Published in final edited form as:

IEEE Trans Med Imaging. 2013 March ; 32(3): 556–564. doi:10.1109/TMI.2012.2229711.

Accelerated Regularized Estimation of MR Coil Sensitivities Using Augmented Lagrangian Methods

Michael J. Allison [Student Member, IEEE], Sathish Ramani [Member, IEEE], and Jeffrey A. Fessler [Fellow, IEEE]

Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109 USA

Michael J. Allison: mjalliso@umich.edu; Sathish Ramani: sramani@umich.edu; Jeffrey A. Fessler: fessler@umich.edu

Abstract

Several magnetic resonance (MR) parallel imaging techniques require explicit estimates of the receive coil sensitivity profiles. These estimates must be accurate over both the object and its surrounding regions to avoid generating artifacts in the reconstructed images. Regularized estimation methods that involve minimizing a cost function containing both a data-fit term and a regularization term provide robust sensitivity estimates. However, these methods can be computationally expensive when dealing with large problems. In this paper, we propose an iterative algorithm based on variable splitting and the augmented Lagrangian method that estimates the coil sensitivity profile by minimizing a quadratic cost function. Our method, ADMM-Circ, reformulates the finite differencing matrix in the regularization term to enable exact alternating minimization steps. We also present a faster variant of this algorithm using intermediate updating of the associated Lagrange multipliers. Numerical experiments with simulated and real data sets indicate that our proposed method converges approximately twice as fast as the preconditioned conjugate gradient method (PCG) over the entire field-of-view. These concepts may accelerate other quadratic optimization problems.

Index Terms

Augmented Lagrangian; coil sensitivity; finite differences; parallel imaging; quadratic minimization

I. Introduction

Accurate radio-frequency coil sensitivity profiles are required in many parallel imaging applications (e.g., sensitivity encoding (SENSE) [1], simultaneous acquisition of spatial harmonics (SMASH) [2], and k - t SENSE [3]). Due to coil deformation during patient setup and dielectric coupling, these profiles must be determined at the time of acquisition [4]. One common approach is to perform a calibration scan prior to the parallel imaging acquisition in which images from a large body coil and multiple surface coils are acquired and reconstructed. Since the body coil has near uniform sensitivity, its image can be used in conjunction with a surface coil image to estimate the surface coil sensitivity profile.

The most straightforward method to estimate the coil sensitivity is to compute the ratio of the surface coil image voxel values (z_i) to the body coil image voxel values (y_i), z_i/y_i . However, ratio estimates can be corrupted by measurement noise, particularly in low signal regions. Furthermore, such estimates can have sharp discontinuities at object edges, contrary to the smooth nature of true coil sensitivity profiles [5]. It is also desirable to have reasonable sensitivity estimates in any low signal regions surrounding the object to avoid reconstruction artifacts that could arise due to patient motion [6]. The ratio estimator, however, does not extrapolate; thus, improved estimation methods can be beneficial.

One approach to generate smooth sensitivity estimates is to measure only the center of k -space [6]. Although simple, this approach does not accurately estimate sensitivities near object edges and can introduce Gibbs ringing artifacts. Filtering procedures have also been proposed including polynomial fitting [1], [7]–[9], wavelet denoising [10], and using thin-plate splines [11]. These methods do not completely eliminate the Gibbs ringing, while selecting a particular basis function is complicated by the varying size of low signal regions within the images [5], [12]. Furthermore, many of these methods disregard the non-stationary variance of the noise in the ratio estimates. In contrast, regularized estimation methods [5], [13], [14] provide smooth sensitivity estimates and are capable of extrapolation without explicit basis function selection or filtering. These methods, however, can be computationally expensive for large problems [5] and this cost is compounded by the large number of coils in some arrays [15]. Although sensitivity estimation can be performed off-line, the computational costs of regularized methods can increase the overall compute times of parallel imaging.

In this paper, we take a regularized approach and pose sensitivity estimation as the minimization of a quadratic cost function like in [5]. The large matrices in the cost function prevent one from computing a simple, non-iterative solution. Instead, iterative methods must be used for large data sets; however, traditional methods like conjugate gradient (CG) converge slowly for this problem [5], [16]. Augmented Lagrangian (AL) based minimization techniques [17], and the related Bregman iterations method [18], have been used to accelerate convergence in imaging problems such as denoising [19] and reconstruction [19]–[27]. Those papers primarily focus on problems that contain non-differentiable regularization terms such as those based on the ℓ_1 -norm. However, the underlying theory applies to a wide variety of optimization problems, including the quadratic problem considered here. We therefore propose a new AL based method for estimating sensitivity profiles. To derive this method, we introduce a reformulation of the finite differencing matrix and a subsequent variable splitting that lead to an algorithm with exact alternating minimization steps. This algorithm is equivalent to an alternating direction method of multipliers (ADMM) [28] formulation, which provides a guarantee of convergence. We also explore a variation of this algorithm that updates the Lagrange multipliers between alternating minimization steps. Such variations have been found to improve the convergence rates of other AL based algorithms [29].

Section II presents the derivation of our ADMM algorithm and its intermediate updating variant. Section III compares the convergence speeds of these algorithms with those of CG based methods by performing experiments on both simulated and real data. Section IV discusses the results of these experiments and additional properties of the algorithms. Section V concludes by discussing other problems that have quadratic cost functions where our methods may provide an improvement over the traditional techniques.

II. Materials and Methods

This section introduces our proposed methods for MR coil sensitivity estimation. We begin by posing the estimator as an optimization problem. We then outline the general approach used to solve this problem and present our specific algorithm, with variations, in detail.

A. Cost Function Formulation

Regularized methods for MR coil sensitivity estimation are both robust to noise and effective at extrapolating the estimate in regions of low signal [5], [14]. These methods avoid computing the quotient (z_i/y_i) by finding the minimizer of a cost function containing a data-fidelity term and a regularization term that promotes smoothness in the estimate. Similar to [5], [13], we estimate the sensitivity profile by minimizing a weighted sum of quadratic terms:

$$\hat{\mathbf{s}} \triangleq \arg \min_{\mathbf{s}} \frac{1}{2} \|\mathbf{z} - \mathbf{D}\mathbf{s}\|_{\mathbf{W}}^2 + \frac{\lambda}{2} \|\mathbf{R}\mathbf{s}\|_2^2, \quad (1)$$

where $\mathbf{s} = [s_1, \dots, s_N]^T$ with $s_i \in \mathbb{C}$ denoting the desired coil sensitivity at the i th voxel and N denoting the number of voxels, $\mathbf{z} = [z_1, \dots, z_N]^T$ with $z_i \in \mathbb{C}$ denoting the surface coil image value at the i th voxel, $\mathbf{D} = \text{diag}\{y_i\}$ is a diagonal matrix containing the body coil image voxel values ($y_i \in \mathbb{C}$), $\mathbf{R} \in \mathbb{R}^{M \times N}$ is a finite differencing matrix for the case of non-periodic boundary conditions with M sets of finite differences, and $\lambda > 0$ is a regularization coefficient. Additionally, $\mathbf{W} = \text{diag}\{w_i\}$ is a diagonal weighting matrix (with $w_i \in [0, 1]$) that allows us to ensure that the estimate is based primarily on voxels that provide meaningful sensitivity information. Note that a finite differencing matrix with non-periodic boundary conditions is necessary as periodic boundary conditions introduce errors at the edges of the image that can propagate and corrupt the estimate near the object voxels.¹

Equation (1) has a quadratic cost function and therefore has the closed-form solution $\hat{\mathbf{s}} = [\mathbf{D}^H \mathbf{W} \mathbf{D} + \lambda \mathbf{R}^H \mathbf{R}]^{-1} \mathbf{D}^H \mathbf{W} \mathbf{z}$ where $\mathbf{X}^H$ denotes the Hermitian transpose of $\mathbf{X}$; however, computing this solution is impractical due to the size and complexity of $\mathbf{R}$. Memory constraints further restrict the use of other direct methods, such as Cholesky factorization, for large problems like 3D data sets. Furthermore, standard iterative solution methods, such as CG, exhibit slow converge for this problem even when using carefully selected preconditioners [30]. To address this, we propose an augmented Lagrangian method to minimize the cost function, the development of which consists of three stages [20]. First, we use variable splitting [25], [31] to convert the unconstrained optimization problem into an equivalent constrained problem, thereby decoupling the effects of the matrices in (1). Second, we introduce vector Lagrange multipliers and express the constrained problem in an AL framework. Third, we solve the resulting AL problem using an alternating minimization scheme.

B. ADMM–Circ: ADMM Sensitivity Estimation Algorithm with Circulant Substeps

Directly applying variable splitting to (1) results in an AL algorithm requiring an approximate solution for one of the alternating minimization steps due to the complexity of the finite differencing matrix $\mathbf{R}$ [16]. The supplementary material for this paper presents one such algorithm, ADMM–CG. We can avoid this complication if we focus on traditional finite differencing matrices (those with spatially invariant stencils). For such regularizers, we can express the finite differencing matrix as $\mathbf{R} = \mathbf{B}\mathbf{C}$ where $\mathbf{C} \in \mathbb{R}^{M \times N}$ is a typical finite differencing matrix for the case of periodic boundary conditions, containing additional non-

¹See the supplementary material for additional details.

zero rows that penalize the differences between voxels on opposing boundaries of the image, and $\mathbf{B} \in \{0, 1\}^{M \times M}$ is a diagonal matrix that contains a mask to eliminate the effects of the added rows. The additional non-zero rows in $\mathbf{C}$ ensure that $\mathbf{C}^H \mathbf{C}$ is block circulant with circulant blocks unlike $\mathbf{R}$. Fig. 1 illustrates these matrices for the case of 1D second-order finite differences. We then write the estimation problem in (1) as

$$\hat{\mathbf{s}} = \arg \min_{\mathbf{s}} \frac{1}{2} \|\mathbf{z} - \mathbf{D}\mathbf{s}\|_{\mathbf{w}}^2 + \frac{\lambda}{2} \|\mathbf{B}\mathbf{C}\mathbf{s}\|_2^2. \quad (2)$$

We introduce two splitting variables, $\mathbf{u}_0 \in \mathbb{C}^M$ and $\mathbf{u}_1 \in \mathbb{C}^N$, to this new formulation to decouple the matrices $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{C}$. The resulting equivalent constrained optimization problem is

$$\hat{\mathbf{s}} = \arg \min_{\mathbf{s}, \mathbf{u}_0, \mathbf{u}_1} \frac{1}{2} \|\mathbf{z} - \mathbf{D}\mathbf{u}_1\|_{\mathbf{w}}^2 + \frac{\lambda}{2} \|\mathbf{B}\mathbf{u}_0\|_2^2 \text{ s.t. } \mathbf{u}_1 = \mathbf{s} \text{ and } \mathbf{u}_0 = \mathbf{C}\mathbf{s}. \quad (3)$$

Solving this constrained optimization problem is exactly equivalent to solving the unconstrained problem (1).

We express (3) in the more concise notation:

$$\arg \min_{\mathbf{s}, \mathbf{u}} \frac{1}{2} \|\mathbf{h} - \mathbf{A}\mathbf{u}\|_2^2 \text{ s.t. } \mathbf{u} = \mathbf{T}\mathbf{s}, \quad (4)$$

where

$$\mathbf{u} \triangleq \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_0 \end{bmatrix}, \mathbf{T} \triangleq \begin{bmatrix} \mathbf{I} \\ \mathbf{C} \end{bmatrix}, \mathbf{h} \triangleq \begin{bmatrix} \mathbf{W}^{\frac{1}{2}}\mathbf{z} \\ \mathbf{0} \end{bmatrix}, \mathbf{A} \triangleq \begin{bmatrix} \mathbf{W}^{\frac{1}{2}}\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \sqrt{\lambda}\mathbf{B} \end{bmatrix},$$

and $\mathbf{W}^{\frac{1}{2}} \triangleq \text{diag}\{\sqrt{w_i}\}$.

We then introduce two vectors of Lagrange multipliers, $\boldsymbol{\eta}_0 \in \mathbb{C}^M$ and $\boldsymbol{\eta}_1 \in \mathbb{C}^N$, and express (4) as an AL problem. We use the general AL formulation outlined in [20] that incorporates the Lagrange multiplier into the quadratic penalty term. This formulation is a natural extension of the traditional AL to the case of complex values and it simplifies the derivation of the subsequent alternating minimization steps. The resulting AL function-based minimization problem is

$$\arg \min_{\mathbf{s}, \mathbf{u}} \frac{1}{2} \|\mathbf{h} - \mathbf{A}\mathbf{u}\|_2^2 + \frac{1}{2} \|\mathbf{u} - \mathbf{T}\mathbf{s} - \boldsymbol{\eta}\|_{\mathbf{V}}^2, \quad (5)$$

where

$$\boldsymbol{\eta} \triangleq \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_0 \end{bmatrix}, \mathbf{V} \triangleq \begin{bmatrix} \nu_1 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \nu_0 \mathbf{I} \end{bmatrix},$$

and $\nu_0, \nu_1 > 0$ are AL penalty parameters that influence the convergence rate of the algorithm but do not affect the final estimate [26].

Traditional AL methods would require jointly minimizing (5) over the vectors $\mathbf{s}$ and $\mathbf{u}$; however, such an approach is computationally expensive for typical image sizes. Instead, we use a block Gauss–Seidel type alternating minimization strategy that has been effective in solving other AL problems [19], [28] in which we alternate between minimizing (5) independently with respect to $\mathbf{s}$ and $\mathbf{u}$ as follows:

$$\mathbf{s}^{(j+1)} = \arg \min_{\mathbf{s}} \frac{1}{2} \|\mathbf{u}^{(j)} - \mathbf{T}\mathbf{s} - \eta^{(j)}\|_{\mathbf{V}}^2, \quad (6)$$

$$\mathbf{u}^{(j+1)} = \arg \min_{\mathbf{u}} \frac{1}{2} \|\mathbf{h} - \mathbf{A}\mathbf{u}\|_2^2 + \frac{1}{2} \|\mathbf{u} - \mathbf{T}\mathbf{s}^{(j+1)} - \eta^{(j)}\|_{\mathbf{V}}^2. \quad (7)$$

Update (7) has a simple closed-form solution:

$$\mathbf{u}^{(j+1)} = (\mathbf{A}^H \mathbf{A} + \mathbf{V})^{-1} [\mathbf{A}^H \mathbf{h} + \mathbf{V}(\mathbf{T}\mathbf{s}^{(j+1)} + \eta^{(j)})]. \quad (8)$$

In fact, the block diagonal structures of $\mathbf{A}$ and $\mathbf{V}$ decouple the update of $\mathbf{u}$ into two parallel updates in terms of $\mathbf{u}_1$ and $\mathbf{u}_0$:

$$\mathbf{u}_1^{(j+1)} = \mathbf{D}_2^{-1} [\mathbf{D}^H \mathbf{W} \mathbf{z} + \nu_1 (\mathbf{s}^{(j+1)} + \eta_1^{(j)})], \quad (9)$$

$$\mathbf{u}_0^{(j+1)} = \mathbf{B}_2^{-1} (\mathbf{C}\mathbf{s}^{(j+1)} + \eta_0^{(j)}), \quad (10)$$

where $\mathbf{B}_2 \triangleq \frac{\lambda}{\nu_0} \mathbf{B}^H \mathbf{B} + \mathbf{I}$ and $\mathbf{D}_2 \triangleq \mathbf{D}^H \mathbf{W} \mathbf{D} + \nu_1 \mathbf{I}$ are both diagonal matrices that are trivial to invert. The closed-form update for $\mathbf{s}$ may at first appear more complicated to compute:

$$\begin{aligned} \mathbf{s}^{(j+1)} &= (\mathbf{T}^H \mathbf{V} \mathbf{T})^{-1} \mathbf{T}^H \mathbf{V} (\mathbf{u}^{(j)} - \eta^{(j)}) \\ &= (\nu_1 \mathbf{I} + \nu_0 \mathbf{C}^H \mathbf{C})^{-1} [\nu_0 \mathbf{C}^H (\mathbf{u}_0^{(j)} - \eta_0^{(j)}) + \nu_1 (\mathbf{u}_1^{(j)} - \eta_1^{(j)})]. \end{aligned} \quad (11)$$

However, since $\mathbf{C}^H \mathbf{C}$ is block circulant with circulant blocks, $\mathbf{C}^H \mathbf{C} = \mathbf{Q}^H \Phi \mathbf{Q}$ where $\mathbf{Q}$ is a (multidimensional) discrete Fourier transform (DFT) matrix and Φ is a diagonal matrix containing the spectrum of the convolution kernel of $\mathbf{C}^H \mathbf{C}$. Substituting this decomposition into (11) yields:

$$\mathbf{s}^{(j+1)} = \mathbf{Q}^H \Phi_2^{-1} \mathbf{Q} [\nu_0 \mathbf{C}^H (\mathbf{u}_0^{(j)} - \eta_0^{(j)}) + \nu_1 (\mathbf{u}_1^{(j)} - \eta_1^{(j)})], \quad (12)$$

where $\Phi_2 \triangleq \nu_1 \mathbf{I} + \nu_0 \Phi$. This formulation is simpler to compute since Φ_2 is a diagonal matrix and we implement $\mathbf{Q}$ efficiently using fast Fourier transforms (FFTs).

Fig. 2 summarizes the resulting sensitivity profile estimation algorithm, ADMM–Circ. Each stage of the proposed algorithm consists of an exact, non-iterative update. Furthermore, it can be shown that the steps in this formulation are identical to those of an ADMM algorithm applied to the real valued case where we treat the complex valued terms as a stack of their real and imaginary components. As discussed in Section IV, this equivalence allows us to

conclude that the ADMM–Circ algorithm converges to the solution of (1). In contrast, the parallel imaging reconstruction algorithm in [20] is an AL method that lacks a convergence proof due to the type of splitting used.

C. Alternating Minimization with Intermediate Updating

Updating the Lagrange multipliers η between each alternating minimization step has been shown to increase the convergence rates of several AL based algorithms [29]. We also explore this variation in our proposed algorithm by updating the relevant Lagrange multipliers after each alternating minimization step, Fig. 3. The resulting algorithm, ADMM–Circ–IU, requires no additional variables and the added updates (Step 2) are computationally inexpensive. Section IV describes the convergence properties of such adaptations.

D. Parameter Selection

Any regularized method requires the selection of the regularization parameter, λ in (1), which controls the smoothness of the sensitivity profile. We discuss how λ is selected for typical problems in Section III-C.

In addition, our proposed AL methods require that we specify values for the AL penalty parameters ν_0 and ν_1 . Following [20], we determine the parameter values using the condition numbers of the matrices requiring inversion in the alternating minimization steps. For both the ADMM–Circ method and its variation, we consider the matrices $\mathbf{B}_2$, Φ_2 , and $\mathbf{D}_2$. We normalize the coil images before performing the estimate; thus, the condition number of $\mathbf{D}_2$, $\kappa(\mathbf{D}_2)$, is $(1+\nu_1)/(d_{\min}^2+\nu_1)$ where $d_{\min}^2$ is the smallest diagonal element of $\mathbf{D}^H\mathbf{W}\mathbf{D}$. Furthermore, effective weighting matrices should have near zero values to remove the effects of noise in the low signal regions of the body coil image. Thus, $d_{\min}^2 \approx 0$ and $\kappa(\mathbf{D}_2)$ does not typically depend on the data. We therefore set our parameters by considering the condition numbers of the other two matrices, $\kappa(\mathbf{B}_2)$ and $\kappa(\Phi_2)$. Through extensive numerical simulation, we found that setting ν_0 such that $\kappa(\mathbf{B}_2) \in [225, 400]$ and then ν_1 such that $\kappa(\Phi_2) \in [200, 1000]$ provided good convergence rates for a variety of data sets.

III. Results

We evaluated our proposed sensitivity estimation methods using two very different data sets. The first experiment used simulated brain data whereas the second used real breast phantom data. Previous publications investigated the accuracy of similar regularized estimators [5]; however, there have been few comparisons with other methods concerning their effects on SENSE reconstruction quality. We therefore included an illustration of the improved SENSE reconstruction quality obtained from using regularized sensitivity estimates over standard techniques in the supplementary material. The focus of this paper is on accelerating these algorithms and thus, in this section, we compare the convergence speeds of our AL algorithms with those of conventional CG and PCG with the following circulant preconditioner (PCG–Circ):

$$\mathbf{P}_c = \mathbf{Q}^H (\mathbf{I} + \lambda \mathbf{\Omega}) \mathbf{Q}, \quad (13)$$

where $\mathbf{\Omega}$ is a diagonal matrix containing the spectrum of the convolution kernel of $\mathbf{R}^H\mathbf{R}$ [32].

We initialized each algorithm with a sensitivity profile comprising the standard ratio estimate over the object voxels and the mean magnitude and phase of these values over the non-object voxels. All of the algorithms were implemented in MATLAB (The MathWorks,

Natick, MA, USA) and the experiments were run on a PC with a 2.66 GHz, quad-core Intel Xeon CPU.

We compared the convergence properties of the algorithms using the normalized ℓ_2 -distance between the current estimate, $\mathbf{s}^{(j)}$, and the minimizer of (1), $\hat{\mathbf{s}}$:

$$\mathcal{D}(\mathbf{s}^{(j)}) = \frac{\|\mathbf{s}^{(j)} - \hat{\mathbf{s}}\|_2}{\|\hat{\mathbf{s}}\|_2}. \quad (14)$$

We focused on 2D estimation problems so that we could use Cholesky factorization to determine a non-iterative “exact” solution to (1). Using this non-iterative solution for $\hat{\mathbf{s}}$ avoids favoring a specific iterative algorithm.

A. Cost Function Setup

In defining the estimation problem (1), we chose a second-order finite differencing matrix for $\mathbf{R}$ as it resulted in more accurate sensitivity estimates than both first-order and fourth-order finite differences (results not shown). We used a binary mask, created by thresholding the body coil image, for the weighting matrix $\mathbf{W}$. This ensured that the majority of voxels in the object support were included in the estimate, while limiting the number of noisy, non-object voxels.

We selected the AL penalty parameters ν_0 and ν_1 for both experiments using the same set of condition numbers. In particular, we selected ν_0 and ν_1 such that $\kappa(\mathbf{B}_2) = 255$ and $\kappa(\Phi_2) = 650$ for both the ADMM-Circ and ADMM-Circ-IU algorithms.

B. Simulated Brain Data

Our first experiment used a 256×192 pixel, T1 weighted, transverse plane brain image from the BrainWeb database [33] (1 mm isotropic in-plane resolution, slice thickness = 1 mm, no noise). To create a more realistic MR image, we added a slowly varying phase component to the brain image. We then added complex random Gaussian noise to create a body coil image, $\mathbf{y}$, with a signal-to-noise ratio² (SNR) of 10. Fig. 4 presents the magnitude and phase of the resulting body coil image.

We simulated sensitivity profiles for four circular coils placed just outside the field-of-view (FOV) using an analytic method [34]. These sensitivities were then combined with our complex brain image and complex random Gaussian noise to create four surface coil images, $\mathbf{z}$, with SNRs of approximately 10. Fig. 5 presents the true sensitivities and their corresponding surface coil images.

We estimated the coil sensitivities using our proposed AL methods and the two CG methods. We set $\lambda = 2^5$ as this value produced accurate estimates (compared to the truth) over both the high intensity voxels and their surrounding regions. We ran 20 000 iterations of each method to ensure convergence. All of the algorithms converged to a normalized ℓ_2 -distance of less than -200 dB from, and appeared nearly identical to, the Cholesky based solution $\hat{\mathbf{s}}$. Fig. 6 presents the estimated coil sensitivities as well as their percentage difference to the truth. The convergence rates of the algorithms were similar for all four coils so we present the results for one representative coil. Fig. 7 plots $\mathcal{D}(\mathbf{s}^{(j)})$ with respect to both iteration and time for the bottom left coil in Fig. 5. ADMM-Circ-IU was the fastest

²SNR = μ_o/σ_b where μ_o is the mean of the magnitudes of the nonzero object pixels in the true image and σ_b is the standard deviation of the background pixels in the noisy image.

algorithm, converging within $\mathcal{D}(\mathbf{s}^{(j)}) = 0.1\%$ in approximately 85 seconds. PCG–Circ was faster than ADMM–Circ with convergence times of nearly 130 and 165 seconds, respectively. Conventional CG took by far the longest time at 535 seconds.

C. Breast Phantom Data

Our second experiment used a breast phantom consisting of two containers plastered with vegetable shortening and filled with “Super Stuff” bolus material (Radiation Products Design Inc., Albertville, MN, USA). Calibration data was acquired using four surface coils and one body coil on a Philips 3T scanner ($T_R = 4.6$ ms, $T_E = 1.7$ ms, matrix = 384×96). We reconstructed four surface coil images and one body coil image, each 384×96 pixels, using an inverse FFT. Figs. 8 and 9(a) show the magnitudes of the body coil image and surface coil images, respectively. This data set presents several challenges for sensitivity estimation due to the placement of coils near the center of the FOV and because of large regions of low signal both within and outside the object.

To determine a suitable regularization parameter, λ , we first estimated the coil sensitivities using the CG method for several values of λ . We then performed two-fold accelerated SENSE reconstructions [1] using each set of estimated sensitivities and compared the resulting images to the body coil image (not shown). We selected $\lambda = 2^7$ as its corresponding reconstructed image had minimal artifacts and matched closely to the body coil image.

We estimated the coil sensitivities using our proposed AL methods and the two CG methods. We ran 20 000 iterations of each algorithm to ensure that convergence was achieved. Again, the resulting estimates all converged to a normalized ℓ_2 -distance of less than -200 dB from the Cholesky based solution $\hat{\mathbf{s}}$. Fig. 9(b) presents the estimated coil sensitivities. The convergence rates of the algorithms were similar for all four coils so we present the results for one representative coil. Fig. 10 plots $\mathcal{D}(\mathbf{s}^{(j)})$ with respect to both iteration and time for the bottom left coil in Fig. 9. ADMM–Circ–IU was again the fastest algorithm, converging within $\mathcal{D}(\mathbf{s}^{(j)}) = 0.1\%$ in approximately 50 seconds. Unlike in the brain data experiment, ADMM–Circ had a similar convergence rate to PCG–Circ with both algorithms requiring approximately 100 seconds. Conventional CG again took the longest time at 445 seconds.

IV. Discussion

The sensitivity estimates generated by minimizing the cost function in (1) are smooth like true coil sensitivity profiles. As further discussed in the supplementary material, the sensitivity estimates of the brain data are highly accurate over the object and surrounding pixels. The largest errors are at the extreme corners of the image where there is no information about the true sensitivities. The flexibility of the regularized estimation method is highlighted in the breast phantom experiment by its ability to simultaneously estimate the sensitivity within both breasts and smoothly extrapolate over the regions in-between. This is particularly evident for the coils that have near uniform sensitivity over a single breast (the top right and bottom left coils in Fig. 9). As illustrated in the supplementary material, SENSE reconstructions performed with these sensitivity profiles were artifact free unlike those created using low-pass filter techniques.

ADMM–Circ–IU was the fastest method in all experiments requiring as little as half the time of PCG–Circ and a ninth the time of conventional CG. ADMM–Circ, although much faster than the CG based methods over the first few iterations, had similar convergence times to PCG–Circ in our breast experiment and was slower in our simulated brain experiment. Thus, using intermediate updating significantly accelerated our ADMM

algorithm. The CG algorithm remained the slowest method in all experiments. Interestingly, the relative convergence rate of the PCG–Circ algorithm depended on the experimental data. This behavior is partly a result of the varying accuracy of the preconditioner used in the PCG algorithm. Specifically, the circulant preconditioner used an identity matrix in place of the weighted body coil image voxel intensities (i.e., $\mathbf{I} + \lambda \mathbf{R}^H \mathbf{R}$ for $\mathbf{D}^H \mathbf{W} \mathbf{D} + \lambda \mathbf{R}^H \mathbf{R}$). This approximation works best for images that have few low signal voxels as is apparent from the decreased performance of the PCG–Circ algorithm on the breast data compared to the simulated brain data which has a higher percentage of voxels with significant signal. In contrast, our proposed ADMM algorithms do not require such approximations and their convergence speeds are therefore more robust to differences in the data.

Table I presents the approximate number of complex multiplication and addition operations required by an iteration of each algorithm. For typical finite differencing matrices, ADMM–Circ–IU, ADMM–Circ, and PCG–Circ require a similar number of operations, whereas traditional CG requires the fewest number of operations per iteration. The effect of these varying costs per iteration is highlighted by contrasting the convergence rates of each algorithm in terms of iteration and time as seen in Figs. 7 and 10. As with time, ADMM–Circ–IU needed approximately half as many iterations as PCG–Circ and ADMM–Circ. CG required significantly more iterations to converge than the other algorithms, offsetting any savings in cost per iteration.

The convergence curves for our ADMM methods exhibited non-monotonic behavior with respect to $\mathcal{D}(\mathbf{s}^{(j)})$. We found that the degree of non-monotonicity was influenced by the choice of AL penalty parameters, ν_0 and ν_1 . In fact, the parameter settings that provided the fastest convergence rates typically resulted in non-monotonicity in the $\mathcal{D}(\mathbf{s}^{(j)})$ plots.

All of our proposed algorithms converged to the solution of (1) in every experiment. As discussed after (12), our ADMM–Circ algorithm is equivalent to an ADMM algorithm with exact update steps. We can therefore conclude that this algorithm converges to the solution of (1) as per [28, Th. 8]. Our intermediate updating variant, ADMM–Circ–IU, does not have the exact formulation outlined in the hypotheses of [28, Th. 8]. However, a guarantee of convergence exists for similar ADMM variants with symmetric Lagrange multiplier updating [29]. We are currently investigating an extension of this proof to ADMM–Circ–IU.

The convergence rates of our proposed algorithms were robust to the particular choice of condition numbers used to determine the AL penalty parameters ν_0 and ν_1 . In fact, we used the same condition numbers for our two very different experiments. Furthermore, our fastest algorithm, ADMM–Circ–IU, remained faster than PCG–Circ for $(\mathbf{B}_2)$ values nearly two times larger and smaller than the optimal value and for $\kappa(\Phi_2)$ values three times larger or smaller than optimal. We also explored varying the λ value in our experiments and found that this set of condition numbers consistently worked well. The choice of the best condition numbers does not depend on the surface coil image. Therefore, if one wanted to fine-tune the convergence rate of the algorithms, a single coil of a multi-coil array would suffice.

It is a common practice in medical imaging to restrict estimates and reconstructions to within masked regions to improve both their computation time and quality over the region. If this is done for simple problems like our simulated brain data, which requires minimal interpolation within the object support, then the PCG–Circ algorithm estimating within a masked region will converge faster than our ADMM–Circ methods estimating over the entire FOV. However, this is not the case for more complicated problems like our breast phantom data. In particular, we found that our ADMM–Circ–IU algorithm, estimating over a full FOV, converged to $\mathcal{D}(\mathbf{s}^{(j)}) = 0.1\%$ at the same speed or faster than a PCG–Circ algorithm estimating within a masked region consisting of a convex hull³ surrounding the

object support, Fig. 11. Furthermore, the quality of the unmasked ADMM–Circ–IU estimates was similar to that of the masked PCG estimates over the masked region. This is partially because the weighting matrix $\mathbf{W}$ minimizes the impact of noisy voxels outside of the object support. A major disadvantage of masking is that the lack of an estimate outside the mask can lead to significant SENSE reconstruction artifacts if the object moves into this region during acquisition [6]. Thus, the mask would have to be carefully selected with this in mind. We therefore followed existing work [5] and focused on algorithms without support masks.

In addition to the algorithms presented in this work and the supplementary material, we also explored AL algorithms that incorporated simpler variable splittings. For instance, we introduced the single splitting variable $\mathbf{u}_0 = \mathbf{R}\mathbf{s}$ to (1) and similarly, $\mathbf{u}_0 = \mathbf{C}\mathbf{s}$ to (2). The AL formulations used to minimize the resulting cost functions had only two update equations. However, one of these equations required an approximate iterative solution and the resulting AL algorithms were highly sensitive to inaccuracies in the approximation. In fact, when using PCG for the approximate update step, the optimal number of inner PCG iterations was so large that the overall algorithms were slower than regular CG. Curiously, this is the type of splitting that is used in the popular split Bregman approaches [19], although there it is used in cases where $\mathbf{R}^H\mathbf{R}$ is circulant.

If the body coil data $\mathbf{y}$ is not available, one could use the square-root of the sum-of-squares of the surface coil images in its place [8], [35], [36]. Our algorithms would remain the same and only the elements of $\mathbf{D}$ would change. However, it may be more desirable in this situation to perform joint estimation of the final image and the sensitivity profiles (e.g., [8]). Such algorithms are more complicated to compute than (1) and might also benefit from an ADMM reformulation.

V. Conclusions

We developed a new iterative method, ADMM–Circ, using variable splitting and AL strategies that accelerates the regularized estimation of MR coil sensitivities. By separating the finite differencing matrix for the case of non-periodic boundary conditions into a finite differencing matrix for the case of periodic boundary conditions and a diagonal masking matrix, we were able to find a variable splitting strategy that resulted in an algorithm with exact update steps. Additionally, we demonstrated that intermediate updating of the Lagrange multipliers significantly accelerated our proposed AL algorithm. Our fastest method, ADMM–Circ–IU, had convergence speeds up to twice those of the PCG method with a circulant preconditioner.

More generally, this work illustrates how AL methods can be used to accelerate convergence for imaging problems with certain classes of quadratic cost functions. There are many areas in MR imaging where similar cost functions are used. For instance, B_0 and B_1 map estimation can be performed by minimizing cost functions over the image domain with quadratic regularization terms [37]–[39]. The application of similar acceleration techniques to these problems is currently being investigated [40].

Supplementary Material

Refer to Web version on PubMed Central for supplementary material.

³See the supplementary material for an illustration of why a convex hull is required.

Acknowledgments

We would like to thank the anonymous reviewers for their insightful comments regarding practical aspects of the algorithms. We would also like to thank Dr. T. Chenevert and Dr. M. Ivancevic for providing the breast phantom data.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and in part by the National Institutes of Health (NIH) under Grant P01 CA87634.

References

1. Pruessmann KP, Weiger M, Scheidegger MB, Boesiger P. SENSE: sensitivity encoding for fast MRI. *Mag Res Med*. Nov; 1999 42(5):952–62.
2. Sodickson DK, Manning WJ. Simultaneous acquisition of spatial harmonics (SMASH): Fast imaging with radiofrequency coil arrays. *Mag Res Med*. Oct; 1997 38(4):591–603.
3. Tsao J, Boesiger P, Pruessmann KP. k-t BLAST and k-t SENSE: Dynamic MRI with high frame rate exploiting spatiotemporal correlations. *Mag Res Med*. Nov; 2003 50(5):1031–42.
4. Schönberg, SO.; Dietrich, O.; Reiser, MF.; Baert, AL. *Medical Radiology Diagnostic Imaging Series*. Springer; 2007. *Parallel Imaging in Clinical MR Applications*.
5. Keeling SL, Bammer R. A variational approach to magnetic resonance coil sensitivity estimation. *Applied Mathematics and Computation*. Nov; 2004 158(2):53–82.
6. Larkman DJ, Nunes RG. Parallel magnetic resonance imaging. *Phys Med Biol*. Apr; 2007 52(7):R15–R55. [PubMed: 17374908]
7. Bammer R, Stollberger R, Augustin MJ, Seifert T, Strasser-Fuchs S, Wach P, Hartung HP, Fazekas F. Parallel imaging strategies for high-speed magnetic resonance diffusion imaging. *Proc SPIE* 3978. 2000; 12
8. Ying L, Sheng J. Joint image reconstruction and sensitivity estimation in SENSE (JSENSE). *Mag Res Med*. Jun; 2007 57(6):1196–1202.
9. Vemuri P, Kholmovski EG, Parker DL, Chapman BE. Coil sensitivity estimation for optimal SNR reconstruction and intensity inhomogeneity correction in phased array MR imaging. *Information Processing in Medical Im*. 2005:603–14.
10. Lin FH, Chen YJ, Belliveau JW, Wald LL. A wavelet-based approximation of surface coil sensitivity profiles for correction of image intensity inhomogeneity and parallel imaging reconstruction. *Hum Brain Map*. Mar; 2003 19(2):96–111.
11. Ballester MAG, Machida Y, Kassai Y, Hamamura Y, Sugimoto H. Robust estimation of coil sensitivities for RF subencoded acquisition techniques. *Proc Intl Soc Mag Res Med*. 2001:799.
12. Jin J, Liu F, Weber E, Li Y, Crozier S. An electromagnetic reverse method of coil sensitivity mapping for parallel MRI - theoretical framework. *J Magn Reson*. 2010; 207(1):59–68. [PubMed: 20833091]
13. Bammer R, Auer M, Keeling SL, Augustin M, Stables LA, Prokesch RW, Stollberger R, Moseley ME, Fazekas F. Diffusion tensor imaging using single-shot SENSE-EPI. *Mag Res Med*. Jul; 2002 48(1):128–36.
14. Huang F, Chen Y, Duensing GR, Akao J, Rubin A, Saylor C. Application of partial differential equation-based inpainting on sensitivity maps. *Mag Res Med*. Feb; 2005 53(2):388–97.
15. Wald LL. Parallel Imaging Update: How Many Elements Do We Need? *Proc Intl Soc Mag Res Med*. 2006:202.
16. Allison MJ, Fessler JA. An augmented Lagrangian method for MR coil sensitivity estimation. *Proc Intl Soc Mag Res Med*. 2011:2881.
17. Bertsekas DP. Multiplier methods: A survey. *Automatica*. Mar; 1976 12(2):133–45.
18. Bregman LM. The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR Comp Math and Math Phys*. 1967; 7:200–17.
19. Goldstein T, Osher S. The split Bregman method for L1-regularized problems. *SIAM J Imaging Sci*. 2009; 2(2):323–43.

20. Ramani S, Fessler JA. Parallel MR image reconstruction using augmented Lagrangian methods. *IEEE Trans Med Imag.* Mar; 2011 30(3):694–706.
21. Liu B, King K, Steckner M, Xie J, Sheng J, Ying L. Regularized sensitivity encoding (SENSE) reconstruction using Bregman iterations. *Mag Res Med.* Jan; 2009 61(1):145–52.
22. Ye X, Chen Y, Huang F. Computational acceleration for MR image reconstruction in partially parallel imaging. *IEEE Trans Med Imag.* May; 2011 30(5):1055–63.
23. Ma S, Yin W, Zhang Y, Chakraborty A. An efficient algorithm for compressed MR imaging using total variation and wavelets. *Proc IEEE Conf on Comp Vision and Pattern Recognition.* 2008:1–8.
24. Afonso MV, Bioucas-Dias JM, Figueiredo MAT. An augmented Lagrangian approach to the constrained optimization formulation of imaging inverse problems. *IEEE Trans Im Proc.* 2011; 20(3):681–695.
25. Afonso MV, JoséBioucas-Dias M, Mário A, Figueiredo T. Fast image recovery using variable splitting and constrained optimization. *IEEE Trans Im Proc.* Sep; 2010 19(9):2345–56.
26. Ramani S, Fessler JA. A splitting-based iterative algorithm for accelerated statistical X-ray CT reconstruction. *IEEE Trans Med Imag.* Mar; 2012 31(3):677–88.
27. Figueiredo MAT, JoséBioucas-Dias M. Restoration of Poissonian images using alternating direction optimization. *IEEE Trans Im Proc.* Dec; 2010 19(12):3133–45.
28. Eckstein J, Bertsekas DP. On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Mathematical Programming.* Apr; 1992 55(1–3):293–318.
29. Glowinski R, Tallec PL. Augmented Lagrangian and operator-splitting methods in nonlinear mechanics. *Soc Indust Appl Math.* 1989
30. Allison MJ, Ramani S, Fessler JA. Regularized MR coil sensitivity estimation using augmented Lagrangian methods. *Proc IEEE Intl Symp Biomed Imag.* 2012:394–7.
31. Wang Y, Yang J, Yin W, Zhang Y. A new alternating minimization algorithm for total variation image reconstruction. *SIAM J Imaging Sci.* 2008; 1(3):248–72.
32. Chan RH. Circulant preconditioners for Hermitian Toeplitz systems. *SIAM J Matrix Anal Appl.* 1989; 10(4):542–50.
33. Collins DL, Zijdenbos AP, Kollokian V, Sled JG, Kabani NJ, Holmes CJ, Evans AC. Design and construction of a realistic digital brain phantom. *IEEE Trans Med Imag.* Jun; 1998 17(3):463–8.
34. Grivich MI, Jackson DP. The magnetic field of current-carrying polygons: An application of vector field rotations. *Amer J Phys.* May; 2000 68(5):469–74.
35. Roemer PB, Edelstein WA, Hayes CE, Souza SP, Mueller OM. The NMR phased array. *Mag Res Med.* Nov; 1990 16(2):192–225.
36. McKenzie CA, Yeh EN, Ohliger MA, Price MD, Sodickson DK. Self-calibrating parallel imaging with automatic coil sensitivity extraction. *Mag Res Med.* Mar; 2002 47(3):529–38.
37. Funai AK, Fessler JA, Yeo DTB, Olafsson VT, Noll DC. Regularized field map estimation in MRI. *IEEE Trans Med Imag.* Oct; 2008 27(10):1484–94.
38. Funai A, Fessler JA, Grissom W, Noll DC. Regularized B1+ map estimation in MRI. *Proc IEEE Intl Symp Biomed Imag.* 2007:616–9.
39. Olafsson VT, Noll DC, Fessler JA. Fast joint reconstruction of dynamic R2* and field maps in functional MRI. *IEEE Trans Med Imag.* Sep; 2008 27(9):1177–88.
40. Allison MJ, Fessler JA. Accelerated computation of regularized field map estimates. *Proc Intl Soc Mag Res Med.* 2012:0413.

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix}}_{\mathbf{R}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & -1 \\ -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ \vdots & & & \ddots & & \\ -1 & 0 & 0 & 0 & & 2 \end{bmatrix}}_{\mathbf{C}}$$

Fig. 1.

The matrices **R**, **B**, and **C** for the case of 1D second-order finite differences. The top and bottom rows of **C** compute the difference between the first and last pixels, hence the need for the mask **B**.

ADMM–Circ

Initialize: $\mathbf{u}_1^{(0)} = \mathbf{s}^{(0)}$, $\mathbf{u}_0^{(0)} = \mathbf{C}\mathbf{s}^{(0)}$, $\boldsymbol{\eta}_0^{(0)} = \mathbf{0}$,
 $\boldsymbol{\eta}_1^{(0)} = \mathbf{0}$ and $j = 0$.

Set $\mathbf{D}_2^{-1} = [\mathbf{D}^H \mathbf{W} \mathbf{D} + \nu_1 \mathbf{I}]^{-1}$ and $\mathbf{z}_2 = \mathbf{D}^H \mathbf{W} \mathbf{z}$.

Set $\mathbf{B}_2^{-1} = [\frac{\lambda}{\nu_0} \mathbf{B}^H \mathbf{B} + \mathbf{I}]^{-1}$.

Set $\boldsymbol{\Phi}_2^{-1} = [\nu_1 \mathbf{I} + \nu_0 \boldsymbol{\Phi}]^{-1}$.

Repeat until stop criterion is achieved:

- 1) $\mathbf{s}^{(j+1)}$ from (12) exactly,
- 2) $\mathbf{u}_1^{(j+1)} = \mathbf{D}_2^{-1} [\mathbf{z}_2 + \nu_1 (\mathbf{s}^{(j+1)} + \boldsymbol{\eta}_1^{(j)})]$,
 $\mathbf{u}_0^{(j+1)} = \mathbf{B}_2^{-1} (\mathbf{C}\mathbf{s}^{(j+1)} + \boldsymbol{\eta}_0^{(j)})$,
- 3) $\boldsymbol{\eta}_1^{(j+1)} = \boldsymbol{\eta}_1^{(j)} - (\mathbf{u}_1^{(j+1)} - \mathbf{s}^{(j+1)})$,
 $\boldsymbol{\eta}_0^{(j+1)} = \boldsymbol{\eta}_0^{(j)} - (\mathbf{u}_0^{(j+1)} - \mathbf{C}\mathbf{s}^{(j+1)})$,
- 4) $j = j + 1$.

Fig. 2.

Overview of the ADMM–Circ algorithm. Note that $\mathbf{C}\mathbf{s}^{(j+1)}$ only needs to be computed once per iteration.

ADMM–Circ–IU

Initialize: $\mathbf{u}_1^{(0)} = \mathbf{s}^{(0)}$, $\mathbf{u}_0^{(0)} = \mathbf{C}\mathbf{s}^{(0)}$, $\boldsymbol{\eta}_0^{(0)} = \mathbf{0}$,
 $\boldsymbol{\eta}_1^{(0)} = \mathbf{0}$ and $j = 0$.

Set $\mathbf{D}_2^{-1} = [\mathbf{D}^H \mathbf{W} \mathbf{D} + \nu_1 \mathbf{I}]^{-1}$ and $\mathbf{z}_2 = \mathbf{D}^H \mathbf{W} \mathbf{z}$.

Set $\mathbf{B}_2^{-1} = \left[\frac{\lambda}{\nu_0} \mathbf{B}^H \mathbf{B} + \mathbf{I} \right]^{-1}$.

Set $\boldsymbol{\Phi}_2^{-1} = [\nu_1 \mathbf{I} + \nu_0 \boldsymbol{\Phi}]^{-1}$.

Repeat until stop criterion is achieved:

- 1) $\mathbf{s}^{(j+1)}$ from (12) exactly,
- 2) $\boldsymbol{\eta}_1^{(j+1/2)} = \boldsymbol{\eta}_1^{(j)} - (\mathbf{u}_1^{(j)} - \mathbf{s}^{(j+1)})$,
 $\boldsymbol{\eta}_0^{(j+1/2)} = \boldsymbol{\eta}_0^{(j)} - (\mathbf{u}_0^{(j)} - \mathbf{C}\mathbf{s}^{(j+1)})$,
- 3) $\mathbf{u}_1^{(j+1)} = \mathbf{D}_2^{-1} \left[\mathbf{z}_2 + \nu_1 (\mathbf{s}^{(j+1)} + \boldsymbol{\eta}_1^{(j+1/2)}) \right]$,
 $\mathbf{u}_0^{(j+1)} = \mathbf{B}_2^{-1} (\mathbf{C}\mathbf{s}^{(j+1)} + \boldsymbol{\eta}_0^{(j+1/2)})$,
- 4) $\boldsymbol{\eta}_1^{(j+1)} = \boldsymbol{\eta}_1^{(j+1/2)} - (\mathbf{u}_1^{(j+1)} - \mathbf{s}^{(j+1)})$,
 $\boldsymbol{\eta}_0^{(j+1)} = \boldsymbol{\eta}_0^{(j+1/2)} - (\mathbf{u}_0^{(j+1)} - \mathbf{C}\mathbf{s}^{(j+1)})$,
- 5) $j = j + 1$.

Fig. 3.

The ADMM–Circ algorithm with intermediate Lagrange multiplier updating (ADMM–Circ–IU). Note that $\mathbf{C}\mathbf{s}^{(j+1)}$ only needs to be computed once per iteration.

Fig. 4. The (a) magnitude and (b) phase (masked) of the body coil image for the simulated brain data.

Fig. 5.
The magnitudes of the (a) simulated sensitivity profiles and the (b) simulated surface coil images for the brain data.

Fig. 6.
The magnitudes of the (a) estimated sensitivity profiles and (b) their percentage difference to the true sensitivities for the simulated brain data.

Fig. 7. Plots of the normalized ℓ_2 -distance between $\mathbf{s}^{(j)}$ and $\hat{\mathbf{s}}$, $\mathcal{D}(\mathbf{s}^{(j)})$, with respect to iteration (top) and time (bottom) for the bottom left brain data surface coil image in Fig. 5.

Fig. 8.
The magnitude of the breast phantom body coil image.

Fig. 9.
The magnitudes of the (a) breast phantom surface coil images and the (b) corresponding estimated sensitivity profiles.

Fig. 10.

Plots of the normalized ℓ_2 -distance between $\mathbf{s}^{(j)}$ and $\hat{\mathbf{s}}$, $\mathcal{D}(\mathbf{s}^{(j)})$, with respect to iteration (top) and time (bottom) for the bottom left breast data surface coil image in Fig. 9.

Fig. 11.

Plots of $\mathcal{D}(\mathbf{s}^{(j)})$ with respect to time for ADMM-Circ-IU and PCG-Circ without masks, as well as PCG-Circ using masks with various degrees of dilation (5, 10, 20 pixels), applied to the bottom left breast data surface coil image in Fig. 9. For each case, the $\hat{\mathbf{s}}$ used in $\mathcal{D}(\mathbf{s}^{(j)})$ is the regularized solution for the appropriate mask.

TABLE I

Approximate number of complex arithmetic operations per iteration for the case of second-order finite differences

Estimator	Number of Operations
ADMM–Circ–IU	$19N + 13M^a + 2 \cdot O_{\text{FFT}}^b$
ADMM–Circ	$17N + 11M + 2 \cdot O_{\text{FFT}}$
PCG–Circ	$23N + 11M + 2 \cdot O_{\text{FFT}}$
CG	$22N + 11M$

^a $M \approx 4N$ for 2D problems.

^b O_{FFT} denotes the cost of the FFT operations ($O(N \log(N))$).