Published in final edited form as:

J Biomech. 2009 April 16; 42(6): 781–785. doi:10.1016/j.jbiomech.2009.01.015.

A FRAME-INVARIANT FORMULATION OF FUNG ELASTICITY

GERARD A. ATESHIAN^{1,2} and KEVIN D. COSTA²

¹Department of Mechanical Engineering, Columbia University, New York, NY

²Department of Biomedical Engineering, Columbia University, New York, NY

Abstract

Fung elasticity refers to the hyperelasticity constitutive relation proposed by Y.C. Fung and co-workers for describing the pseudo-elastic behavior of biological soft tissues undergoing finite deformation. A frame-invariant formulation of Fung elasticity is provided for material symmetries ranging from orthotropy to isotropy, which uses Lamé-like material constants. In the orthotropic case, three orthonormal vectors are used to defined mutually orthogonal planes of symmetry and associated texture tensors. The strain energy density is then formulated as an isotropic function of the Lagrangian strain and texture tensors. The cases of isotropy and transverse isotropy are derived from the orthotropic case. Formulations are provided for both material and spatial frames. These formulations are suitable for implementation into finite element codes. It is also shown that the strain energy function can be naturally uncoupled into a dilatational and a distortional part, to facilitate the computational implementation of incompressibility.

1. Introduction

The constitutive relation proposed by Fung (1979; 1993) for the pseudo-elastic behavior of soft biological tissues has been used extensively (Sacks and Sun, 2003). The strain energy density function for this hyperelastic constitutive relation may be expressed in the form

$$W = \frac{1}{2} c (e^Q - 1) \quad (1)$$

where

$$Q = \frac{1}{2} a_{KLMN} E_{KL} E_{MN} \quad (2)$$

c is a material coefficient with units of stress, a_{KLMN} are dimensionless material parameters, E_{KL} are components of the Green-Lagrange strain tensor $\mathbf{E}$ (Holzapfel, 2000), and summations over $K, L, M, N = 1, 2, 3$ are implicit (Humphrey, 2002). Based on the form of these relations and the symmetry of $\mathbf{E}$, it follows that a_{KLMN} represents a set of at most 21 distinct material constants.

© 2009 Elsevier Ltd. All rights reserved.

Address all correspondence to Gerard Ateshian, Department of Mechanical Engineering, Columbia University, 500 West 120th St, MC4703, New York, NY 10027, USA, ateshian@columbia.edu.

Publisher's Disclaimer: This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

Implementing this finite deformation constitutive relation into a 3D finite element formulation can be achieved more conveniently when using a frame-invariant formulation. This frame-invariant form is presented here for orthotropic, transversely isotropic, and isotropic symmetry, in material and spatial frames.

2. Material and Spatial Frames

The second Piola-Kirchhoff stress is given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \frac{1}{2} c e^Q \frac{\partial Q}{\partial \mathbf{E}} \quad (3)$$

and the material elasticity tensor is¹

$$\begin{aligned} \mathbb{C} &= \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial^2 W}{\partial \mathbf{E}^2} \\ &= \frac{1}{2} c e^Q \left(\frac{\partial Q}{\partial \mathbf{E}} \otimes \frac{\partial Q}{\partial \mathbf{E}} + \frac{\partial^2 Q}{\partial \mathbf{E}^2} \right) \\ &= 2c^{-1} e^{-Q} \mathbf{S} \otimes \mathbf{S} + \frac{1}{2} c e^Q \frac{\partial^2 Q}{\partial \mathbf{E}^2} \end{aligned} \quad (4)$$

The Cauchy stress is obtained from $\mathbf{S}$ using the Piola transformation (Bonet and Wood, 1997),

$$\mathbf{T} = J^{-1} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T = \frac{1}{2} J^{-1} c e^Q \mathbf{F} \cdot \frac{\partial Q}{\partial \mathbf{E}} \cdot \mathbf{F}^T \quad (5)$$

where $\mathbf{F}$ is the deformation gradient and $J = \det \mathbf{F}$. Similarly, the spatial elasticity tensor is given by

$$\begin{aligned} \mathcal{C} &= J^{-1} (\mathbf{F} \otimes \mathbf{F}) : \mathbb{C} : (\mathbf{F}^T \otimes \mathbf{F}^T) \\ &= 2Jc^{-1} e^{-Q} \mathbf{T} \otimes \mathbf{T} + \frac{1}{2} J^{-1} c e^Q (\mathbf{F} \otimes \mathbf{F}) : \frac{\partial^2 Q}{\partial \mathbf{E}^2} : (\mathbf{F}^T \otimes \mathbf{F}^T) \end{aligned} \quad (6)$$

3. Frame-Invariant Forms

3.1. Orthotropic Symmetry

For orthotropic and higher symmetries, we can express the strain energy density in a frame-invariant form as

$$Q = c^{-1} \sum_{a=1}^3 \left[2\mu_a \mathbf{A}_a^0 : \mathbf{E}^2 + \sum_{b=1}^3 \lambda_{ab} (\mathbf{A}_a^0 : \mathbf{E}) (\mathbf{A}_b^0 : \mathbf{E}) \right] \quad (7)$$

where

$$\mathbf{A}_a^0 = \mathbf{a}_a^0 \otimes \mathbf{a}_a^0 \quad (8)$$

¹The tensor dyadic products $\otimes$, $\underline{\otimes}$ and $\overline{\otimes}$ are described by Curnier et al. (1994):

$(\mathbf{M} \otimes \mathbf{N})_{ijkl} = M_{ij} N_{kl}$, $(\mathbf{M} \underline{\otimes} \mathbf{N})_{ijkl} = M_{ik} N_{jl}$, $(\mathbf{M} \overline{\otimes} \mathbf{N})_{ijkl} = \frac{1}{2} (M_{ik} N_{jl} + M_{il} N_{jk})$ for arbitrary second-order tensors $\mathbf{M}$ and $\mathbf{N}$.

In this expression, $\mathbf{a}_a^0$ are the preferred directions of material texture in the reference configuration, representing unit normal vectors to the orthotropic planes of symmetry, and satisfying $\mathbf{a}_a^0 \cdot \mathbf{a}_b^0 = \delta_{ab}$. The associated tensor $\mathbf{A}_a^0$ given in the above equation may be called texture tensors in the reference configuration. Note that $\lambda_{ba} = \lambda_{ab}$, which implies that there are 6 distinct coefficients λ_{ab} in this expression. Thus there are ten material constants in this constitutive relation (c, λ_{ab}, μ_a), all having units of stress.

Example 1. Consider texture vectors aligned with the basis vectors,

$\mathbf{a}_a^0 = \mathbf{e}_a$, then $\mathbf{A}_a^0 : \mathbf{E} = (\mathbf{e}_a \otimes \mathbf{e}_a) : \mathbf{E} = E_{aa}$ (no sum implied) and

$$Q = c^{-1} \left[(\lambda_{11} + 2\mu_1) E_{11}^2 + (\lambda_{22} + 2\mu_2) E_{22}^2 + (\lambda_{33} + 2\mu_3) E_{33}^2 + 2\lambda_{23} E_{22} E_{33} + 2\lambda_{31} E_{33} E_{11} + 2\lambda_{12} E_{11} E_{22} + 2(\mu_2 + \mu_3) E_{23} E_{32} + 2(\mu_3 + \mu_1) E_{13} E_{31} + 2(\mu_1 + \mu_2) E_{12} E_{21} \right] \quad (a)$$

This expression can be used to find the equivalence between the material coefficients c, λ_{ab} and μ_a used here, and those employed in other notations. For example, Humphrey (2002) uses

$$Q = c_1 E_{RR}^2 + c_2 E_{\Theta\Theta}^2 + c_3 E_{ZZ}^2 + 2c_4 E_{RR} E_{\Theta\Theta} + 2c_5 E_{\Theta\Theta} E_{ZZ} + 2c_6 E_{ZZ} E_{RR} + c_7 (E_{R\Theta}^2 + E_{\Theta R}^2) + c_8 (E_{\Theta Z}^2 + E_{Z\Theta}^2) + c_9 (E_{ZR}^2 + E_{RZ}^2) \quad (b)$$

in a cylindrical coordinate system. Letting the cylindrical basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ be equivalent to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we find that

$$\begin{aligned} c_1 &= c^{-1}(\lambda_{11} + 2\mu_1) & c_2 &= c^{-1}(\lambda_{22} + 2\mu_2) & c_3 &= c^{-1}(\lambda_{33} + 2\mu_3) \\ c_4 &= c^{-1}\lambda_{12} & c_5 &= c^{-1}\lambda_{23} & c_6 &= c^{-1}\lambda_{13} \\ c_7 &= c^{-1}(\mu_1 + \mu_2) & c_8 &= c^{-1}(\mu_2 + \mu_3) & c_9 &= c^{-1}(\mu_3 + \mu_1) \end{aligned} \quad (c)$$

or equivalently,

$$\begin{aligned} \mu_1 &= \frac{c}{2}(c_7 - c_8 + c_9) & \mu_2 &= \frac{c}{2}(c_7 + c_8 - c_9) & \mu_3 &= \frac{c}{2}(-c_7 + c_8 + c_9) \\ \lambda_{11} &= c(c_1 - c_7 + c_8 - c_9) & \lambda_{22} &= c(c_2 - c_7 - c_8 + c_9) & \lambda_{33} &= c(c_3 + c_7 - c_8 - c_9) \\ \lambda_{12} &= cc_4 & \lambda_{23} &= cc_5 & \lambda_{13} &= cc_6 \end{aligned} \quad (d)$$

Humphrey (2002) has summarized values of c and $c_1 - c_6$ from various experimental studies on arteries and noted that only one study (Deng et al., 1994) has reported a representative value for the shear moduli (related to $c_7 - c_9$).

From the frame-invariant representation of Eq.(7) it follows that

$$\frac{\partial Q}{\partial \mathbf{E}} = c^{-1} \sum_{a=1}^3 \left[2\mu_a (\mathbf{A}_a^0 \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{A}_a^0) + \sum_{b=1}^3 \lambda_{ab} [(\mathbf{A}_a^0 \cdot \mathbf{E}) \mathbf{A}_b^0 + (\mathbf{A}_b^0 \cdot \mathbf{E}) \mathbf{A}_a^0] \right] \quad (9)$$

and

$$\frac{\partial^2 Q}{\partial \mathbf{E}^2} = c^{-1} \sum_{a=1}^3 \left[2\mu_a (\mathbf{A}_a^0 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_a^0) + \sum_{b=1}^3 \lambda_{ab} (\mathbf{A}_a^0 \otimes \mathbf{A}_b^0 + \mathbf{A}_b^0 \otimes \mathbf{A}_a^0) \right] \quad (10)$$

Thus, using Eq.(9) in Eq.(3), the second Piola-Kirchhoff stress is given by

$$\mathbf{S} = e^Q \sum_{a=1}^3 \left[\mu_a (\mathbf{A}_a^0 \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{A}_a^0) + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} [(\mathbf{A}_a^0 \cdot \mathbf{E}) \mathbf{A}_b^0 + (\mathbf{A}_b^0 \cdot \mathbf{E}) \mathbf{A}_a^0] \right] \quad (11)$$

Similarly, using Eq.(10) in Eq.(4) produces the material elasticity tensor

$$\mathbb{C} = 2e^{-1} e^{-Q} \mathbf{S} \otimes \mathbf{S} + e^Q \sum_{a=1}^3 \left[\mu_a (\mathbf{A}_a^0 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_a^0) + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} (\mathbf{A}_a^0 \otimes \mathbf{A}_b^0 + \mathbf{A}_b^0 \otimes \mathbf{A}_a^0) \right] \quad (12)$$

To evaluate the stress and elasticity tensor in the spatial frame, recognize that upon deformation the vectors $\mathbf{a}_a^0$ transform to $\mathbf{F} \cdot \mathbf{a}_a^0 = \lambda_a \mathbf{a}_a$, where λ_a is the stretch along $\mathbf{a}_a^0$, and $\mathbf{a}_a$ is a unit vector. Thus $\lambda_a^2 = (\mathbf{F} \cdot \mathbf{a}_a^0) \cdot (\mathbf{F} \cdot \mathbf{a}_a^0) = \mathbf{A}_a^0 : \mathbf{C} \equiv K_a$ where $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ is the right Cauchy-Green tensor. We can similarly define $L_a \equiv \mathbf{A}_a^0 : \mathbf{C}^2$ so that

$$\begin{aligned} \mathbf{A}_a^0 : \mathbf{E} &= \frac{1}{2} (K_a - 1) \\ \mathbf{A}_a^0 : \mathbf{E}^2 &= \frac{1}{4} (L_a - 2K_a + 1) \end{aligned} \quad (13)$$

where we used the identities $\mathbf{E} = \frac{1}{2} (\mathbf{C} - \mathbf{I})$ and $\mathbf{A}_a^0 : \mathbf{I} = 1$. Now the expression for Q in Eq.(7) can be rewritten as

$$Q = \frac{1}{4} c^{-1} \sum_{a=1}^3 \left[2\mu_a (L_a - 2K_a + 1) + \sum_{b=1}^3 \lambda_{ab} (K_a - 1)(K_b - 1) \right] \quad (14)$$

Substituting Eq.(13) into Eq.(11), and the resulting expression into Eq.(5) yields

$$\mathbf{T} = \frac{1}{2} J^{-1} e^Q \left[\sum_{a=1}^3 \mu_a K_a [\mathbf{A}_a \cdot (\mathbf{B} - \mathbf{I}) + (\mathbf{B} - \mathbf{I}) \cdot \mathbf{A}_a] + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} [(K_a - 1)K_b \mathbf{A}_b + (K_b - 1)K_a \mathbf{A}_a] \right] \quad (15)$$

where $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$ is the left Cauchy-Green tensor and $\mathbf{A}_a = K_a^{-1} \mathbf{F} \cdot \mathbf{A}_a^0 \cdot \mathbf{F}^T = \mathbf{a}_a \otimes \mathbf{a}_a$. The elasticity tensor in the spatial frame is obtained by substituting Eq.(12) into Eq.(6),

$$\mathbb{C} = 2Jc^{-1} e^{-Q} \mathbf{T} \otimes \mathbf{T} + J^{-1} e^Q \sum_{a=1}^3 \left[\mu_a K_a (\mathbf{A}_a \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}_a) + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} K_a K_b (\mathbf{A}_a \otimes \mathbf{A}_b + \mathbf{A}_b \otimes \mathbf{A}_a) \right] \quad (16)$$

where we have used the identities

$$\begin{aligned} (\mathbf{F} \otimes \mathbf{F}) : (\mathbf{M} \otimes \mathbf{N}) : (\mathbf{F}^T \otimes \mathbf{F}^T) &= (\mathbf{F} \cdot \mathbf{M} \cdot \mathbf{F}^T) \otimes (\mathbf{F} \cdot \mathbf{N} \cdot \mathbf{F}^T) \\ (\mathbf{F} \otimes \mathbf{F}) : (\mathbf{M} \otimes \mathbf{N}) : (\mathbf{F}^T \otimes \mathbf{F}^T) &= (\mathbf{F} \cdot \mathbf{M} \cdot \mathbf{F}^T) \otimes (\mathbf{F} \cdot \mathbf{N} \cdot \mathbf{F}^T) \end{aligned} \quad (17)$$

for arbitrary second-order tensors $\mathbf{M}$ and $\mathbf{N}$.

Example 2. In the strain-free reference configuration, $\mathbf{B} = \mathbf{I}$, $J = 1$, $K_a = 1$ and $L_a = 1$ ($a = 1, 2, 3$), so that $\mathbf{T} = \mathbf{0}$ and

$$\mathcal{C}|_{\mathbf{B}=\mathbf{I}} = \sum_{a=1}^3 \left[\mu_a (\mathbf{A}_a^0 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_a^0) + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} (\mathbf{A}_a^0 \otimes \mathbf{A}_b^0 + \mathbf{A}_b^0 \otimes \mathbf{A}_a^0) \right] \quad (a)$$

which is the elasticity tensor for a linear orthotropic elastic material (Curnier et al., 1994). Thus we can interpret λ_{ab} and μ_a as Lamé-like constants, where μ_a represent shear moduli. An added advantage of this representation is that it clarifies the stability constraints on the material coefficients λ_{ab} and μ_a , since the 6×6 stiffness matrix representation of $\mathcal{C}|_{\mathbf{B}=\mathbf{I}}$ in the coordinate basis aligned with the preferred directions,

$$[\mathcal{C}|_{\mathbf{B}=\mathbf{I}}] = \begin{bmatrix} \lambda_{11}+2\mu_1 & \lambda_{12} & \lambda_{13} & 0 & 0 & 0 \\ \lambda_{12} & \lambda_{22}+2\mu_2 & \lambda_{23} & 0 & 0 & 0 \\ \lambda_{13} & \lambda_{23} & \lambda_{33}+2\mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\mu_2+\mu_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(\mu_3+\mu_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\mu_1+\mu_2) \end{bmatrix} \quad (b)$$

must be positive semi-definite. Thus $\mu_a > 0$ ($a = 1, 2, 3$), and the sub-matrix consisting of the first three rows and columns must have positive eigenvalues. Also note that $c > 0$ to produce a positive strain energy density W , as evident from Eq.(1).

Example 3. In the limit as $c \rightarrow \infty$ we find that Q in Eq.(14) reduces to zero. Then, the constitutive relations for the stress, Eq.(15), and elasticity tensor, Eq.(16), reduce to

$$\mathbf{T} = \frac{1}{2} J^{-1} \left[\sum_{a=1}^3 \mu_a K_a [\mathbf{A}_a \cdot (\mathbf{B} - \mathbf{I}) + (\mathbf{B} - \mathbf{I}) \cdot \mathbf{A}_a] + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} [(K_a - 1)K_b \mathbf{A}_b + (K_b - 1)K_a \mathbf{A}_a] \right] \quad (a)$$

and

$$\mathcal{C} = J^{-1} \sum_{a=1}^3 \left[\mu_a K_a (\mathbf{A}_a \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}_a) + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} K_a K_b (\mathbf{A}_a \otimes \mathbf{A}_b + \mathbf{A}_b \otimes \mathbf{A}_a) \right] \quad (b)$$

These represent the expressions that would be achieved with a strain energy density that is a polynomial function of the strains, as used for example by Tong and Fung (1976).

3.2. Isotropic Symmetry

In the case of isotropic symmetry, let $\lambda_{ab} = \lambda$ and $\mu_a = \mu$ ($a, b = 1, 2, 3$) and recognize that

$\sum_{a=1}^3 \mathbf{A}_a^0 = \mathbf{I}$, so that Eqs.(7), (11), and (12) become

$$Q = c^{-1} [\lambda (\mathbf{I} : \mathbf{E})^2 + 2\mu \mathbf{I} : \mathbf{E}^2] \quad (18)$$

$$\mathbf{S} = e^Q [\lambda (\mathbf{I} : \mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}] \quad (19)$$

$$\mathbb{C} = 2c^{-1} e^{-Q} \mathbf{S} \otimes \mathbf{S} + e^Q (\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{I} \otimes \mathbf{I}) \quad (20)$$

in the material frame. Also recognize that the scalar invariants of $\mathbf{B}$ and $\mathbf{C}$ are the same so

that $\sum_{a=1}^3 K_a = \mathbf{I} : \mathbf{B}$, $\sum_{a=1}^3 L_a = \mathbf{I} : \mathbf{B}^2$, and $\sum_{a=1}^3 K_a \mathbf{A}_a = \mathbf{B}$, and Eqs.(14), (15), and (16) reduce to

$$Q = \frac{1}{4} c^{-1} \left[\lambda (\mathbf{I} : \mathbf{B} - 3)^2 + 2\mu (\mathbf{I} : \mathbf{B}^2 - 2\mathbf{I} : \mathbf{B} + 3) \right] \quad (21)$$

$$\mathbf{T} = \frac{1}{2} J^{-1} e^Q \left[\lambda (\mathbf{I} : \mathbf{B} - 3) \mathbf{B} + 2\mu (\mathbf{B}^2 - \mathbf{B}) \right] \quad (22)$$

$$\mathcal{C} = 2Jc^{-1} e^{-Q} \mathbf{T} \otimes \mathbf{T} + J^{-1} e^Q (\lambda \mathbf{B} \otimes \mathbf{B} + 2\mu \mathbf{B} \otimes \mathbf{B}) \quad (23)$$

in the spatial frame.

3.3. Transversely Isotropic Symmetry

For transverse isotropy, let the orthogonal vectors $\mathbf{a}_1^0$ and $\mathbf{a}_2^0$ define the transverse plane of isotropy, so that $\mathbf{a}_3^0 \equiv \mathbf{a}^0$ is normal to that plane. Also let $\mu_1 = \mu_2 \equiv \mu_T$ and $\mu_3 \equiv \mu$; $\lambda_{11} = \lambda_{22} = \lambda_{12} \equiv \lambda_T$, $\lambda_{13} = \lambda_{23} \equiv \lambda_L$, $\lambda_{33} \equiv \lambda$. Then, in the material frame,

$$Q = c^{-1} \left[\lambda_T (\mathbf{I} : \mathbf{E})^2 + (\lambda + \lambda_T - 2\lambda_L) (\mathbf{A}^0 : \mathbf{E})^2 + 2(\lambda_L - \lambda_T) (\mathbf{I} : \mathbf{E}) (\mathbf{A}^0 : \mathbf{E}) + 2\mu_T \mathbf{I} : \mathbf{E}^2 + 2(\mu - \mu_T) \mathbf{A}^0 : \mathbf{E}^2 \right] \quad (24)$$

$$\mathbf{S} = e^Q \left[\lambda_T (\mathbf{I} : \mathbf{E}) \mathbf{I} + (\lambda_T + \lambda - 2\lambda_L) (\mathbf{A}^0 : \mathbf{E}) \mathbf{A}^0 + (\lambda_L - \lambda_T) [(\mathbf{I} : \mathbf{E}) \mathbf{A}^0 + (\mathbf{A}^0 : \mathbf{E}) \mathbf{I}] + 2\mu_T \mathbf{E} + (\mu - \mu_T) (\mathbf{A}^0 \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{A}^0) \right] \quad (25)$$

$$\mathbb{C} = 2c^{-1} e^{-Q} \mathbf{S} \otimes \mathbf{S} + e^Q \left[\lambda_T \mathbf{I} \otimes \mathbf{I} + (\lambda_T + \lambda - 2\lambda_L) \mathbf{A}^0 \otimes \mathbf{A}^0 + (\lambda_L - \lambda_T) (\mathbf{I} \otimes \mathbf{A}^0 + \mathbf{A}^0 \otimes \mathbf{I}) + 2\mu_T \mathbf{I} \otimes \mathbf{I} + (\mu - \mu_T) (\mathbf{A}^0 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}^0) \right] \quad (26)$$

where $\mathbf{A}^0 \equiv \mathbf{A}_3^0$, and in the spatial frame

$$Q = \frac{1}{2} c^{-1} \left[\lambda_T (\mathbf{I} : \mathbf{B} - 3)^2 + (\lambda + \lambda_T - 2\lambda_L) (K - 1)^2 + (\lambda_L - \lambda_T) (\mathbf{I} : \mathbf{B} - 3) (K - 1) + \mu_T (\mathbf{I} : \mathbf{B}^2 - 2\mathbf{I} : \mathbf{B} + 3) + (\mu - \mu_T) (L - 2K + 1) \right] \quad (27)$$

$$\mathbf{T} = J^{-1} e^Q \left[\lambda_T (\mathbf{I} : \mathbf{B} - 3) \mathbf{B} + (\lambda_T + \lambda - 2\lambda_L) (K - 1) K \mathbf{A} + (\lambda_L - \lambda_T) [(\mathbf{I} : \mathbf{B} - 3) K \mathbf{A} + (K - 1) \mathbf{B}] + \mu_T (\mathbf{B}^2 - \mathbf{B}) + \frac{1}{2} (\mu - \mu_T) K [\mathbf{A} \cdot (\mathbf{B} - \mathbf{I}) + (\mathbf{B} - \mathbf{I}) \cdot \mathbf{A}] \right] \quad (28)$$

$$\mathcal{C} = 2Jc^{-1} e^{-Q} \mathbf{T} \otimes \mathbf{T} + J^{-1} e^Q \left[\lambda_T \mathbf{B} \otimes \mathbf{B} + (\lambda_T + \lambda - 2\lambda_L) K^2 \mathbf{A} \otimes \mathbf{A} + (\lambda_L - \lambda_T) K (\mathbf{B} \otimes \mathbf{A} + \mathbf{A} \otimes \mathbf{B}) + 2\mu_T \mathbf{B} \otimes \mathbf{B} + (\mu - \mu_T) K (\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}) \right] \quad (29)$$

where $\mathbf{A} \equiv \mathbf{A}_3$, $K \equiv K_3$ and $L \equiv L_3$.

Example 4. Considering again that the texture vectors align with the basis vectors, with $\mathbf{e}_3$ representing the preferred material axis, Eq.(24) becomes

$$Q = c^{-1} \left[(\lambda_T + 2\mu_T) (E_{11}^2 + E_{22}^2) + (\lambda + 2\mu) E_{33}^2 + 2\lambda_L (E_{22}E_{33} + E_{11}E_{33}) + 2\lambda_T E_{11}E_{22} + 2(\mu_T + \mu) (E_{23}E_{32} + E_{13}E_{31}) + 4\mu_T E_{12}E_{21} \right] \quad (a)$$

which allows the above material coefficients $c, \lambda_T, \lambda_L, \lambda, \mu_T$ and μ to be related to those used in other notations, such as the form proposed by Guccione and McCulloch (1991) and widely used to model passive myocardium (Humphrey, 2002; Costa et al., 2001),

$$Q = b_1 E_{FF}^2 + b_2 (E_{RR}^2 + E_{CC}^2 + E_{RC}^2 + E_{CR}^2) + b_3 (E_{RF}^2 + E_{FR}^2 + E_{CF}^2 + E_{FC}^2) \quad (b)$$

Equating $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with the local muscle fiber coordinate basis $\{\mathbf{e}_R, \mathbf{e}_C, \mathbf{e}_F\}$ aligned with the radial, in-plane cross-fiber, and in-plane fiber axes, respectively, yields

$$\begin{aligned} b_1 &= c^{-1}(\lambda + 2\mu) & b_2 &= c^{-1}(\lambda_T + 2\mu_T) = c^{-1}(2\mu_T) & b_3 &= c^{-1}(\mu_T + \mu) \\ 0 &= c^{-1}(2\lambda_T) & 0 &= c^{-1}(2\lambda_L) \end{aligned} \quad (c)$$

or equivalently

$$\begin{aligned} \lambda_T &= 0 & \lambda_L &= 0 & \lambda &= c(b_1 + b_2 - 2b_3) \\ \mu_T &= \frac{c}{2}b_2 & \mu &= c\left(b_3 - \frac{1}{2}b_2\right) \end{aligned} \quad (d)$$

By neglecting the interaction terms $E_{22}E_{33}$, $E_{11}E_{33}$ and $E_{11}E_{22}$, Guccione's formulation implicitly assumes that the constants λ_T and λ_L must be zero. This is in contrast to studies of the arterial wall, which typically neglect the shear moduli as seen in Example 1.

4. Incompressibility

In past applications, Fung's constitutive relation has been used mostly under the assumption that the solid is incompressible.² This implies that $J = 1$ and a Lagrange multiplier p must be incorporated in the stress and elasticity tensors to enforce incompressibility. In finite element implementations, it is often convenient to assume that the solid is slightly compressible, by uncoupling the strain energy function into a dilatational (volumetric) and a distortional (isochoric) part (Holzapfel, 2000; Bonet and Wood, 1997; Simo et al., 1985; Weiss et al., 1996), as first proposed by Flory (1961),

$$W(\mathbf{E}) = U(J) + \tilde{W}(\tilde{\mathbf{E}}) \quad (30)$$

where $\tilde{\mathbf{E}} = \frac{1}{2}(\tilde{\mathbf{C}} - \mathbf{I})$, $\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{F}}$, and $\tilde{\mathbf{F}} = J^{-1/3}\mathbf{F}$ is the distortional part of the deformation gradient ($\det \tilde{\mathbf{F}} = 1$). Then it can be shown (Holzapfel, 2000) that

$$\mathbf{S} = -Jp\mathbf{C}^{-1} + J^{-2/3}\text{Dev } \tilde{\mathbf{S}} \quad (31)$$

where $\tilde{\mathbf{S}} = \tilde{W}'(\tilde{\mathbf{E}})$ and the deviatoric operator in the material frame is

$$\text{Dev}(\cdot) = (\cdot) - \frac{1}{3}[(\cdot):\mathbf{C}]\mathbf{C}^{-1}. \text{ Similarly,}$$

$$\mathbf{T} = -p\mathbf{I} + \text{dev } \tilde{\mathbf{T}} \quad (32)$$

²It can be shown that Fung elasticity is not well suited for compressible solids. For example, under uniaxial confined compression, the compressive stress tends to zero as the material is compressed to zero thickness; a more appropriate behavior would have the stress rise to infinite magnitude in this limit.

where $\tilde{\mathbf{T}} = J^{-1} \tilde{\mathbf{F}} \cdot \tilde{\mathbf{S}} \cdot \tilde{\mathbf{F}}^T$ and the deviatoric operator in the spatial frame is

$\text{dev}(\cdot) = (\cdot) - \frac{1}{3} [(\cdot) : \mathbf{I}] \mathbf{I}$. In this approach the dilatational part of the stress is interpreted as a pressure term, $p \equiv -dU/dJ$ ³

By the chain rule of differentiation, the material elasticity tensor is given by (Simo et al., 1985; Weiss et al., 1996)

$$\begin{aligned} \mathbb{C} = & 2Jp\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - J \left(p + J \frac{dp}{dJ} \right) \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \\ & + J^{-4/3} \left[\tilde{\mathbf{C}} - \frac{1}{3} [\tilde{\mathbf{C}} : (\tilde{\mathbf{C}} \otimes \tilde{\mathbf{C}}^{-1}) + (\tilde{\mathbf{C}}^{-1} \otimes \tilde{\mathbf{C}}) : \tilde{\mathbf{C}} - \frac{1}{3} (\tilde{\mathbf{C}} : \tilde{\mathbf{C}} : \tilde{\mathbf{C}}) (\tilde{\mathbf{C}}^{-1} \otimes \tilde{\mathbf{C}}^{-1}) \right] \\ & + \frac{2}{3} \left[(\tilde{\mathbf{C}} : \tilde{\mathbf{S}}) \left(\tilde{\mathbf{C}}^{-1} \otimes \tilde{\mathbf{C}}^{-1} + \frac{1}{3} \tilde{\mathbf{C}}^{-1} \otimes \tilde{\mathbf{C}}^{-1} \right) - (\tilde{\mathbf{S}} \otimes \tilde{\mathbf{C}}^{-1} + \tilde{\mathbf{C}}^{-1} \otimes \tilde{\mathbf{S}}) \right] \end{aligned} \quad (33)$$

where $\tilde{\mathbf{C}} = 2\tilde{W} \tilde{\mathbf{E}}^2$, and the spatial elasticity tensor by

$$\begin{aligned} \mathcal{C} = & 2p\mathbf{I} \otimes \mathbf{I} - \left(p + J \frac{dp}{dJ} \right) \mathbf{I} \otimes \mathbf{I} \\ & + \tilde{\mathcal{C}} - \frac{1}{3} \left[\tilde{\mathcal{C}} : (\mathbf{I} \otimes \mathbf{I}) + (\mathbf{I} \otimes \mathbf{I}) : \tilde{\mathcal{C}} - \frac{1}{3} (\mathbf{I} : \tilde{\mathcal{C}} : \mathbf{I}) (\mathbf{I} \otimes \mathbf{I}) \right] \\ & + \frac{2}{3} \left[(\mathbf{I} : \tilde{\mathbf{T}}) \left(\mathbf{I} \otimes \mathbf{I} + \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) - (\tilde{\mathbf{T}} \otimes \mathbf{I} + \mathbf{I} \otimes \tilde{\mathbf{T}}) \right] \end{aligned} \quad (34)$$

where $\tilde{\mathcal{C}} = J^{-1} (\tilde{\mathbf{F}} \otimes \tilde{\mathbf{F}}) : \tilde{\mathbf{C}} : (\tilde{\mathbf{F}}^T \otimes \tilde{\mathbf{F}}^T) \tilde{\mathbf{F}} : \tilde{\mathbf{C}} : (\tilde{\mathbf{F}}^T \otimes \tilde{\mathbf{F}}^T)$ (Simo et al., 1985; Weiss et al., 1996; Ateshian et al., 2007).

Though it does not appear that this approach has been previously adopted, it is natural to consider functional forms for $\tilde{W}(\tilde{\mathbf{E}})$ which are similar to Fung's strain energy density. Thus,

$$\tilde{W}(\tilde{\mathbf{E}}) = \frac{1}{2} c \left(e^{\tilde{Q}} - 1 \right) \quad (35)$$

where, by analogy to Eqs.(7) and (14) for example,

$$\begin{aligned} \tilde{Q} = & c^{-1} \sum_{a=1}^3 \left[2\mu_a \mathbf{A}_a^0 : \tilde{\mathbf{E}}^2 + \sum_{b=1}^3 \lambda_{ab} (\mathbf{A}_a^0 : \tilde{\mathbf{E}}) (\mathbf{A}_b^0 : \tilde{\mathbf{E}}) \right] \\ = & \frac{1}{4} c^{-1} \sum_{a=1}^3 \left[2\mu_a (\tilde{L}_a - 2\tilde{K}_a + 1) + \sum_{b=1}^3 \lambda_{ab} (\tilde{K}_a - 1) (\tilde{K}_b - 1) \right] \end{aligned} \quad (36)$$

in the case of orthotropic symmetry, with $\tilde{K}_a = \mathbf{A}_a^0 : \tilde{\mathbf{C}}$ and $\tilde{L}_a = \mathbf{A}_a^0 : \tilde{\mathbf{C}}^2$. Then, according to Eqs. (11) and (15),

$$\tilde{\mathbf{S}} = e^{\tilde{Q}} \sum_{a=1}^3 \left[\mu_a (\mathbf{A}_a^0 \cdot \tilde{\mathbf{E}} + \tilde{\mathbf{E}} \cdot \mathbf{A}_a^0) + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} [(\mathbf{A}_a^0 : \tilde{\mathbf{E}}) \mathbf{A}_b^0 + (\mathbf{A}_b^0 : \tilde{\mathbf{E}}) \mathbf{A}_a^0] \right] \quad (37)$$

and

³For example, $U(J) = \kappa(\ln J)^2/2$ (Weiss et al., 1996; Ateshian et al., 2007) where κ is a bulk modulus which serves as a penalty parameter, such that $p = -\kappa (\ln J)/J$.

$$\begin{aligned} \tilde{\mathbf{T}} = & \frac{1}{2} J^{-1} e^{\tilde{Q}} \left[\sum_{a=1}^3 \mu_a \tilde{K}_a [\mathbf{A}_a \cdot (\tilde{\mathbf{B}} - \mathbf{I}) + (\tilde{\mathbf{B}} - \mathbf{I}) \cdot \mathbf{A}_a] \right. \\ & \left. + \frac{1}{2} \sum_{b=1}^3 \lambda_{ab} [(\tilde{K}_a - 1) \tilde{K}_b \mathbf{A}_b + (\tilde{K}_b - 1) \tilde{K}_a \mathbf{A}_a] \right] \end{aligned} \quad (38)$$

where $\tilde{\mathbf{B}} = \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}^T$. These expressions for $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$ must be substituted into Eqs.(31) and (32), respectively, to yield $\mathbf{S}$ and $\mathbf{T}$. Corresponding expressions for $\tilde{\mathbf{C}}$ and $\tilde{\mathcal{C}}$ can be obtained similarly from Eqs.(12) and (16), and substituted into Eqs.(33) and (34), respectively, to yield $\mathbf{C}$ and $\mathcal{C}$. Isotropic and transversely isotropic constitutive relations for nearly incompressible solids can be derived from this orthotropic form in the same manner as described above.

5. Conclusion

This paper summarizes frame-invariant formulations of Fung elasticity for orthotropic, transversely isotropic and isotropic symmetries. These formulations facilitate implementations in finite element codes, where the coordinate system used in an analysis is arbitrary and need not align with planes of material symmetry. For each finite element, the local preferred directions $\{\mathbf{a}_1^0, \mathbf{a}_2^0, \mathbf{a}_3^0\}$ can be specified relative to the global coordinate system, thus facilitating the evaluation of the stress and elasticity tensors in the global frame using the formulas presented in this paper. These formulations also establish a simple relationship between the material coefficients usually employed in Fung's constitutive relation, and the familiar Lamé constants. As a result, it becomes straight-forward to enforce positive-definiteness of the elasticity tensor at the strain origin, thereby placing constraints on the material coefficients.

Acknowledgments

This study was supported by the National Institutes of Health (AR46532) and the National Science Foundation (CAREER award to KDC, BES-02-39138).

References

- Ateshian GA, Ellis BJ, Weiss JA. Equivalence between short-time biphasic and incompressible elastic material responses. *J Biomech Eng.* 2007; 129(3):405–412. [PubMed: 17536908]
- Bonet, J.; Wood, RD. *Nonlinear continuum mechanics for finite element analysis.* Cambridge ; New York, NY: Cambridge University Press; 1997.
- Costa KD, Holmes JW, McCulloch AD. Modeling cardiac mechanical properties in three dimensions. *Phil Trans R Soc Lond A.* 2001; 359:1233–1250.
- Curnier A, Qi-Chang H, Zysset P. Conewise linear elastic materials. *J Elasticity.* 1994; 37(1):1–38.
- Deng SX, Tomioka J, Debes JC, Fung YC. New experiments on shear modulus of elasticity of arteries. *Am J Physiol.* 1994; 266(1 Pt 2):H1–H10. [PubMed: 8304490]
- Flory PJ. Thermodynamic relations for high elastic materials. *Trans Faraday Soc.* 1961; 57:829–838.
- Fung, YC. *Biomechanics: mechanical properties of living tissues.* 2nd Edition. New York: Springer-Verlag; 1993.
- Fung YC, Fronek K, Patitucci P. Pseudoelasticity of arteries and the choice of its mathematical expression. *Am J Physiol.* 1979; 237(5):H620–H631. [PubMed: 495769]
- Guccione, JM.; McCulloch, AD. Finite element modeling of ventricular mechanics. In: Glass, L.; Hunter, P.; McCulloch, A., editors. *Theory of Heart: Biomechanics, Biophysics, and Nonlinear Dynamics of Cardiac Function.* Springer; 1991. p. 121-144.

- Holzappel, GA. Nonlinear solid mechanics: a continuum approach for engineering. Chichester: Wiley; 2000.
- Humphrey, JD. Cardiovascular solid mechanics: cells, tissues, and organs. New York: Springer; 2002.
- Sacks MS, Sun W. Multiaxial mechanical behavior of biological materials. Annu Rev Biomed Eng. 2003; 5:251–284. [PubMed: 12730082]
- Simo J, Taylor R, Pister K. Variational and projection methods for the volume constraint in finite deformation elasto-plasticity. Comput Methods Appl Mech Eng. 1985; 51(1–3):177–208.
- Tong P, Fung YC. The stress-strain relationship for the skin. J Biomech. 1976; 9(10):649–657. [PubMed: 965417]
- Weiss J, Maker B, Govindjee S. Finite element implementation of incompressible, transversely isotropic hyperelasticity. Comput Methods Appl Mech Eng. 1996; 135(1–2):107–128.