Published in final edited form as:

Microfluid Nanofluidics. 2013 January 1; 14(1-2): 371–382. doi:10.1007/s10404-012-1058-8.

Scaling Law for Cross-stream Diffusion in Microchannels under Combined Electroosmotic and Pressure Driven Flow

Hongjun Song^{*}, Yi Wang, and Kapil Pant

CFD Research Corporation, 215 Wynn Drive, Huntsville, AL, 35805, U.S.A.

Abstract

This paper presents an analytical study of the cross-stream diffusion of an analyte in a rectangular microchannel under combined electroosmotic flow (EOF) and pressure driven flow to investigate the heterogeneous transport behavior and spatially-dependent diffusion scaling law. An analytical model capable of accurately describing 3D steady-state convection-diffusion in microchannels with arbitrary aspect ratios is developed based on the assumption of the thin Electric Double Layer (EDL). The model is verified against high-fidelity numerical simulation in terms of flow velocity and analyte concentration profiles with excellent agreement (<0.5% relative error). An extensive parametric analysis is then undertaken to interrogate the effect of the combined flow velocity field on the transport behavior in both the positive pressure gradient (PPG) and negative pressure gradient (NPG) cases. For the first time, the evolution from the spindle-shaped concentration profile in the PPG case, via the stripe-shaped profile (pure EOF), and finally to the butterfly-shaped profile in the PPG case is obtained using the analytical model along with a quantitative depiction of the spatially-dependent diffusion layer thickness and scaling law across a wide range of the parameter space.

Keywords

cross stream; combined electroosmotic and pressure driven flow; microfluidic; diffusion; analytical model; scaling law

1 Introduction

Lab-on-a-chip systems hold great promise for myriad applications in biology, medicine, and chemistry (Aurouz et al. 2002; Reyes et al. 2002; Whitesides 2006). In microfluidic applications, ‘transverse’ or ‘cross-stream’ diffusion under laminar flow conditions plays an essential role in chemical and biological assays and is key to sample preparation such as mixing and separation (Hatch et al. 2004), concentration gradient generation (Dertinger et al. 2001; Jeon et al. 2000), and molecular interactions (Hatch et al. 2004). In contrast to macro-scale devices, the cross-stream transport in the low-Reynolds (typical $Re < 1$) microchannel flows is governed by molecular diffusion at the interface between two co-flowing streams.

Pressure driven flow or electroosmotic flow (EOF) are the two most routinely used mechanisms to pump the analyte-carrying medium in the microchannel. EOF is generated by the tangential electrical field in the electric double layer (EDL) formed due to the interaction of the electrolyte solution with the dielectric surface. In many cases, the thickness of the EDL is negligible relative to the transverse size of the microchannels, therefore the velocity profile of EOF is plug-shaped except in the EDL. In contrast pressure driven flow is driven

^{*}Corresponding author, hjs@cfdr.com, Phone: +01-256-327-5294, Fax: +01-256-327-0985.

by the pressure difference along the longitudinal direction of the channel that can be externally applied or internally induced, producing a parabolic velocity profile. In practice, EOF is inevitably accompanied by pressure-driven flow to some extent, due to (1) the presence of additional pumping mechanisms; (2) local variations of the electrokinetic properties (e.g., zeta potential); or (3) difference in the column height between the reservoirs (Dutta and Beskok 2001; Horiuchi and Dutta 2006). The combination of EOF and pressure-driven flow leads to a non-uniform velocity profile (i.e., convex and concave profile) along the microchannel cross-section, which in turn results in unique species transport phenomena such as heterogeneous transport rate and direction- and position-dependent diffusion scaling law, arising from the difference in the residence time of the analyte at various cross-sectional positions.

The heterogeneous transport phenomena under pressure driven flow have been extensively studied. Kamholz *et al.* (Kamholz et al. 1999) were the first to interrogate a unique butterfly-shaped concentration profile (termed “butterfly effect”) along the cross section. By using confocal fluorescent microscopy, Ismagilov *et al.* (Ismagilov et al. 2000) found that the diffusion broadening region near the top and bottom walls of the channel is significantly wider than that at the half-depth plane of the channel, and the thickness of the diffusion zone scales approximately as one-third power of (zH^2/P_e) at the walls and as one-half power of (zH/P_e) at the half-depth plane, where H is the channel depth height, z is the axial distance from the channel inlet, and P_e is the Péclet number defined using H . Subsequently, Kamholz *et al.* (Kamholz and Yager 2001, 2002) presented optical measurements and theoretical analysis to investigate transverse molecular diffusion in microchannels and found similar diffusion scaling law. Chen *et al.* (Chen et al. 2006) used conventional microscopy to measure the mixing efficiency along the channel, which also scales as a power law of the ratio of the normalized downstream distance to the average flow velocity. As a non-invasive technique, magnetic resonance imaging (MRI) has also been used to visualize the cross-stream diffusion. The velocity and analyte concentration profile were captured and exhibited the butterfly effect (Akpa et al. 2007; Sullivan et al. 2007).

Concurrently, high-fidelity numerical modeling and simulation analysis have been actively pursued by researchers to develop an understanding of the unique transport behavior observed in the experiments. Numerical methods such as finite difference method (Kamholz and Yager 2001; Salmon and Ajdari 2007a), finite element method (Beard 2001), method of lines (Chen et al. 2006) and lattice Boltzmann (LB) method (Ayodele et al. 2009; Sullivan et al. 2007) have been harnessed to quantitatively describe the analyte concentration profile in rectangular microchannels with arbitrary aspect ratios. However, numerical simulation inherently suffers from several limitations. First, it is difficult to provide direct, physical insights into the underlying transport mechanism. In addition to the prohibitive computational costs, large amount of reliable data are necessary to deduce and generalize the governing law (Ayodele et al. 2009). Furthermore, numerical diffusion (or so-called pseudo-diffusion) caused by the discretization of the governing equations induces additional, artificial broadening of the diffusion zone leading to error in the analysis, which becomes more appreciable at high Péclet number regime (e.g., small diffusion coefficients and fast flow velocity). This can be even further exacerbated in the scaling law analysis, where the scaling coefficient is susceptible to the diffusion flux and becomes increasingly dependent on the choice of the computational mesh. One way to alleviate the issue of numerical diffusion is to employ very fine mesh size for the simulation, which can further overburden the task of acquiring the large, representative data pool as discussed above.

Despite salient prior efforts, research dedicated to investigating unique heterogeneous transport phenomena under combined EOF and pressure-driven flow by the virtue of accurate 3D analytical models is indeed scarce. In this paper, we present an analytical model

to investigate the heterogeneous transport phenomena and scaling law of the cross-stream diffusion in rectangular microchannels with arbitrary aspect ratios under combined EOF and pressure-driven flow. The three-dimensional steady-state convection-diffusion equation is solved in terms of a Fourier series. Based on a double integral transformation (de Almeida et al. 2008; Moreira et al. 2005; Wortmann et al. 2005) of the governing equation, the Fourier coefficients are obtained analytically via eigensystem calculation (Song et al. 2012). The analytical model is verified against high-fidelity numerical analysis in addition to relevant experimental data reported in the literature, including our prior publication for pressure driven flows (Song et al. 2012). Our model enables fully-resolved, three-dimensional insight into the unique transport behavior of the cross-stream diffusion under combined EOF and pressure driven flow, and is thus well suited for the design of microfluidic assays for efficient accurate liquid handling and analyte manipulation.

In contrast to previous studies solely focusing on the pressure driven flow (Song et al. 2012), our research features two salient novelties: (1) Consideration of the direction- and position-dependent scaling law and the heterogeneous transport rate under the combined EOF and pressure driven flow, which involves two additional parameters absent in prior analysis, i.e., the direction (aligning or opposing) and the extent of the pressure driven flow relative to the background EOF. Their effects on the position-dependent scaling law are extensively investigated. (2) The analysis is undertaken by virtue of accurate 3D analytical models applicable to microchannels with arbitrary transverse aspect ratios, which eliminates errors due to numerical diffusion and the approximation of the depth-wise average in prior analytical models (Holden et al. 2003; Lam et al. 2005; Wang et al. 2007; Wang et al. 2006) and provides direct physical insight into the underlying transport mechanism.

This paper is organized as follows: the description of the governing equations, model formulation, and the analytical solution are first described in Section 2; results and discussion of the heterogeneous diffusion behavior under the combined EOF and pressure driven flow, and the effect of both velocity ratio and channel aspect ratio are presented in Section 3; and a summary of our technical findings gained from the analytical parametric investigation is concluded in Section 4.

2 Model Formulation and Method

In this section, we present the governing equations and analytical solution to the cross-stream diffusion under combined EOF and pressure-driven flow. Figure 1 illustrates the coordinates of a Y-shaped microchannel with a rectangular cross-section used in the present study. The microchannel has two inlet channels and a main diffusion channel for analyte supply and cross-stream diffusion, respectively. x , y , and z denote the channel's cross-stream, depth-wise, and axial coordinates, respectively. The coordinate origin is located at the center of the plane where the inlet channels and the main channel merge. The length, width and height of the main channel are L , W and H , respectively. The flow within the channels is driven by the combined EOF and pressure-driven flow, arising from the externally applied or internally induced electric potentials and pressure head (see discussion above). Our model focuses on the species (analyte) transport in the main diffusion channel. The following assumptions are made to simplify the development of the analytical model:

1. The flow in the main microchannel is steady and laminar featuring a low Reynolds number.
2. The EDL is very thin (typically 100 nm) relative to the transverse dimension of the channel.
3. The length of the channel is significantly larger than its width and height (i.e., $L \gg W$ and $L \gg H$). Note that in distinct contrast to prior models (Holden et al. 2003;

Lam et al. 2005; Wang et al. 2007; Wang et al. 2006), no constraint of the relative magnitude of W and H is imposed in the present model. The flow entry effect after the merging junction is neglected as its length is negligible compared to the main channel length.

4. Convective mass transport in the axial direction dominates over the axial diffusion, and the latter is neglected in the model (refer to (Song et al. 2012) for detailed discussion about validity of this assumption in microfluidics.)

Based on these assumptions, the governing equation of fluid flow in the main channel is:

$$\mu \nabla^2 u = -\rho_e E_z + \frac{\partial P}{\partial x} \quad (1)$$

where u is the axial flow velocity of the buffer solution, P is the pressure, μ the dynamic viscosity of the liquid, ρ_e is the net electric charge density in the EDL and E_z is the axial electric strength. For uniform and straight rectangular microchannel, the axial flow velocity profile u can be obtained by superposing the individual components of the EOF and pressure driven flow (Dutta and Beskok 2001; Horiuchi and Dutta 2006):

$$u = -\epsilon \zeta E_z / \mu + [(1 - 4(y/h)^2) + 4 \sum_{s=1}^{\infty} \frac{(-1)^s}{p_s^3 \cosh(p_s \gamma)} \cosh(2p_s x/h) \cos(2p_s y/h)] \Delta P / RA \quad (2)$$

where $p_s = \pi(2s - 1)/2$, $A = \frac{2wh}{3} + 4 \sum_{s=1}^{\infty} \frac{(-1)^{2s-1} h^2 \tanh(p_s \gamma)}{p_s^5}$, $\gamma = w/h$ is the channel aspect ratio, ϵ is the buffer electric permittivity, ζ is zeta potential of the channel surface, E_z is the axial electric strength, ΔP is the total pressure drop across the channel length and R is relative hydrodynamic resistance, which is given by (Wang et al. 2007; Wang et al. 2006)

$$R = 12\gamma l \mu / \left\{ w^4 [1 - 192\gamma \sum_{s=1}^{\infty} \tanh(p_s \gamma) / p_s^5] \right\} \quad (3)$$

The first term of equation (2) is the EOF velocity based on the Helmholtz-Smoluchowski assumption in a thin EDL (Probstein 1994). The second term is the hydrodynamic velocity caused by the pressure gradient along the channel, which is derived from the analytical solution of the steady state axial Poiseuille flow in the rectangular channel (Stroock and McGraw 2004). As shown in Figure 1b and Figure 1c, the case for $\Delta P < 0$ and $\Delta P > 0$, corresponds to flow with negative pressure gradient (NPG) and positive pressure gradient (PPG) respectively. It should be pointed out that the scenario involving reversed flow at the channel center and bi-directional analyte transport due to strong NPG, viz., the maximum velocity of the pressure-driven flow opposes and exceeds the EOF velocity, is not considered in the present study (see below).

With the 1st and the 4th assumption above, the analyte transport is described by

$$u \frac{\partial c}{\partial z} = D \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) \quad (4)$$

where c is the analyte concentration, u is the flow velocity in Eq. (2) and D is the diffusivity. The reader is referred to (Song et al. 2012) for a detailed discussion about the validity of these assumptions in microfluidic applications. Equation (4) is subject to the following boundary conditions for cross-stream diffusion:

$$\left. \frac{\partial c}{\partial x} \right|_{x=\pm W/2} = 0, c|_{z=0} = c_0 = \begin{cases} 0 & -W/2 \leq x < 0 \\ C_0 & 0 \leq x \leq W/2 \end{cases}, \left. \frac{\partial c}{\partial y} \right|_{y=\pm H/2} = 0 \quad (5)$$

where c_0 is the initial analyte concentration at the merging junction ($z=0$). By defining $C=c/C_0$, $X=x/H$, $Y=y/H$, $Z=z/(\gamma Pe H)$, $Pe=U_{avg}H/D$, and $U=u/U_{avg}$, Equations (4) and (5) can be cast into a non-dimensional form

$$U \frac{\partial C}{\partial Z} = \gamma \left(\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} \right) \quad (6)$$

$$\left. \frac{\partial C}{\partial X} \right|_{X=\pm \gamma/2} = 0, C|_{Z=0} = \begin{cases} 0 & -1/2 \leq X < 0 \\ 1 & 0 \leq X \leq 1/2 \end{cases}, \left. \frac{\partial C}{\partial Y} \right|_{Y=\pm 1/2} = 0 \quad (7)$$

where Pe is the Péclet number defined with the channel width and U_{avg} is the average axial velocity along the cross section, which can be calculated by

$$U_{avg} = \bar{U}_{EOF} + \bar{U}_p = -\varepsilon \zeta E_z / \mu + \Delta P / Rwh \quad (8)$$

Here $\bar{U}_{EOF}$ and $\bar{U}_p$ are the average velocity components of the EOF and pressure-driven flow. The velocity ratio $\eta = \bar{U}_p / \bar{U}_{EOF}$ is defined to quantitatively characterize the relative strength/extent of the pressure driven flow relative to the EOF. $\eta > 0$ and $\eta < 0$ corresponds to the case of PPG and NPG, respectively, and the cases of $\eta = 0$ and $\eta = \infty$ are reduced to pure EOF and pure pressure driven flow. The Fourier series solution to the three-dimensional convection-diffusion equation (6) subject to appropriate boundary conditions (Equation (7)) is obtained using a double integral transformation method and associated eigensystem analysis developed in (Song et al. 2012) (see Appendix A for additional details), which is given by

$$C = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} K_{mn}(Z) \phi_m(X) \varphi_n(Y) \quad (9)$$

where $\phi_m(X)$ and $\varphi_n(Y)$ are the eigenfunctions in X and Y . $K_{mn}(Z)$ is the coefficient of the Fourier series and is dependent on Z .

Note that the heterogeneous transport behavior and scaling law cannot be properly defined for the case involving reversed flow along the cross-section of the channel, i.e., the maximum velocity caused by the negative pressure gradient opposes and exceeds the EOF velocity. Therefore our present analysis focuses on the scenario of the unidirectional flow. Accordingly the range of η in our analysis needs to be constrained and depends on the aspect ratio of the channel (e.g., $\eta = -0.48$ for the aspect ratio $\gamma = 1$, see Appendix B for more details).

3 Results and Discussion

Our analytical model for cross stream diffusion under pressure driven flow has been widely validated in (Song et al. 2012) against experimental data extracted from the literature and numerical analysis in broad parameter space of Péclet number and aspect ratios. In this paper, high-fidelity numerical simulation conducted with the commercial finite volume-based simulation software CFD-ACE+ (ESI-CFD, Inc.) is primarily utilized to verify our analytical models for combined EOF and pressure-driven flow. The computational domain

is meshed by hexahedral grids using the preprocessor module (CFDGEOM) available within CFD-ACE+. The software solves the 3D incompressible Navier-Stokes equations with the Helmholtz-Smoluchowski boundary condition for the flow velocity and the convection-diffusion equation for the analyte concentration in the microfluidic channels, respectively. The CFD-ACE+ solver uses the SIMPLEC algorithm for pressure-velocity coupling. A second order scheme is used for spatial discretization to obtain the analyte distribution in the entire computational domain. A grid independence check for the high fidelity numerical simulation was undertaken by refining the computational mesh, and all the numerical results presented in this paper converge to an accuracy level of <0.1% between two grid settings. To obtain accurate results for comparison, 60 terms in the Fourier series along both x and y direction in the analytical model (Equation (9)) are used.

3.1 Model Validation

The analytical model was validated using a rectangular microchannel whose width and depth are both 100 μm (i.e., the aspect ratio $\gamma=1$) where the heterogeneous transport phenomenon is the most salient. The EDL is assumed to be thin (<100 nm) and the zeta potential of the channel surface is taken to be -50 mV, typical of microfluidic substrates (Kirby and Hasselbrink Jr 2004). Both PPG and NPG cases were considered. In the former, the applied electric field is 290 V/cm and the pressure gradient ($\Delta P/l$) is 2845 Pa/m, thus both the EOF and the pressure driven flow components of the average velocity are the same -1 mm/s (i.e., $\eta=1$). In the NPG case, the applied electric field is 868.8 V/cm and the pressure gradient ($\Delta P/l$) is -2845 Pa/m, yielding an average EOF velocity of 3 mm/s and an average pressure driven velocity of -1 mm/s (i.e., $\eta=-0.333$). The density and the dynamic viscosity of the buffer is 1000 kg/m³ and 0.001 kg/(m·s), respectively, and the diffusivity of the analyte is 1×10^{-10} m²/s, corresponding to a Péclet number of 2000 for both cases.

Figure 2a shows the comparison of axial velocity profile between the analytical model and CFD-ACE+ simulation for both PPG and NPG cases. We can see that both match very well except at the region very close to the channel walls due to the use of the Helmholtz-Smoluchowski assumption in our analytical model, which essentially treats the wall as slip boundaries. However, this assumption has a negligible impact on analyte transport as the cross-stream diffusion primarily occurs outside of the EDL and in the bulk of the channel. The quantitative comparison of the concentration profile at $Y=0$ (i.e., half-depth plane) is depicted in Figure 2b. Excellent agreement between the analytical and numerical result is observed. Figure 3 shows the comparison between the analytical and numerical results on the cross-sectional concentration profile at $Z=0.025$ exhibiting the excellent accord again.

For quantitative comparison, the transverse concentration profiles extracted at the top/bottom ($|Y|=1/2$) and the half-depth plane ($Y=0$) of the channel at a downstream distance $Z=0.025$ are presented in Figure 4. It shows that the present analytical solution matches CFD-ACE+ result very well at the sampling lines for both PPG and NPG cases with virtually indistinguishable difference. The result shows diffusion near the wall is faster than that at the half-depth centerline in the PPG case, and slower in the NPG case, which can be attributed to the distinct difference in the flow velocity (see Figure 1) and the residence time of the analyte at both locations (see the next section). It should be pointed out that prior analytical models based on the depth-wise averaging (Holden et al. 2003; Lam et al. 2005; Wang et al. 2007; Wang et al. 2006) are incapable of capturing such differential transport behavior.

To quantitatively characterize the discrepancy between the high-fidelity CFD-ACE+ simulation and the analytical model, the relative error of the analytical model relative to the

numerical simulation, $\Delta(X) = \frac{|C - C_N|}{\langle C \rangle} \times 100\%$, is introduced. Here C and C_N are the

normalized transverse concentration, respectively, obtained from the analytical model and numerical CFD-ACE+ simulation, and $\langle C \rangle$ is the normalized average concentration value (i.e., 0.5). Figure 4 also displays the relative error Δ at the half-depth and the top/bottom for both PPG (Figure 4a) and NPG case (Figure 4b). The maximum relative error is less than 0.3% and 0.5%, respectively. A careful comparison of the concentration profile across the whole cross section was also carried out. Excellent agreement between our model and the numerical simulation was observed with the maximum relative error less than 0.5 %.

3.2 Heterogeneous Transport and Scaling Law Analysis

Our validated analytical model is then used to investigate the heterogeneous analyte transport under combined EOF and pressure driven flow. Figure 5 depicts the cross sectional concentration profiles at three downstream locations from the merging junction ($Z=0.001$, 0.01, 0.05) for both PPG and NPG cases with the same velocity ratios as before ($\eta=1$ and $\eta=-0.333$, respectively). We can see that the concentration distribution varies along the channel depth, which can be attributed to the position- and direction- dependent velocity distribution and non-uniform residence time of the analyte along the channel cross section in the presence of the pressure driven flow. In the PPG case, the pressure gradient aligns with the EOF, and hence, the velocity at the center is faster than that near the wall. It leads to slower advection and longer residence time of the analyte at the wall exhibiting the well-known “butterfly effect” (Figure 5a) along the cross section and is similar to that under purely pressure driven flow (Kamholz et al. 1999). In contrast, the pressure gradient is against the EOF in the NPG case, viz., the velocity at the center is slower than that near the walls. Thus the residence time of analyte at the center is longer, giving rise to faster diffusion therein. As a result, the concentration reveals a spindle-like profile (Figure 5b).

Next the analytical model is used to investigate the diffusion scaling law in the microchannel and identify the differences in the transverse diffusion rate along the channel depth. The general scaling law states that the thickness δ of the cross-stream diffusion layer is position-dependent and proportional to a power law of the normalized axial distance, i.e., $\delta \sim Z^{\lambda(Y,Z)}$, where λ is the local scaling exponent and is a function of the axial (Z) and depth-wise (Y) position due to the non-uniform velocity and transport (Ismagilov et al. 2000). For example, in pure pressure driven flow, $\lambda = 1/3$ and $\lambda=1/2$, respectively, near the wall and at the half-depth plane. Similar to (Ayodele and Raabe 2009; Salmon and Ajdari 2007b), in this paper δ is defined as the transverse distance along X direction between the channel centerline and a location where the concentration is reduced to 80% of the maximum value or risen to 20% of its minimum value at the inlet. Thus, $\lambda(Y, Z)$ is obtained by calculating the local slope of the curve $\text{Log}_{10}(\delta)$ vs. $\text{Log}_{10}Z$. The investigation in the present paper focuses on the regime $5 \times 10^{-5} < Z < 2.5 \times 10^{-2}$ (i.e., $-4.3 < \text{Log}_{10}Z < -1.6$), similar to prior studies on purely pressure driven flows (Salmon and Ajdari 2007b). Beyond $Z > 2.5 \times 10^{-2}$, the analyte could touch the side walls and the species transport will enter a new regime in bound domain, which will be analyzed in the future.

Figure 6 shows the log-log plots of the transverse diffusion thickness δ vs. the axial distance Z at different depth-wise position ($|Y|=0, 1/8, 1/4, 1/2$). We can see that the relationship between $\text{Log}_{10}(\delta)$ and $\text{Log}_{10}Z$ is approximately linear in this regime. In the PPG case, the diffusion layer thickness δ at the top/bottom wall is always thicker than that at the half-depth plane due to the slow flow and long residence time of the analyte at the top/bottom wall, which however is opposite to the NPG case where δ at the half-depth plane is larger.

In order to precisely capture the dependence of the exponential coefficients on the spatial positions, Figure 7 displays the contour plot of λ in the $\text{Log}_{10}Z$ and Y plane for both the PPG and NPG cases. It is found that $\lambda(Y, Z)$ is close to $1/2$ across the entire channel depth in the proximity of the inlet regardless of the direction of the pressure gradient. In the PPG

case (Figure 7a), $\lambda(Y, Z)$ decreases from 1/2 to 0.44 at the top/bottom wall where the flow velocity is slower and the residence time of the analyte is longer in contrast to the other regions along the cross section, which essentially signifies that the transport rates decrease along the axial direction. Further, $\lambda(Y, Z)$ remains 1/2 around the half-depth plane and then increases to 0.56 at a large axial distance (e.g., $\text{Log}_{10}Z > -2.523$, i.e., $Z > 3 \times 10^{-3}$), exhibiting a continuously increasing transport rate along the axial direction, which is similar to the observation in purely pressure driven flow reported in the literature (Ayodele and Raabe 2009; Salmon and Ajdari 2007b). On the contrary, in the NPG case $\lambda(Y, Z)$ decreases from 1/2 down to 0.38 at the half-depth plane of the channel and grows slowly from 1/2 up to 0.56 at the wall along the axial direction.

To interrogate the depth-wise dependence of the scaling law, the average scaling coefficient obtained by integrating λ along the axial direction, i.e., $\bar{\lambda}(Y) = \int_{-5 \times 10^{-5}}^{2.5 \times 10^{-2}} \lambda(Y, Z) dZ / \int_{-5 \times 10^{-5}}^{2.5 \times 10^{-2}} dZ$ was also calculated in both PPG and NPG cases as shown in Figure 8. The scaling exponent in the PPG case takes on a bell-shaped distribution. It stays slightly above 1/2 ($\bar{\lambda} = 0.51$) over a wide region ($|Y| < 0.25$) around the half-depth plane, and rapidly decreases to $\bar{\lambda} = 0.46$ at the top/bottom wall (i.e., $|Y| = 1/2$). These values correspond to the slope of the blue and red lines shown in Figure 6a, respectively. The latter $\bar{\lambda} = 0.46$ is larger than 1/3, the exponent of the classical scaling law in pure pressure driven flow, which is expected due to the presence of appreciable EOF at the wall leading to faster flow velocity, longer residence time, and thinner diffusion layer thickness therein. The observation qualitatively accords with the result in pure pressure driven flow reported by Kamholz and Yager (Kamholz and Yager 2001). On the other hand, the diffusion behavior for the NPG case is completely opposite. The scaling exponent $\bar{\lambda}$ exhibits a V-shaped profile. It stays at a minimum value of 0.47 at the half-depth plane and gradually increases to 0.52 while approaching the top/bottom wall.

3.3 Effect of the velocity ratio and channel aspect ratio

In combined EOF and pressure driven flow, the velocity ratio η has a significant impact on diffusion behavior and scaling law. Figure 9 illustrates the contour plot of the cross sectional concentration at $Z=0.01$ for various η ranging from -0.474 to 9 and reveals the evolution from the spindle-like shape in the NPG cases ($\eta < 0$), via the straight stripe shape in the pure EOF case ($\eta = 0$), and finally to the butterfly shape in the PPG case ($\eta > 0$). The image of $\eta = 9$ and beyond virtually converges to the pure pressure driven flow case (Song et al. 2012). Note that in all the simulation, Pe and ($P + P_{EOF}$) is held constant. Accordingly, a large η yields lower EOF velocity.

The effect of η on the diffusion scaling law is also quantitatively analyzed. The relationship between the diffusion layer thickness δ with the axial distance Z at the top/bottom wall and at the half-depth plane is depicted in Figure 10a and Figure 10b, respectively. As shown above in the PPG case given a constant Pe a large η representing a relatively lower EOF velocity at the wall prolongs the residence time of the analyte and broadens the diffusion layer δ therein (see Figure 10a). As a result its diffusion layer becomes thinner at the half-depth plane in contrast to the small η case subject to the mass balance constraint (see Figure 10b). The difference in the diffusion thickness caused by η diminishes along the axial direction as the analyte concentration becomes more uniform.

The dependence of the average scaling exponents $\bar{\lambda}$ (see definition above) at the top/bottom wall and at the half-depth plane on the velocity ratio η was also analyzed in Figure 11, where the diagram is divided into three regimes: $\eta = 0$, $\eta > 0$, and $\eta < 0$. In the first regime ($\eta = 0$), in absence of pressure driven flow the EOF carries a uniform velocity profile, leading to the exact 1/2 power law (i.e., $\bar{\lambda} = 1/2$) everywhere along the cross section including the

top/bottom wall and the half-depth plane and the intersection of both curves. In the 2nd regime ($\eta > 0$), the pressure driven flow aligns with the EOF and brings about broader diffusion layer δ at the wall (see Figure 10a), which consequently attenuates the transport rate therein and leads to continuously decreasing $\bar{\lambda}$ (from 0.5 to 0.38) with η (from 0 to 9). Eventually $\bar{\lambda}$ approaches $\sim 1/3$ asymptotically (i.e., the classical 1/3 power law—dashed line in Figure 11) for pure pressure driven flow as η approaches ∞ . As a result of the overall mass balance, the scaling exponent and transport rate at the half-depth plane slightly grows with η . In the 3rd regime ($\eta < 0$), the pressure driven flow acts against the EOF giving rise to the concave flow velocity. The region around the half-depth plane ($Y=0$) experiences slow flow and produces a thicker diffusion layer (see Figure 10b), where the diffusive transport rate slows down and $\bar{\lambda}$ decreases from 0.5 to 0.267 when η drops from 0 to -0.474 . As a result of mass balance, $\bar{\lambda}$ at the half-depth plane increases from 0.5 to 0.53 (see the left region in Figure 11). We also conducted the similar studies at other aspect ratios (e.g., $\gamma = 0.5, 2$, and 4). Similar behavior of the scaling exponent was observed except for the different values for the curves.

The effect of channel aspect ratio γ on the diffusion behavior and scaling law under the combined EOF and pressure driven flow was also investigated, and the results are quantitatively summarized in Figure 12. Figure 12a and b illustrate the dependence of the scaling exponent on the aspect ratio γ at the top/bottom wall and the half depth plane, respectively, for different velocity ratios. In the NPG case (i.e., $\eta = -0.412$) the average scaling exponents $\bar{\lambda}$ increases along with γ , which, however, behaves in an opposite manner in the PPG case that $\bar{\lambda}$ decreases at both the top/bottom wall and the half depth plane. We also observed that the variation in $\bar{\lambda}$ is more salient at the low range of the aspect ratio (e.g., $\gamma < 2$), and tends to saturate in the high range of $2 < \gamma < 4$, in particular, for the PPG case. In the pure EOF case ($\eta = 0$), $\bar{\lambda}$ keeps 0.5 regardless of the aspect ratio due to the uniform, plug-like EOF flow profile.

4 Conclusions

This paper presented an analytical model to study the heterogeneous transport behavior of an analyte during cross-stream diffusion in rectangular microchannels under combined electroosmotic flow (EOF) and pressure driven flow. An analytical model capable of describing 3D steady-state convection-diffusion in a microchannel with arbitrary aspect ratios was developed to interrogate the effect of the combined velocity field that is obtained by superposing the Poiseuille flow onto the Helmholtz-Smoluchowski EOF (i.e., thin EDL). Two flow scenarios: the positive pressure gradient (PPG) and the negative pressure gradient (NPG) were analyzed. In addition to the experimental verification for the pressure driven flow case (Song et al. 2012), the analytical model was extensively validated against high-fidelity numerical simulation for the combined EOF and pressure-driven flow.

Thorough studies and comparison in terms of the 3D concentration profile, spatially-dependent diffusion layer thickness, and scaling law were undertaken for both PPG and NPG cases, indicating distinctly different transport behavior of both. The concentration profiles exhibit the well-known butterfly shape and the spindle shape, respectively, in the PPG case and the NPG case due to the convex or concave velocity distribution. To the best of our knowledge, it is the first time that such heterogeneous diffusion effects were analyzed using a 3D analytical model. The evolution from the spindle-shaped concentration profile, via the stripe-shaped concentration profile, and finally to the butterfly-shaped concentration profile obtained by our analytical model was clearly captured over a wide range of the velocity ratios (η ranging from -0.474 to 9). It substantiates that η plays a key role in shaping the concentration distribution and determining the diffusion behavior in the combined EOF and pressure-driven flow.

In the PPG case, the diffusion layer at the half-depth plane is thinner and the transport characteristics is similar to the classical 1/2 power law in pure pressure-driven flow. At the top/bottom wall, the diffusion layer is thicker but features slower transport rate manifested by the slop of $\text{Log}_{10}(\delta)$ vs. $\text{Log}_{10}Z$ with the scaling exponent reduced from 0.5 to 0.38 in the range of $0 < \eta < 9$. It approaches the classical 1/3 power law in pure pressure-driven flow (i.e., $\eta \rightarrow \infty$). The diffusion behavior is opposite in the NPG case. The diffusion layer at the half-depth plane is thicker than that at the top/bottom wall. Accordingly the scaling exponent drops down from 0.5 to 0.265 at the half-depth plane and increases from 0.5 to 0.53 at the top/bottom wall in the range of $-0.474 < \eta < 0$. In the pure EOF case ($\eta=0$), the thickness of the diffusion layer and the scaling exponent at the half-depth plane and wall are the same. The results show that the scaling exponent increases in NPG case (i.e. $-0.474 < \eta < 0$) and decreases in PPG case (i.e. $0 < \eta < 9$) with the aspect ratio. The change of scaling exponent is more dramatic at the low range of the aspect ratio (e.g., $0.5 < \gamma < 2$), in particular, for the PPG case. The scaling exponent in the pure EOF case ($\eta=0$) is independent of the aspect ratio due to the uniform flow profile.

Acknowledgments

This research is sponsored by NIH/NHGRI under grant number 5R44HG004290-03.

Appendix A

To solve Equation (6), we assume the solution takes the form of a Fourier series expansion of normalized eigenfunctions $\phi_m(X)$ and $\varphi_n(Y)$:

$$C = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} K_{mn}(Z) \phi_m(X) \varphi_n(Y) \quad (10)$$

where the eigenfunctions and related eigenvalues can be structured from Sturm–Liouville auxiliary problems satisfying the related boundary conditions:

$$\left\{ \begin{array}{l} \frac{d^2 \phi_m(X)}{dX^2} + \alpha_m^2 \phi_m(X) = 0 \\ \left. \frac{d\phi_m(X)}{dX} \right|_{X=\pm\gamma/2} = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \frac{d^2 \varphi_n(Y)}{dY^2} + \beta_n^2 \varphi_n(Y) = 0 \\ \left. \frac{d\varphi_n(Y)}{dY} \right|_{Y=\pm 1/2} = 0 \end{array} \right. \quad (11)$$

and the solution can be expressed as

$$\left\{ \begin{array}{l} \phi_m(X) = \cos(\alpha_m(X + \gamma/2)) / N_m \\ \alpha_m = m\pi/\gamma; m=0, 1, 2, \dots \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \varphi_n(Y) = \cos(\beta_n(Y + 1/2)) / N_n \\ \beta_n = n\pi; n=0, 1, 2, \dots \end{array} \right. \quad (12)$$

where $N_m = \int_{-\gamma/2}^{\gamma/2} \phi_m^2(X) dX$ and $N_n = \int_{-1/2}^{1/2} \varphi_n^2(Y) dY$. By applying the double integration $\int_{-\gamma/2}^{\gamma/2} \int_{-1/2}^{1/2} \phi_i(X) \varphi_j(Y) dX dY$ to Equation (6), an infinite system for the coefficients $K_{mn}(Z)$ is obtained:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{dK_{mn}(Z)}{dZ} M_{mnij} = R_{ij} K_{ij}(Z) \quad (13)$$

where $M_{mnij} = \int_{-\gamma/2}^{\gamma/2} \int_{-1/2}^{1/2} U(X, Y) \phi_m(X) \phi_i(X) \varphi_n(Y) \varphi_j(Y) dX dY$ and $R_{ij} = -(\alpha_i^2 + \beta_j^2)$. For convenience, Equation (13) can be rewritten in a single sum form (by replacing m, n with k and i, j with q)

$$\sum_{p=0}^{\infty} \frac{dK_p(\tilde{z})}{d\tilde{z}} M_{pq} = R_q K_q(\tilde{z}), q=0, 1, 2, \dots \quad (14)$$

Using the compact matrix expression, Equation (14) is given by

$$[M][K'(Z)] = [R][K(Z)] \quad (15)$$

The analytical solution of Equation (15) can be found as

$$[K(Z)] = [K(0)] \bullet e^{[Q]Z} \quad (16)$$

where $[Q] = [M]^{-1}[R]$ and $K[0]$ is obtained by applying the integral transformation to the initial boundary condition at $Z=0$. Upon determining the eigenvalues and eigenvectors of matrix $[Q]$, the solution in Equation (16) in the decoupled form is

$$[K(Z)] = [K(0)] \bullet [V]^{-1} e^{[D]Z} [V] \quad (17)$$

where $[V]$ is the eigenvector matrix and $[D]$ is the diagonal eigenvalue matrix, satisfies $[V][Q][V]^{-1} = [D]$.

Appendix B

Since the diffusion scaling law cannot be properly defined for the case involving reversed flow in the microchannel, there is a constraint for the NPG case, i.e., the maximum velocity $U_p^{\max}$ caused by the negative pressure gradient should not exceed the EOF velocity U_{EOF} (see Figure 13).

The constraint is mathematically expressed by

$$U_{EOF} + U_p^{\max} \geq 0 \quad (18)$$

As the maximum velocity of the pressure driven flow occurs at the channel center, according to Eq. (2) and (8), $U_p^{\max}$ can be obtained as:

$$U_p^{\max} = \bar{U}_p B w h / A \quad (19)$$

where $\bar{U}_p$ is the average velocity of the pressure driven flow component;

$A = \frac{2wh}{3} + 4 \sum_{s=1}^{\infty} \frac{(-1)^{2s-1} h^2 \tanh(p_s \gamma)}{p_s^5}$ and $B = \left[1 + 4 \sum_{s=1}^{\infty} \frac{(-1)^s}{p_s^3 \cosh(p_s \gamma)} \right]$. In addition, given U_{EOF} in the uniform EOF flow, Eq. (18) yields

$$\bar{U}_{EOF} + \bar{U}_p B w h / A \geq 0 \quad (20)$$

Thus the constraint of the velocity ratio η is given by

$$\eta = \bar{U}_p / \bar{U}_{EOF} \geq -A / w h B \quad (21)$$

As the values of A and B are dependent on the aspect ratio γ , the constraint of velocity ratio also varies with aspect ratios. The value of A/whB is 0.48 for the case $\gamma=1$.

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Figure 1.
 (a) Geometry and coordinate of the microchannel in analysis; schematics of velocity profile with (b) positive pressure gradient (PPG); and (c) negative pressure gradient (NPG)

Figure 2.
The comparison of velocity between the present analytical model and CFD-ACE+ simulation: (a) velocity profile for both PPG (left) and NPG (right) cases; (b) velocity distribution at $Y=0$ (half-depth plane)

Figure 3.
Comparison of the cross-sectional concentration profile between the analytical results and CFD-ACE+ simulation for both cases: (a) PPG ($\eta > 0$); (b) NPG ($\eta < 0$);

Figure 4.
The comparison of concentration profile between the analytical model and CFD-ACE+ simulation: (a) PPG case; and (b) NPG case

Figure 5.
Cross sectional concentration profile at three different downstream locations ($Z=0.001$, 0.01 , 0.05) : (a) PPG ($\eta > 0$); (b) NPG ($\eta < 0$);

Figure 6.
 $\text{Log}_{10}(\delta)$ vs. $\text{Log}_{10}Z$ for different depths ($|Y|=0, 1/8, 1/4, 1/2$): (a) PPG case; (b) NPG case

Figure 7.

The contour plot of the local scaling exponents $\lambda(Y, Z)$ in the $\text{Log}_{10}Z$ - Y plane for both cases: (a) PPG ($\eta > 0$); (b) NPG ($\eta < 0$);

Figure 8.
The relationship between $\bar{\lambda}$ and depth-wise position Y for both PPG and NPG cases

Figure 9.
The cross-sectional images of the analyte concentration at $Z=0.01$ for different velocity ratio η .

Figure 10.
 $\text{Log}_{10}(\delta)$ vs. $\text{Log}_{10}Z$ for different velocity ratio η : (a) the top/bottom wall; (b) the half-depth plane

Figure 11.
The relationship between the overall scaling exponents $\bar{\lambda}$ and the velocity ratio η

Figure 12.
The effect of channel aspect ratio γ : (a) at the top/bottom wall, and (b) at the half-depth plane

Figure 13.
Schematic of the constraint of non-reversed flow for the model