t + 2)}{[\varepsilon_t(1 + \xi_t)]^{3/2}} = 0, \quad (21)$$

$$\frac{\omega r^2}{12} [\varepsilon_b(6 - \Gamma_b) - \varepsilon_t(6 - \Gamma_t)] = J_{total}, \quad (22)$$

where J_{total} is the net fluid flux. For geometry Type II (see Figure 2(b)), $J_{total} = 0$, but for geometry Type III (see Figure 2(c)), the net fluid flux J_{total} should be equal to the flux in the circular channel between the concentric cylinders (caused by the pressure gradient built up on the two sides of the rotating cylinder). The net fluid flux for geometry Type III can be approximately written as

$$J_{total} = \frac{\ell^3(p(+\infty) - p(-\infty))}{3\pi\mu\mathcal{R}},$$

where $\mathcal{R}$ is the radius of the middle plane, $\mathcal{R} = (R_b + R_t)/2$. After the integration constants are determined from the resultant system of equations (Eqs. (21) and (22)), the force and torque can be determined according to Eqs. (14) and (15). Therefore, the total force on the rotating cylinder for geometry Type II can be written as

$$\mathcal{F} = \frac{\sqrt{2}\mu\omega r}{Q}(S(b, t) - S(t, b)) \quad (23)$$

and, by the same token, the total force on the rotating cylinder for geometry Type III is

$$\mathcal{F} = \mu\omega r(N(b, t) - N(t, b)), \quad (24)$$

where the formulae for the functions $S(i, j)$, Q , and $N(i, j)$ can be found in Appendix B. The torque on the rotating cylinder for geometry Type II is

$$\mathcal{T} = \frac{-\pi\sqrt{2}\mu\omega r^2 Z}{Q} \quad (25)$$

and the torque on the rotating cylinder for geometry Type III is

$$\mathcal{T} = \frac{-2\pi\mu\omega r^2}{\Lambda} \left[3\sqrt{2}\ell^3 \left(\frac{1 + \xi_b}{\sqrt{\varepsilon_t(1 + \xi_t)}} + \frac{1 + \xi_t}{\sqrt{\varepsilon_b(1 + \xi_b)}} \right) + r^2\mathcal{R}Z \right], \quad (26)$$

where the functions Z and Λ are given in Appendix B.

IV. RESULTS AND DISCUSSION

The solutions from scaling analysis and lubrication analysis can be compared with full numerical solutions in the corresponding geometries. The finite-element method software COMSOL Multiphysics (COMSOL, Inc., Burlington, Massachusetts, USA) was used to obtain the numerical

solutions. No-slip boundary conditions are applied on the parallel walls, upstream and downstream of the channel, as well as on the rotating cylinder. Triangular meshes are adopted for spatial discretization, quadratic and linear basis functions are applied to velocity and pressure fields, respectively, and a special COMSOL script was designed to change the geometry automatically for different eccentricities. Meshes are generated by COMSOL's built-in mesh generator with default settings that resolve the geometry adaptively. There are approximately 20 000 triangular elements used in the simulation. For geometries Types I and II, the channels are extended to infinity. For our numerical work, however, we follow the heuristic laid out by Dvinsky and Popel:¹⁷ "In the numerical solution, the infinity boundary conditions are imposed at a finite distance from the cylinder. Typically, 1.5-3 channel widths on each side of the particle are sufficient for the calculated characteristics to be independent of channel length." A few cases were implemented to validate the numerical results presented in Champmartin *et al.*,¹³ and consistent results were obtained. The numerical solutions obtained with COMSOL were therefore employed as a reference solution for comparison with the corresponding lubrication and scaling solutions.

A. Scaling analysis

The force or torque on the cylinder varies with its position relative to the walls of the channel. The effect of the displacement of the cylinder from the middle plane of the channel, the effect of the radius of the cylinder, and the effect of the curvature of the channel walls on the force and torque are investigated in this paper. If the distance between the cylinder's axis and the symmetry plane of the parallel walls is c and the distance between the top and bottom walls is 2ℓ , the eccentricity is defined as $e = c/\ell$.

The force and torque from the scaling analysis for geometry Type I are compared with the corresponding numerical solutions at different confinement parameters k (see Figures 6(a) and 6(b)). It was found that the scaling results with proportionality coefficients $C_F = -1.14$ (for force: obtained from best fit of the scaling solution to the corresponding numerical solution at different confinement parameters k), and $C_T = -\pi\sqrt{2}$ (for torque: obtained from comparing the scaling solution Eq. (3) with the zeroth order term of the lubrication solution Eq. (17), see Appendix C) match well with the numerical solutions for most k values. Scaling solutions agreed well with the corresponding numerical solutions over a larger range of k compared with those from lubrication analysis. Furthermore, the force and torque curves from the scaling analysis at different k values can be collapsed onto one universal curve for geometry Type I using the scaling factors $(1/k - 1)^{3/2}$ for force and $(1/k - 1)^{1/2}$ for torque.

Symmetry is broken for the cases with curved walls, i.e., geometry Types II and III, and the eccentricity changes sign when the cylinder cross the middle plane of the channel. Since the scaling analysis of geometry Type II is essentially the same as that of geometry Type I, only the latter are

FIG. 6. Dimensionless (a) force and (b) torque on the rotating cylinder as a function of eccentricity at different confinement parameter k values (scaling analysis).

FIG. 7. Dimensionless force on the rotating-only cylinder as a function of eccentricity at different confinement parameter values k for the geometry Type III from scaling analysis when the dimensionless curvature $\xi_b = 0.05$.

shown here. Figure 7 shows the dimensionless force as a function of eccentricity at different k values for geometry Type III with fixed dimensionless curvature $\xi_b = 0.05$. The scaling coefficient C_F keeps the same value as in the cases with flat walls, but the force curves at different confinements k can no longer be collapsed onto one universal curve as before. Scaling-analysis solutions agree qualitatively with the corresponding numerical solutions, though some subtle features are missed by the scaling analysis, such as the force divergence when the cylinder approaches the convex or concave walls, and the non-zero force when the cylinder cross the middle plane of the channel. Since the torque results from the scaling analysis for all the geometries in Figure 2 are essentially the same, the Type III scaling-analysis torques are not presented here.

B. Lubrication analysis

1. Geometry Type I

Figure 8(a) shows the force on the rotating cylinder as a function of eccentricity at different confinements k when the cylinder is rotating without translation. The symbols (cross, circle, square, and plus) in the figure represent the numerical solutions based on COMSOL's finite-element method, and the lines (dotted, dotted-dashed, dashed, and solid line) represent the corresponding asymptotic solutions from lubrication analysis. The force on the cylinder increases with eccentricity and reaches a plateau after a certain eccentricity. Also, the results from lubrication analysis agree better with the

FIG. 8. Dimensionless (a) force and (b) torque on the rotating cylinder as a function of eccentricity at different confinement parameter k values for the geometry Type I (lubrication analysis).

numerical solutions at high k values, i.e., when the gaps between the cylinder and walls are relatively small. This is understandable due to the assumptions on which lubrication analysis is based. The force curves at confinements k are scaled onto one universal curve by the scaling factor $(1/k - 1)^{3/2}$.

Figure 8(b) shows the corresponding torque on the rotating cylinder as a function of eccentricity for the rotating cylinder. It can be seen that the torque on the cylinder increases with the eccentricity monotonically. The symbol markers in the figure represent the numerical solutions at different confinements k , and the lines represent the corresponding results from the lubrication analysis. Similarly, torque curves are collapsed by the scaling factor $(1/k - 1)^{1/2}$ only at high values of k ; therefore, the torque does not show self-similarity over the whole range of k , but only in the regime $k > 0.5$. Torque resistance on the rotating cylinder is mainly caused by viscous forces from the fluid, and viscous force is a type of short-range force compared to pressure forces. Small k values means that the rotating cylinder diameter is much less than the channel height. Thus, the rotating cylinder can be considered as being immersed in an infinite fluid. The viscous force only becomes important when the rotating cylinder is close to the wall, therefore the eccentricity affects the torque only slightly when the rotating cylinder is around the middle plane of the channel. This explains why the torque curves can be collapsed into one curve only in the high- k regime and not in the small- k regime—the torque is dominated by viscous force only in the high- k value regime.

In order to pin down the underlying physics, the pressure distributions along the top and bottom gaps are also explored. Figure 9 shows typical pressure distributions along the top and bottom gaps. The pressure distributions between the highest peaks and the lowest valleys are very close to linear functions. The force on the cylinder is mainly attributed to the pressure difference across the two sides of the cylinder, which is caused by the rotation of the cylinder, and the fluid flow has to satisfy the continuity requirement. When the cylinder deviates far from the middle plane of the channel, such as the configuration of Figure 9(a), the top gap is much smaller than the bottom gap. The fluid is pushed through the top gap (and the bottom gap) by the rotation of the cylinder. Note that part of the top gap formed between the cylinder and the top wall leads to converging flows, in which the pressure increases, forming the maximum in the pressure distribution, and the other part of the top gap leads to diverging flows, in which the pressure decreases, forming the minimum in the pressure distribution. Since the bottom gap is much wider than the top gap, the converging and diverging parts of the gap do not affect the pressure distribution dramatically, causing only a pressure drop. When the cylinder is moved toward the middle plane of the channel (see Figure 9(b)), the converging and diverging parts of the bottom channel will also create a maximum and a minimum in the pressure distribution. When the cylinder is exactly on the middle plane of the channel (see Figure 9(c)), the pressure distributions in the top and bottom gaps are symmetric—actually, anti-symmetric—so the pressures cancel each other and there is no net force on the cylinder.

For the case of a cylinder translating without rotation, the torque on the translating cylinder as a function of eccentricity is shown in Figure 10(a). Lines represent solutions from the lubrication analysis and the symbol markers represent numerical data. The torque on the translating cylinder increases with eccentricity, and the data can be collapsed with the scaling factor $(1/k - 1)^{3/2}$, identical to that for the force on the rotating cylinder. The numerics for the force on the rotating cylinder

FIG. 9. Typical pressure distributions along the bottom and top gaps for geometry Type I (rotating cylinder with no translation) when the rotating cylinder is located: (a) close to the top gap; (b) around the middle plane of the channel; and (c) on the middle plane of the channel.

FIG. 10. Dimensionless (a) torque and (b) force on the translating cylinder as a function of eccentricity for different confinements k (geometry Type I, lubrication analysis).

are also shown in Figure 10(a) for comparison. It can be seen that the force on the rotating-only cylinder is identical with the torque on the translating-only cylinder, i.e., they satisfy the reciprocity theorem. Figure 10(b) shows the force on the translating-only cylinder as a function of eccentricity at different confinements k . The force decreases with eccentricity, since the deviation of the cylinder from the middle plane of the channel allows one gap to be larger than the other gap; the larger gap acts as a short circuit, with more fluid passing through the larger gap to relieve the pressure, hence reducing the resistance force. For the force on the translating cylinder at zero eccentricity, Faxen's solution¹⁸ works well for small confinement parameter $k < 0.5$ and breaks down for large k , while our solution works better for high k and leads to large errors for small k . These two can be viewed as complementary solutions. The scaling factor for the force on the translating-only cylinder is $(1-k)^{5/2}/(k(1+k))$. The deviation of the numerical solutions from the lubrication solutions at high eccentricity is because the lubrication analysis starts to break down when either of the gaps between the cylinder and walls becomes too large. However, the lubrication results still agree well with the numerical solutions for high confinement parameters k over a broad range of eccentricities.

2. Geometry Types II and III

Figure 11 shows the force and torque on the rotating cylinder for geometry Type II as a function of eccentricity at different confinements k . Here, the main features are that the force increases with eccentricity, changes direction when the eccentricity switches sign, and diverges when the rotating cylinder approaches the walls (this feature is well reproduced in the lubrication analysis).

FIG. 11. Dimensionless (a) force and (b) torque on the rotating cylinder as a function of eccentricity for geometry Type II at fixed dimensionless curvature $\xi = 0.05$ (lubrication analysis).

FIG. 12. The effect of the dimensionless curvature ξ on the (a) force and (b) torque on the rotating cylinder for geometry Type II, with fixed confinement $k = 0.8$.

The divergence of the force can be explained by the leading-order term in an expansion in small gap width between the cylinder and wall. The leading term of this series is $\sim \xi / ((1 + \xi)^{5/2} \epsilon^{1/2})$. Therefore, the force diverges as the gap width goes to zero for a curved wall; however, this term is zero when the wall curvature is zero. The deviations of the lubrication solutions from the numerical solutions decrease with increasing confinement k , and the solutions are almost indistinguishable for $k = 0.9$. The force curves at different k can be collapsed onto one curve by the scaling factor $(1/k - 1)^{3/2}$. The torque on the rotating cylinder is symmetric with respect to eccentricity, and the torque increases with the magnitude of the eccentricity. The torque curves at different k can be collapsed onto one curve by the scaling factor $(1/k - 1)^{1/2}$.

Figure 12 presents the effect of the dimensionless curvature ξ on the force and torque curves for geometry Type II when the confinement k is fixed at 0.8 ($\xi = 0$ represents flat walls, i.e., geometry Type I). It can be seen that the curvature ξ does not dramatically affect the force or torque on the rotating cylinder as there are only slight differences between the curves at different ξ . In contrast with geometry Type I, the force on the rotating cylinder for geometry Type II is no longer zero when the cylinder is situated in the middle of the channel, even though, for the cases shown in Figure 12(a), those forces are relatively small. However, the force on the rotating cylinder *does* vanish on a cylindrical surface slightly offset from the middle of the channel. The positional offset of this surface depends on ξ and k , but it can be described by its eccentricity $e/e_{\max}$ (see Appendix D). For small values of the dimensionless curvature ξ , the eccentricity of the neutral surface is (to leading order) quadratic in ξ over a wide range of confinement parameters k (Figure 13). In the limit $k \rightarrow 1$,

FIG. 13. Neutral-surface eccentricity $(e/e_{\max})_0$ (the surface position where the force on the rotating cylinder is zero) as a function of dimensionless curvature ξ showing data collapse at different confinement parameters k for geometry Type II.

linearization of Eq. (23), leads to this expression for the root

$$(e/e_{\max})_0 = \frac{\xi^2}{2 + 7\xi + 4\xi^2} \equiv E(\xi),$$

which is accurate to $O(\xi^5)$ for small ξ . To leading order, $E(\xi) \simeq \xi^2/2$, and, for comparison, the analogous results for confinement parameters of $k = 1/2$ and $k = 2/3$ are $0.3026 \xi^2$ and $0.3549 \xi^2$ (from Taylor expansion), indicating a more complex k dependence. As Figure 13 suggests, the rescaling of the quadratic term can be extended to other k values, and, as a simple mnemonic, this product of independent functions:

$$(e/e_{\max})_0 \cong \phi(k)E(\xi) = \frac{\phi(k)\xi^2}{2} + \dots, \quad (27)$$

with $\phi(k) = e^{k-1}$, captures the general trend over the range $k = 1$ down to at least $k = 0.4$.¹⁹

Figure 14 shows the force on the rotating cylinder in geometry Type III at different confinements k with fixed dimensionless curvature $\xi = 0.05$. The force curves for geometry Type III have essentially the same features as those for geometry Type II, but tend to be nearly linear as k approaches 1. However, force curves for geometry Type III cannot be scaled onto one universal curve. It can be seen that the forces for both geometries Types II and III diverge as the eccentricity approaches 1. This is because the leading-order term for the force as ε approaches zero is $O(\xi/\sqrt{\varepsilon})$. But for flat walls, $\xi = 0$ (i.e., geometry Type I), and this first term disappears, so there is no force divergence phenomena in the geometry Type I.

The corresponding torques on the rotating cylinder for geometry Type III are shown in Figure 15, and the torque curves at different confinements k can be collapsed onto one curve

FIG. 14. Dimensionless force on the rotating cylinder for geometry Type III at different confinements k : (a) 0.60, (b) 0.74, (c) 0.80, and (d) 0.90 (dimensionless curvature $\xi = 0.05$).

FIG. 15. Dimensionless torque on the rotating cylinder for geometry Type III at different confinements k (dimensionless curvature $\xi = 0.05$).

by using the scaling factor $(1/k - 1)^{1/2}$. The torque curves for geometry Type III are similar to those for geometry Type II.

Figure 16 shows the effect of the dimensionless curvature on the force and torque on the rotating cylinder for geometry Type III for fixed confinement $k = 0.8$. It can be seen that the force on the rotating cylinder decreases as the dimensionless curvature ξ is increased from 0.05 to 0.1 (see Figure 16(a)). However, the dimensionless curvature has no significant effect on the torque on the rotating cylinder (see Figure 16(b)). As the dimensionless curvature ξ decreases, the enclosed circumference in the loop of the channel increases, and the resistance to flow increases, therefore higher pressure is accumulated on the two sides of the rotating cylinder thus increasing the force. On the contrary, when ξ is high, the accumulated pressure on the two sides of the rotating cylinder can be relieved, as fluid flows in the loop of the channel due to its shorter circumference and resistance, thus decreasing the force.

The effect of the overall geometric type on the force and torque on the rotating cylinder under the same dimensionless curvature and confinement parameter is shown in Figure 17. There it can be seen that the forces in geometry Type III are much smaller than in geometries I and II, and that the forces for geometries I and II are nearly equal (see Figure 17(a)). This is because more energy is dissipated in geometry Type III due to the fluid flow circulating in the closed loop of the channel. The torque on the rotating cylinder for geometry Type III is slightly smaller than that for geometries Types I and II in the high-eccentricity region (see Figure 17(b)). However, they are basically equivalent. When the rotating cylinder is centered on the neutral surface of the channel in geometry Type III,

FIG. 16. The effect of the dimensionless curvature ξ on the (a) force and (b) torque on the rotating cylinder for geometry Type III for fixed confinement $k = 0.8$.

FIG. 17. The effect of geometry type on the (a) force and (b) torque on the rotating cylinder with fixed dimensionless curvature ($\xi = 0.05$) and confinement parameter ($k = 0.8$).

the pressure difference between the two sides of the rotating cylinder becomes zero, the force on the rotating cylinder becomes zero, as does the fluid flux in the loop of the channel. Therefore, the torque in this case is equal to that in geometry Type I when the rotating cylinder is located exactly on the middle plane of the channel (or on the neutral surface in geometry Type II). However, in the high-eccentricity regime of geometry Type III, the fluid flux in the loop of the channel (driven by the pressure difference on the two sides of the rotating cylinder) cancels part of the fluid flow in the smaller gap, so as to reduce the velocity gradient in the smaller gap, as well as the torque on the rotating cylinder (since the torque is mainly contributed to by the viscous force in the smaller gap). This is the reason why the torques in Figure 17(b) are almost the same for all geometry types near zero eccentricity, but slightly smaller for geometry Type III at high eccentricities.

V. CONCLUSION

The force and torque on a rotating (or translating) cylinder between two walls have been found using both scaling and lubrication analyses. To gauge the effect of channel shape, three different wall geometries have been explored. The results compare well with the corresponding finite-element method simulation results. The force and torque on the cylinder display a certain degree of self-similarity, with the force on the cylinder being mainly caused by the pressure difference on the two sides of the cylinder, and with the viscous force being the main contributor to the torque on the cylinder. Scaling analysis is useful for capturing the essential physics behind the phenomena, and its results are accurate enough for many practical applications. The solutions for the force and torque can be employed in more realistic simulations of spirochete morphology and motility, as well as for the future design of microfluidic devices.

ACKNOWLEDGMENTS

This research was partially supported by National Institutes of Health Grant No. NIH RO1 GM072004 (C.W.W.). Partial support from the National Science Foundation under Grant Nos. NSF PHY05-51164 and PHY11-25915 (KITP, G.H.) is also gratefully acknowledged, as is support from the Richard Berlin Center for Cell Analysis & Modeling (J.Y. and G.H.).

APPENDIX A: FORCE AND TORQUE RESULTS FOR A ROTATING CYLINDER NEAR A SINGLE SURFACE

The analytic solutions for force and torque on a rotating cylinder of radius r next to or inside another cylinder of radius R by Jeffery,¹¹ Jeffrey and Onishi,¹² and Wannier¹⁴ are summarized in Figure 18. The solid lines in the figures represent the force and torque on the rotating cylinder

FIG. 18. Summary of the solutions for (a) force and (b) torque on a rotating cylinder outside or inside another cylinder. $\xi = 0$ corresponds to a cylinder parallel to a planar wall.

at different dimensionless gaps ε between these two cylinders. When the dimensionless curvature $\xi = r/R$ is negative, the rotating cylinder is outside the other cylinder (to the left of the dashed line in the figures), otherwise it is inside the other cylinder (to the right of the dashed line in the figures). When $\xi = 0$ ($R \rightarrow \infty$), the cylinder is rotating above an infinite plane, and the force on the rotating cylinder is zero.

APPENDIX B: FUNCTION DEFINITIONS

The functions used in Eqs. (23)–(26) are

$$Q = (1 + \xi_b)(1 + \xi_t) \left[\varepsilon_b^{5/2} (1 + \xi_b)^{1/2} + \varepsilon_t^{5/2} (1 + \xi_t)^{1/2} \right],$$

$$\Lambda = 3\ell^3 (1 + \xi_b)(1 + \xi_t) + \sqrt{2}r^2 \mathcal{R}Q,$$

$$Z = \frac{2[\varepsilon_b^3(1 + \xi_b)^2 + \varepsilon_t^3(1 + \xi_t)^2]}{\sqrt{\varepsilon_b \varepsilon_t (1 + \xi_b)(1 + \xi_t)}} + \xi_t(2\varepsilon_b - \varepsilon_t)^2 + \xi_b(2\varepsilon_t - \varepsilon_b)^2 + (4 + \xi_b \xi_t)(\varepsilon_b - \varepsilon_t)^2 + (4 + \varepsilon_t + \varepsilon_b)\varepsilon_b \varepsilon_t,$$

$$S(i, j) = \frac{1}{\varepsilon_i^{1/2}(1 + \xi_i)^{3/2}} \left\{ \frac{1}{1 + \varepsilon_i + \xi_i} \left[4\xi_i \varepsilon_i^{1/2} \varepsilon_j^{5/2} (\xi_j - 1)(1 + \xi_j)^{1/2} (1 + \xi_j)^{3/2} - \varepsilon_i \varepsilon_j (2 + \xi_j) \right. \right. \\ \left. \left. (1 + \xi_i)^2 (3(1 + \varepsilon_i)(1 + \xi_i) - \varepsilon_i \xi_i) + 3\varepsilon_i^2 (1 + \xi_i)(1 + \xi_j)(2\varepsilon_i \xi_i^2 + (1 + \xi_i + \varepsilon_i)(2 + \xi_i)) \right] + \right. \\ \left. \left[8\varepsilon_j^{5/2} (1 + \xi_j)^{3/2} \xi_i + \varepsilon_i^{5/2} (1 + \xi_i)^{1/2} (1 + \xi_j)(2 + 13\xi_i + 2\xi_i^2) - \varepsilon_j \varepsilon_i^{1/2} (2 + \xi_j)(1 + \xi_i)^{3/2} \right. \right. \\ \left. \left. ((3 + \varepsilon_i)(1 + \xi_i) + \varepsilon_i \xi_i) + 3\varepsilon_i^{3/2} (1 + \xi_i)^{3/2} (2 + \xi_i)(1 + \xi_j) \right] \tan^{-1} \left(\sqrt{\frac{1 + \xi_i}{\varepsilon_i}} \right) \right\},$$

$$N(i, j) = \frac{-1}{2\sqrt{2}\varepsilon_i^{3/2} (1 + \xi_i)^{5/2} (1 + \varepsilon_i + \xi_i)} \left\{ \tan^{-1} \left(\sqrt{\frac{1 + \xi_i}{\varepsilon_i}} \right) [(2 + W(i, j))(3 + \varepsilon_i) + (34\varepsilon_i + \right. \\ \left. 3W(i, j)(2 + \varepsilon_i)) \xi_i + (W(i, j) - 2)(3 + 2\varepsilon_i) \xi_i^2] (1 + \varepsilon_i + \xi_i) + \varepsilon_i^{1/2} (1 + \xi_i)^{1/2} [W(i, j) \right. \\ \left. (1 + \xi_i)(3 + 3\varepsilon_i + 3\xi_i + 2\varepsilon_i \xi_i) + 6(1 + \varepsilon_i - 3\varepsilon_i \xi_i + (2\varepsilon_i - 1)\xi_i^2)] \right\},$$

where

$$W(i, j) = \frac{2}{\Lambda} \left\{ 3\ell^3(\xi_i - 1)(1 + \xi_j) + \sqrt{2}r^2\mathcal{R} \left[3\varepsilon_i^{5/2}(1 + \xi_i)^{3/2}(1 + \xi_j) + \varepsilon_j^{5/2}(\xi_i - 1)(1 + \xi_j)^{3/2} - 2\varepsilon_j(\varepsilon_i(1 + \xi_i))^{3/2}(2 + \xi_j) \right] \right\}.$$

APPENDIX C: DETERMINATION OF THE SCALING COEFFICIENT C_T

Set v_f , v_b , and v_t equal to zero in Eq. (17), resulting in the torque on the rotating cylinder:

$$\begin{aligned} T &= -\frac{2\pi\sqrt{2}\mu\omega r^2}{\varepsilon_b^{5/2} + \varepsilon_t^{5/2}} \frac{(\sqrt{\varepsilon_b} + \sqrt{\varepsilon_t})^2}{\sqrt{\varepsilon_b\varepsilon_t}} (\varepsilon_b^2 - \varepsilon_b\varepsilon_t + \varepsilon_t^2) \\ &= -2\pi\sqrt{2}\mu\omega r^2 \left(\frac{1}{\varepsilon_b^{1/2}} + \frac{1}{\varepsilon_t^{1/2}} \right) \frac{\varepsilon_b^2 - \varepsilon_b\varepsilon_t + \varepsilon_t^2}{\varepsilon_b^2 + \varepsilon_b\varepsilon_t + \varepsilon_b^2 - \varepsilon_b^{3/2}\varepsilon_t^{1/2} - \varepsilon_b^{1/2}\varepsilon_t^{3/2}} \\ &= -2\pi\sqrt{2}\mu\omega r^2 \left(\frac{1}{\varepsilon_b^{1/2}} + \frac{1}{\varepsilon_t^{1/2}} \right) \frac{1}{1 - \delta} \\ &\sim -2\pi\sqrt{2}\mu\omega r^2 \left(\frac{1}{\varepsilon_b^{1/2}} + \frac{1}{\varepsilon_t^{1/2}} \right) (1 + \delta), \end{aligned} \quad (C1)$$

where

$$\delta = \frac{\sqrt{\varepsilon_b\varepsilon_t}(\sqrt{\varepsilon_b} - \sqrt{\varepsilon_t})^2}{\varepsilon_b^2 - \varepsilon_b\varepsilon_t + \varepsilon_t^2}.$$

Comparing the zeroth order term of Eq. (C1) with the scaling solution Eq. (3), we find the scaling coefficient for the torque on the rotating cylinder $C_T = -\pi\sqrt{2}$.

APPENDIX D: NEUTRAL SURFACE POSITION

For geometry Type II, the force on the rotating cylinder becomes zero only at a certain position offset from the middle plane of the channel, and this position in terms of eccentricity can be solved for from the linear approximation of the force equation (Eq. (23)). Substituting the following equations into the force equation (Eq. (23)):

$$\varepsilon_b = \frac{(1 + \lambda)(1 - k)}{k}, \quad \varepsilon_t = \frac{(1 - \lambda)(1 - k)}{k}, \quad \xi_t = \frac{\xi_b k}{k + 2\xi_b},$$

where $\lambda = e/e_{max}$, we can obtain the expression for the force on the rotating cylinder in terms of eccentricity λ , confinement parameter k , and dimensionless curvature ξ_b . To obtain the neutral surface position, we have to solve for λ when the force (Eq. (23)) equals zero. However, it is very difficult to solve for λ directly from Eq. (23) due to its strong non-linearity. So, we expanded the force expression into a Taylor series for λ around 0, kept only the zeroth and linear order terms, such as $a_0(k, \xi_b) + a_1(k, \xi_b)\lambda$, and obtained the neutral position as $\lambda = -a_0(k, \xi_b)/a_1(k, \xi_b)$. The specific lengthy expressions for $a_0(k, \xi_b)$ and $a_1(k, \xi_b)$ cannot be presented here; readers can obtain these expressions easily by following the procedure mentioned above. Comparing with numerical solution of the force equation, the error for the linear approximation is $\sim O(\xi_b^4)$.

¹ M. Sen, D. Wajerski, and M. Gad-el-Hak, "A novel pump for MEMS applications," *ASME J. Fluids Eng.* **118**, 624–627 (1996).

² R. F. Day and H. A. Stone, "Lubrication analysis and boundary integral simulations of a viscous micropump," *J. Fluid Mech.* **416**, 197–216 (2000).

³ B. Felderhof, "Swimming and peristaltic pumping between two plane parallel walls," *J. Phys.: Condens. Matter* **21**, 204106 (2009).

- ⁴I. O. Götze and G. Gompper, “Dynamic self-assembly and directed flow of rotating colloids in microchannels,” *Phys. Rev. E* **84**, 031404-1–031404-11 (2011).
- ⁵J. Yang, G. Huber, and C. W. Wolgemuth, “Forces and torques on rotating spirochete flagella,” *Phys. Rev. Lett.* **107**, 268101-1–268101-4 (2011).
- ⁶N. W. Charon, S. F. Goldstein, S. M. Block, K. Curci, J. D. Ruby, J. A. Kreiling, and R. J. Limberger, “Morphology and dynamics of protruding spirochete periplasmic flagella,” *J. Bacteriol.* **174**, 832–40 (1992).
- ⁷C. Dombrowski, W. Kan, M. A. Motaleb, N. W. Charon, R. E. Goldstein, and C. W. Wolgemuth, “The elastic basis for the shape of *Borrelia burgdorferi*,” *Biophys. J.* **96**, 4409–4417 (2009).
- ⁸N. W. Charon, S. F. Goldstein, M. Marko, C. Hsieh, L. L. Gebhardt, M. A. Motaleb, C. W. Wolgemuth, R. J. Limberger, and N. Rowe, “The flat-ribbon configuration of the periplasmic flagella of *Borrelia burgdorferi* and its relationship to motility and morphology,” *J. Bacteriol.* **191**, 600–607 (2009).
- ⁹W. Kan and C. W. Wolgemuth, “The shape and dynamics of the *Leptospiraceae*,” *Biophys. J.* **93**, 54–61 (2007).
- ¹⁰D. K. Vig and C. W. Wolgemuth, “Swimming dynamics of the lyme disease spirochete,” *Phys. Rev. Lett.* **109**, 218104-1–218104-5 (2012).
- ¹¹G. B. Jeffery, “The rotation of two circular cylinders in a viscous fluid,” *Proc. R. Soc. London, Ser. A* **101**, 169–174 (1922).
- ¹²D. J. Jeffrey and Y. Onishi, “The slow motion of a cylinder next to a plane wall,” *Q. J. Mech. Appl. Math.* **34**, 129–137 (1981).
- ¹³S. Champmartin, A. Ambari, and N. Roussel, “Flow around a confined rotating cylinder at small Reynolds number,” *Phys. Fluids* **19**, 103101-1–103101-9 (2007).
- ¹⁴G. H. Wannier, “A contribution to the hydrodynamics of lubrication,” *Q. Appl. Math.* **8**, 1–32 (1950).
- ¹⁵R. C. J. Howland and R. C. Knight, “Slow rotation of a circular cylinder in a viscous fluid bounded by parallel walls,” *Math. Proc. Cambridge Philos. Soc.* **29**, 277–87 (1933).
- ¹⁶W. W. Hackborn, “Asymmetric stokes flow between parallel planes due to a rotlet,” *J. Fluid Mech.* **218**, 531–546 (1990).
- ¹⁷A. S. Dvinsky and A. S. Popel, “Motion of a rigid cylinder between parallel plates in stokes flow: Part 1: Motion in a quiescent fluid and sedimentation,” *Comput. Fluids* **15**, 391–404 (1987).
- ¹⁸J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics: With Special Applications to Particulate Media* (Springer, New York, 1983), p. 345.
- ¹⁹It fails, however, near $k = 1$, since $\phi(1 - \eta) = 1 - \eta + \dots$ whereas the correct behavior is $1 - 5/3\eta + \dots$.