Optimal sampling for “Noquist” reduced-data cine magnetic resonance imaging

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Purpose: To analyze and optimize the signal-to-noise ratio (SNR) for the “Noquist” method for acceleration of cine magnetic resonance imaging in the presence of partially static field of view, designing practical methods for selection of optimal or near-optimal sample sets to allow reliable application of the method for variable image dimensions.

Methods: To investigate the impact of the Noquist method and its experimental parameters on the SNR in the image reconstructed from reduced data, and to explore optimization of methods for highest SNR stability, three different optimization parameters have been selected: the condition of the forward matrix (R_{cond}) as it defines the propagation of noise into the reconstructed image, and the maximum ($\Phi_{\text{max}D}$) and the mean ($\Phi_{\text{mean}D}$) linear noise amplification factor of the dynamic field-of-view (FOV) region. As SNR in a Noquist reconstruction is often not uniform across the FOV and since dynamic regions may contain the part of the image more clinically relevant, primarily these noise levels are targeted for optimization. Using these three optimization parameters, three experiments were conducted: characterization of Noquist SNR properties as a function of important image size parameters; for sufficiently small image dimensions, employment of exhaustive search using lexicographical algorithms to visit all possibilities under the cine imaging constraint that equal numbers of views are acquired at each time point of the sequence; and, departing from an hypothetically optimal pattern, generation and evaluation of SNR characteristics of a series of random variations to that optimal pattern.

Results: The impact of favorable sparse data selection is illustrated, and SNR properties are characterized as a function of relevant acquisition parameters. Optimal data selection is investigated by exhaustive methods for small image sizes, and compared with algorithmic selection patterns. Observations from these experiments are confirmed by further studies on data selection for realistic image dimensions and an optimal selection algorithm is proposed. Sixty-four cases of small image sizes were analyzed through exhaustive search with a total of 527 984 141 matrix inversions called in the process, evaluating several SNR parameters for each case. An algorithm, named “Stairwell,” that permits to design image dimensions with optimal SNR characteristics is presented, evaluated and compared with cases analyzed through exhaustive search. In 71.9% of the cases exhaustively studied, the Stairwell algorithm yielded optimal solutions. For no case did the deviation from optimum exceed 3.2% (R_{cond}), 1.0% ($\Phi_{\text{mean}D}$), and 4.9% ($\Phi_{\text{max}D}$).

Conclusions: We have demonstrated SNR-optimality of the “Stairwell” selection algorithm for small image dimensions, and performed additional experiments which all support hypothesized optimality of the algorithm for any image dimensions that satisfy certain symmetry constraints for Noquist reduced-data cine MR imaging. Furthermore, we have presented overall SNR characteristics associated with use of the Noquist method by this algorithm for practical clinical image dimensions. Additionally, observations from our optimization experiments allow us to formulate

recommendations for dimensioning Noquist image acquisition parameters which guarantee stable inversion. Moreover, these results allow prediction of the anticipated SNR properties of the reconstruction for given image dimensions (S, D, T) , relative to SNR in a conventional full-grid acquisition. © 2013 American Association of Physicists in Medicine. [<http://dx.doi.org/10.1118/1.4770270>]

Key words: MRI, image reconstruction, reduced field of view imaging, optimal sampling, SNR optimization

I. INTRODUCTION

Faster imaging remains of critical importance in cardiovascular magnetic resonance imaging (MRI). Improvements in imaging performance are currently explored through use of prior information about the image and/or instrumental parameters, which allows reconstruction from sparser data. Parallel imaging,¹⁻⁴ which exploits knowledge of coil sensitivity maps in systems with multiple receiver coils, and partial-Fourier techniques,^{5,6} which take advantage of Hermitian symmetry in the k -space data, are now commercially available. Further improvement in acquisition speed may be obtained through knowledge of spatiotemporal correlations in dynamic image sequences. The Noquist method⁷ preserves spatiotemporal resolution by a direct inversion approach, in contrast with techniques that use interpolation or spatiotemporal correlations to recover missing data.⁸⁻¹⁵ With Noquist, a more sparsely sampled dynamic image sequence is reconstructed correctly without Nyquist foldover artifact by reductions in the size of the discrete Fourier model of the k -space data for the dynamic image. This allows imaging, compared to conventional imaging techniques, with a reduced imaging time at the same resolution. Alternatively, the same methods may be used to image at higher temporal and/or spatial resolution than conventional imaging with the same imaging time. Similar to parallel imaging, these advantages are achieved at a penalty in SNR, also compared with conventional imaging. The SNR in Noquist depends on the sparse subset of (k, t) -space data selected from a full Cartesian grid. Noquist does not require synthesis of omitted conventional-grid data or after-the-fact removal of foldover artifacts, and accommodates variations in the dynamic FOV fraction easily and naturally.⁷ This paper presents a study on optimization of the Noquist method with respect to SNR criteria. In the following, a review of the Noquist method and the importance of an optimal subset selection will be introduced.

I.A. Review of the Noquist method

If only part of the FOV in the phase encoding dimension changes over time we can partition a FOV with N equidistant points into a static region with S points and a dynamic part with D points ($N = S + D$). Noquist reduces the size of a dynamic image sequence $\vec{f}$ by formulating an image model which only generates a single reconstruction of static FOV regions, instead of a separate reconstruction in each time sample or “frame” of the sequence. Dynamic FOV regions are reconstructed individually for each frame.

The principle is illustrated in Fig. 1. The k -space data $\vec{F}_t$ for each of T frames in the cine sequence may also be combined into a single, k, t -space vector $\vec{F}$, related to the dynamic image $\vec{f}$ through the regular Fourier coefficients, reorganized into a single forwarding matrix M , following Eqs. (1)–(4). Since the overall size of the image $\vec{f}$ is reduced by elimination of multiple copies of the static FOV region(s), we may accordingly reduce the size of each of the data vectors $\vec{F}_t$ until we eliminate the overdetermination of the inversion problem. An appropriate sparse data set may be obtained by selective deletion of data grid points in each frame of $\vec{F}$. Direct inversion of M yields the reconstruction matrix M^{-1} .

$$\vec{F} = M \vec{f}, \quad (1)$$

$$\vec{f} = \begin{bmatrix} \vec{f}_S \\ \vec{f}_{0,D} \\ \vec{f}_{1,D} \\ \vdots \\ \vec{f}_{T-1,D} \end{bmatrix}, \quad (2)$$

$$\vec{F} = \begin{bmatrix} \vec{F}_0 \\ \vec{F}_1 \\ \vdots \\ \vec{F}_{T-1} \end{bmatrix}, \quad (3)$$

$$M = \begin{bmatrix} M_S & M_D & 0 & \dots & 0 \\ M_S & 0 & M_D & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_S & 0 & 0 & \dots & M_D \end{bmatrix}. \quad (4)$$

For an imaging problem with N pixels in the phase encoding FOV of which S in the static portion and D in the dynamic portion ($N = S + D$) and T frames covering the cardiac cycle, we will henceforth use shorthand notation (S, D, T) . For problem dimensions (S, D, T) the number of image points to be reconstructed for a fully determined system is $N_{\text{eq}} = S + D + T$. A cine sequence with T equidistant time samples spanning the cardiac cycle thus requires acquisition of at least

$$N_{\text{frame}} = N_{\text{eq}}/T = S/T + D \quad (5)$$

samples per frame. The acceleration factor R_{Noquist} , relative to conventional full Fourier encoding for each frame, is thus

$$R_{\text{Noquist}} = \frac{N \times T}{S + D \times T}. \quad (6)$$

FIG. 1. Schematic overview of Noquist image reconstruction. The reduced raw k -space data (a) are first transformed in (horizontal) readout direction x by FFT. (b) All data for a certain column x are then simultaneously reconstructed by the Noquist inversion matrix M^{-1} , simultaneously yielding the reconstructions of the static region and the dynamic region for all time samples (c).

I.B. Data selection

The subset of N_{frame} phase-encoding k -space views for each frame determines the condition of matrix M [4] and thus the SNR in the reconstruction considering unsmoothed unregularized inversion of matrix M . Direct inversion of this matrix is stable and can be inverted depending on the sampling scheme used in the acquisition. Not all patterns will yield a nonsingular and well conditioned forward matrix M , being this the main motivation for the work described in this paper. If a sampling strategy that provides a stable well conditioned inversion problem is proposed, regularization would not then be required. Regularization could be applied to improve the SNR by replacing direct inversion by commonly used iterative conjugate gradient pseudoinversion but this would sacrifice image edge detail, thereby defeating the primary purpose of the method to preserve resolution, being this the reason to avoid regularization in this work.

Figure 2 illustrates the impact of favorable data selection on the SNR in a Noquist reconstruction. It shows side by side two images, both reconstructed from equally sized subsets of

FIG. 2. Impact of data selection for sparse data subsampling in Noquist on signal-to-noise ratio. Both images were created using a Noquist direct inversion reconstruction of a retrospectively selected subset of the exact same size from the same full-grid image. Data sizes: k -space sampling grid of 256×216 , at 18 phases across the cardiac cycle, and Noquist reconstruction assumed a centered 50% dynamic field of view. For image (a) a favorable data selection algorithm was used (the “Stairwell algorithm,” detailed in this paper); image (b) used a different subset, corresponding with an invertible, yet less well conditioned, inversion problem. $\Phi(x)$ represents the noise amplification factor through the FOV in the phase encoding direction (vertical direction).

k -space views for a scenario with a 50% static FOV fraction in a 18-frame movie image, with clear differences in SNR. These differences are associated with the condition of the inversion problem of the corresponding phase encoding reconstruction.

An accurate upper bound of the dynamic field of view is a knowledge required for a successful Noquist reconstruction, and although knowledge of the exact location is also required at the time of reconstruction, it may be corrected without need for new data if not entered correctly. A simple and general method to have the user enter both extent and location of the dynamic field of view would be a simple bounding box drawn in a scout image. Moreover, and as the heart extends about halfway the anterior/posterior field of view, a 50% dynamic FOV would be a safe and simple estimate for most individuals (apical short axis images often have a smaller dynamic FOV but if 50% is applied this shall cause no problems).

The problem addressed in this paper is the design of practical methods for selection of optimal or near-optimal sample sets to allow reliable application of the method for variable image dimensions. The large number of possible selections in the combinatorial problem of selection N_{frame} out of N phase encoding views for each frame of the sequence prohibits evaluating the most favorable choice by exhaustive methods for practical image proportions. Figure 3(a) shows three example patterns for $(S,D,T) = (4,2,4)$, for which we calculated there exist over 160 000 cine-compatible combinations (i.e., with equal acquisition time intervals, thus equal numbers of phase encodings, in each frame). All three patterns are in some fashion spaced regularly across k,t -space. Patterns (a.1) and (a.2) are ill conditioned, despite possible intuitive appeal. In (a.1) half of the phase-encoding views are never acquired. Case (a.2) acquires all phase-encodings at least once, but contains identical columns. Pattern (a.3) was generated by the Stairwell algorithm (introduced below), and of the three is the only well-conditioned pattern for Noquist inversion providing the best (smallest) noise amplification factor of all the possible exhaustively analyzed cases.

I.C. Noquist and other prior information formalisms

The problem of SNR optimization of sparse sampling for Noquist has tangents to spatiotemporal subsampling in

FIG. 3. Sampling patterns for the $(S,D,T) = (4,2,4)$ (a), the $(2,2,2)$ (b), the $(3,3,3)$ (c), and the $(4,4,4)$ (d) cases. Below each pattern the three optimization parameters are displayed: condition number with respect to inversion (R_{cond}), and mean (Φ_{meanD}) and maximum (Φ_{maxD}) noise amplification factor for the dynamic region. One of the optimal sampling patterns obtained through an exhaustive search is also given for the four cases as well as some ill-conditioned and some well-conditioned other combinations.

parallel imaging, where it is also commonly assumed that the best way to choose the data subset is to uniformly reduce the density of k -space sampling while maintaining the extent.^{1,2,4,13,15,16} Nonuniform sampling was proposed by SPACE-RIP,³ and was further investigated for SNR optimization in Ref. 17. Like Noquist, SPACE-RIP employs direct inversion. Direct inversion methods allow greater flexibility toward k -space data selection than explicit removal of aliasing, and the associated matrix inversion process implicitly provides SNR metrics for any such selection, such as the matrix condition number.¹⁸

As multicoil acquisition only changes the reconstruction if parallel imaging is used, the original Noquist formulation holds its validity without changes if multiple coils are only used for SNR gains. The available coil characteristics dictate whether a trade of SNR for imaging speed gains is most favorably done by parallel imaging or by Noquist. Thus optimization of Noquist SNR is equally relevant for scanners with multiple coils as for single-coil configurations. Combined use of parallel imaging and Noquist is described in Hamilton *et al.*,¹⁹ where the “PINOT” method (Parallel Imaging and NOquist in Tandem) was presented. PINOT method combines two prior

information formalisms: the SPACE-RIP implementation of parallel imaging and the Noquist method. The SPACE-RIP image reconstruction³ is a direct inversion of the weighted Fourier transform matrix, combined for all coils, taking into account the sensitivity maps for each coil. As Noquist also uses a direct inversion model, these two model formulations can be combined naturally to implement accelerated imaging using both sources of prior knowledge. The forward matrix M [4] would then depend also on the sensitivity profiles of the coils. As stated in Ref. 19, not all patterns will yield a nonsingular and well-conditioned combined forward matrix: different sampling schemes can be designed for reduced sampling of k - t space for Noquist and, similarly, reduced sampling for parallel imaging may be performed in different ways.^{2,3} The Hamilton et al.¹⁹ paper shows how the results of this work are combined with current heuristic sparse sampling methods for parallel imaging to offer stable (regularized or unregularized, direct or pseudo-) combined inversion. The importance of the optimization of the subset selection method for Noquist was pointed out as it is directly related to the noise amplification penalty which may become a dominant factor in image quality for the cases when images have implicitly low SNR or high-acceleration factors.

I.D. The “Stairwell” algorithm

Earlier work on Noquist investigated various empirically designed sampling patterns,⁷ including an algorithm that we have found to give stable results for a variety of image sizes and static FOV fractions. The selection pattern is illustrated in Fig. 4. The algorithm satisfies several conjectured requirements for a well-posed Noquist inversion:

1. Sub-sampling should be evenly spaced across k - t space. We have purposefully abstained from making any kind of explicit assumptions about the spatiotemporal spectral content of the dynamic FOV. As the Noquist method was designed to be generally valid within the stated constraints, and as we see no compelling reason why variable density sampling should be more beneficial for reduced-data imaging than for conventional imaging on a Cartesian grid, we have limited

our analysis to selection of a subset of equidistant Cartesian samples.

2. All Cartesian k -space grid locations should be acquired for at least one time sample. This condition assures to cover completely all the static and dynamic field of view with the forward matrix M .
3. No two phases must have the same selected subset of phase encodings. If two phases had the same selected subset of phase encodings, the forward matrix M would be under-determined as two rows of the matrix would be linearly dependent.

For problem dimensions (S,D,T) this “Stairwell” algorithm first selects D phase-encoding views per frame, evenly spaced across grid locations, which are acquired for all the frames of the dynamic acquisition. Since we need one view for each dynamic FOV pixel for a fully determined inversion system, we call these the “dynamic phase encodings.” The remaining S “static” k -grid values are visited one by one in order of ascending k -value, cycling in “flights” through the time samples, as illustrated in Fig. 4 for image dimensions $(S,D,T) = (16,4,8)$. The algorithm has produced stable results in all instances in our initial research.

I.E. Objectives

The main purposes of this work were to investigate the impact of the Noquist method and its experimental parameters on the SNR in the image, reconstructed from reduced data, and to explore optimization of methods for highest and more homogeneous SNR across the FOV. Experiments were designed to arrive at k -space view selection patterns with optimal or near-optimal SNR characteristics for normally distributed additive white noise. Phase encoding data selection departs from a full Cartesian sampling grid. Furthermore, experiments were conducted to characterize SNR properties for high-resolution Noquist images with optimal or near-optimal data selection. Optimization of SNR in a large domain of possible data selections is pursued using a working hypothesis that the Stairwell algorithm is optimal under certain symmetry constraints. We test this hypothesis by evaluating the performance of the algorithm in comparison with known optimal solutions, in order to assess its value for a robust, practical implementation of the Noquist method.

II. MATERIALS AND METHODS

II.A. MRI data

In this work only the inversion problem for specific data sets has been considered and the only human subject data that have been used is the one shown in Fig. 2, acquired following a protocol approved by the Institutional Review Board. Raw data (i.e., before reconstruction) were collected with breath-held vector-cardiogram-gated cine-imaging sequences of the heart in a normal volunteer. The MRI data shown in Fig. 2 were acquired on a 1.5T Philips Gyroscan Intera scanner (software release 7.1.2; Philips Medical Systems, Inc.) using a balanced fast field echo steady-state free precession

FIG. 4. Illustration of the “Stairwell” selection algorithm used for the case $(S,D,T) = (16,4,8)$. The location of a “dynamic” phase encoding view is called a “floor,” and a run of “static” phase encoding views, acquired in order through successive phases, is termed a “flight.”

sequence and a five-element phased-array receiver coil designed for cardiac applications, with 216 phase encodings, TR = 3.5 ms, TE = 1.7 ms, flip angle = 60°, FOV = 350 mm with 2× readout oversampling, and 18 frames per cardiac cycle.

II.B. Computing

All reconstructions and matrix analyses were implemented in MATLAB R2006a (The MathWorks, Inc., Natick, MA) on a Pentium IV class computer or newer, running Windows 2000, XP, or Linux. Timed calculations were benchmarked on a Pentium IV 3 GHz computer with 2 GB of RAM, running Windows 2000.

II.C. Optimization criteria

The SNR in a Noquist reconstruction is often not uniform across the FOV, and is typically lower and possibly variable in dynamic regions. Since dynamic regions may contain the part of the image more clinically relevant, primarily these noise levels are targeted for optimization.

The condition of the forward matrix M [Eq. (4)] defines the propagation of noise into the reconstructed image. A linear noise amplification factor $\Phi(x)$ can be formulated⁷ at each phase encoding location x , relative to the noise level in conventional full-grid DFT reconstruction at the same spatiotemporal resolution:

$$\Phi(x) = \sqrt{\frac{1}{N} \sum_{k=0}^{N_{\text{eq}}-1} |m_{xk}|^2}, \quad (7)$$

where m_{xk} are the elements of the reconstruction matrix M^{-1} . Φ may be variable across the FOV. Optimization was investigated independently with respect to three criteria:

1. $\Phi_{\text{mean}D}$: The mean value of Φ across the dynamic FOV region.
2. $\Phi_{\text{max}D}$: The maximum value of Φ encountered in the dynamic FOV region.
3. R_{cond} : The condition number of M .

II.D. Experimental design

The following experiments were conducted.

II.D.1. Characterization of Noquist SNR properties as a function of its most important parameters

Studies were performed to chart noise amplification in Noquist as a function of important image size parameters. Using the Stairwell algorithm for sub-sampling the SNR propagation was calculated as a function of the number of phases, of image size, and of static/dynamic FOV region fractions. Results were analyzed for relevant degrees of symmetry in the Stairwell patterns to formulate recommendations on favorable image dimensions.

II.D.2. Optimization by exhaustive search

For sufficiently small image dimensions exhaustive search is a feasible approach to determine optimal combinations. Lexicographical search algorithms^{20,21} visited all possibilities under the cine imaging constraint that equal numbers of views are acquired at each time point of the sequence. Performance of the Stairwell algorithm was compared with optimal results. The scope of these experiments was limited by the computational burden of exhaustive search. For example, the case $(S,D,T) = (4,4,4)$, with a total of eight phase-encodings, four phases, and 50% dynamic FOV, has 9 834 496 cine-compliant combinations, which required 4 days of CPU time. A cine-compliant combination indicates a sampling pattern that complies with the practical constraint that data acquisition intervals for each phase of the cine acquisition must be of equal length, and thus the same number of phase-encodings is acquired for each phase.

Exhaustive search optimization of $(S,D,T) = (5,5,5)$ would visit 4.08×10^{11} cine-compliant combinations. Evaluation would exceed a human lifetime in CPU time. Exhaustive search optimization was for this reason limited to problem sizes, listed in Table I, with less than 10 000 000 cine-compliant combinations.

II.D.3. Variational changes to the “Stairwell” pattern

Additional experiments were conducted to further challenge the hypothesized optimality of the Stairwell algorithm for realistic cine image dimensions where exhaustive search for optimal solutions is not feasible. Departing from the hypothetically optimal pattern, a series of random variations to the pattern were generated and evaluated for SNR characteristics. Two sets of image dimensions were studied in this fashion: $(S,D,T) = (128,128,16)$, i.e., 256 phase-encodings, 16 cardiac phases, and 50% static FOV, and $(S,D,T) = (96,48,24)$, with 144 total phase-encodings, 24 phases, at a 66.7% static FOV. Variations to the “Stairwell” pattern were generated by randomly selecting 1, 2, 5, 10, or 20 phase-encoding views for each frame, replacing each view by a randomly selected different view. Ten thousand repeat trials for each of these 10 experiments were evaluated on the three SNR propagation criteria discussed earlier.

III. RESULTS

III.A. Characterization of Noquist SNR properties

Lowest observed values, for any static/dynamic FOV ratio or image size, of the mean noise amplification factor in the dynamic region ($\Phi_{\text{mean}D}$) and the condition number (R_{cond}) of the modeling matrix M as a function of the number of frames are shown in Figs. 5(a) and 5(b), respectively. Figure 5(c) shows in a scatter diagram the mean dynamic SNR penalty factor for a large number of Noquist reconstructions as a function of the inverse scan acceleration. As a reference, the regular square-root law for SNR vs. acquisition time for multiple-NEX scans is shown.

TABLE I. Results of optimization of data selection by exhaustive search.

<i>S</i>	<i>D</i>	<i>T</i>	R_{Noquist}	No. of cine comb	Optimum				Stairwell algorithm			OPT
					No. of opt comb.	R_{cond}	$\Phi_{\text{mean}D}$	$\Phi_{\text{max}D}$	R_{cond}	$\Phi_{\text{mean}D}$	$\Phi_{\text{max}D}$	
2	1	2	1.5	9	6	2.7578	1.5275	1.5275	2.7578	1.5275	1.5275	+
4	1	2	1.6667	100	30	3.6923	1.8439	1.8439	3.6923	1.8439	1.8439	+
6	1	2	1.75	1225	140	4.4321	2.1044	2.1044	4.4321	2.1044	2.1044	+
8	1	2	1.8	15876	630	5.0645	2.3333	2.3333	5.0645	2.3333	2.3333	+
10	1	2	1.8333	213444	2772	5.6261	2.5406	2.5406	5.6261	2.5406	2.5406	+
2	2	2	1.3333	16	4	2.4142	1.4142	1.4142	2.4142	1.4142	1.4142	+
4	2	2	1.5	225	6	2.7578	1.5275	1.5275	2.7578	1.5275	1.5275	+
6	2	2	1.6	3136	8	3.2703	1.7019	1.7019	3.2703	1.7019	1.7019	−
8	2	2	1.6667	44100	30	3.6923	1.8439	1.8439	3.6923	1.8439	1.8439	+
10	2	2	1.7143	627264	24	4.0791	1.9791	1.9791	4.0812	1.9806	1.9806	−
12	2	2	1.75	9018009	140	4.4321	2.1044	2.1044	4.4321	2.1044	2.1044	+
2	3	2	1.25	25	10	3.3781	1.4511	1.4511	3.3781	1.4511	1.4511	−
4	3	2	1.4	441	14	3.3406	1.5096	1.5148	3.4191	1.5237	1.5626	−
6	3	2	1.5	7056	6	2.7578	1.5275	1.5275	2.7578	1.5275	1.5275	+
8	3	2	1.5714	108900	22	3.6166	1.6760	1.6852	3.6293	1.6760	1.6852	−
10	3	2	1.625	1656369	26	3.9767	1.7736	1.7793	3.9806	1.7736	1.7793	−
2	4	2	1.2	36	6	3.1463	1.4142	1.4142	3.1463	1.4142	1.4142	+
4	4	2	1.3333	784	14	2.4142	1.4142	1.4142	2.4142	1.4142	1.4142	+
6	4	2	1.4286	14400	20	3.1373	1.5280	1.5280	3.1852	1.5365	1.5921	−
8	4	2	1.5	245025	6	2.7578	1.5275	1.5275	2.7578	1.5275	1.5275	+
10	4	2	1.5556	4008004	14	3.4999	1.6485	1.6508	3.4999	1.6485	1.6508	−
2	5	2	1.1667	49	14	3.9156	1.4325	1.4325	3.9156	1.4325	1.4325	−
4	5	2	1.2857	1296	18	3.7633	1.4705	1.5350	3.7801	1.4726	1.5535	−
6	5	2	1.375	27225	22	3.6394	1.4931	1.4992	3.7454	1.5081	1.5471	−
8	5	2	1.4444	511225	26	3.4005	1.5432	1.5621	3.5005	1.5519	1.6387	−
10	5	2	1.5	9018009	6	2.7578	1.5275	1.5275	2.7578	1.5275	1.5275	+
2	6	2	1.1429	64	8	3.7321	1.4142	1.4142	3.7321	1.4142	1.4142	+
4	6	2	1.25	2025	10	3.3781	1.4511	1.4511	3.3781	1.4511	1.4511	+
6	6	2	1.3333	48400	40	2.4142	1.4142	1.4142	2.4142	1.4142	1.4142	+
8	6	2	1.4	1002001	14	3.3406	1.5096	1.5148	3.4191	1.5237	1.5626	−
2	7	2	1.125	81	18	4.3919	1.4252	1.4252	4.3919	1.4252	1.4252	−
4	7	2	1.2222	3025	22	3.7082	1.4469	1.4607	3.7435	1.4489	1.4847	−
6	7	2	1.3	81796	26	4.0067	1.4707	1.5742	4.0372	1.4740	1.6113	−
8	7	2	1.3636	1863225	30	3.8447	1.4824	1.496	3.9682	1.4969	1.5385	−
2	8	2	1.1111	100	10	4.2361	1.4142	1.4142	4.2361	1.4142	1.4142	+
4	8	2	1.2	4356	18	3.1463	1.4142	1.4142	3.1463	1.4142	1.4142	+
6	8	2	1.2727	132496	14	3.6142	1.4683	1.515	3.6152	1.4685	1.515	−
8	8	2	1.3333	3312400	140	2.4142	1.4142	1.4142	2.4142	1.4142	1.4142	+
2	9	2	1.1	121	22	4.8230	1.4215	1.4215	4.823	1.4215	1.4215	−
4	9	2	1.1818	6084	26	4.1671	1.4376	1.4845	4.1775	1.4383	1.4914	−
6	9	2	1.25	207025	10	3.3781	1.4511	1.4511	3.3781	1.4511	1.4511	−
8	9	2	1.3077	5664400	34	4.1851	1.4678	1.5983	4.2265	1.4717	1.6510	−
2	10	2	1.0909	144	12	4.6856	1.4142	1.4142	4.6856	1.4142	1.4142	+
4	10	2	1.1667	8281	14	3.9156	1.4325	1.4325	3.9156	1.4325	1.4325	+
6	10	2	1.2308	313600	32	3.5866	1.4513	1.4665	3.6118	1.4526	1.5046	−
8	10	2	1.2857	9363600	18	3.7633	1.4705	1.5350	3.7801	1.4726	1.5535	−
3	1	3	2	216	24	3.7025	1.8708	1.8708	3.7025	1.8708	1.8708	+
6	1	3	2.3333	42875	630	5.0723	2.3604	2.3604	5.0723	2.3604	2.3604	−
3	2	3	1.6667	1000	30	3.8511	1.6990	1.7819	3.8511	1.6990	1.8181	−
6	2	3	2	343000	24	3.7025	1.8708	1.8708	3.7025	1.8708	1.8708	+
3	3	3	1.5	3375	12	3.2255	1.5275	1.5275	3.2255	1.5275	1.5275	+
6	3	3	1.8	2000376	54	3.6648	1.7375	1.7496	3.7394	1.7477	1.7843	−
3	4	3	1.4	9261	42	4.4586	1.5942	1.6366	4.4586	1.5942	1.6366	−
6	4	3	1.6667	9261000	60	3.8511	1.6990	1.7462	3.8511	1.6990	1.8181	−
3	5	3	1.3333	21952	48	4.7064	1.5767	1.6605	4.7064	1.5767	1.6605	−
3	6	3	1.2857	46656	18	4.3911	1.5275	1.5275	4.3911	1.5275	1.5275	+
3	7	3	1.25	91125	60	5.2857	1.5567	1.6092	5.2857	1.5567	1.6092	−

TABLE I. (Continued.)

<i>S</i>	<i>D</i>	<i>T</i>	<i>R</i> _{Noquist}	No. of cine comb	Optimum			Stairwell algorithm			OPT	
					No. of opt comb.	<i>R</i> _{cond}	$\Phi_{\text{mean}D}$	$\Phi_{\text{max}D}$	<i>R</i> _{cond}	$\Phi_{\text{mean}D}$		$\Phi_{\text{max}D}$
3	8	3	1.2222	166375	66	5.5351	1.5513	1.6063	5.5351	1.5513	1.6063	–
3	9	3	1.2	287496	24	5.3117	1.5275	1.5275	5.3117	1.5275	1.5275	+
3	10	3	1.1818	474552	78	6.0363	1.544	1.5805	6.0363	1.544	1.5805	–
4	1	4	2.5	10000	120	4.6659	2.1448	2.1448	4.6659	2.1448	2.1448	+
4	2	4	2	160000	72	4.4243	1.8257	1.8257	4.4243	1.8257	1.8257	+
4	3	4	1.75	1500625	168	5.3922	1.7582	1.8320	5.3922	1.7582	1.8320	–
4	4	4	1.6	9834496	48	3.9921	1.5811	1.5811	3.9921	1.5811	1.5811	+

For each image size (S, D, T) the table shows the Noquist acceleration factor (R_{Noquist}), the number of cine combinations visited, and the optimal parameter values: forward matrix condition (R_{cond}) and mean ($\Phi_{\text{mean}D}$) and maximum ($\Phi_{\text{max}D}$) linear noise amplification factors. The table also shows optimal values found, and the number of times these optima were encountered. In addition, the table shows the values obtained with the data selection algorithm. Shown in bold are combinations for which the algorithm was optimal, and italic typeface indicates combinations with variable noise levels in the dynamic region. The rightmost column labeled “OPT” lists if this problem size does (+) or does not (–) comply with the algorithm, described in the Appendix, for finding image dimensions for which the Stairwell pattern yields optimal data selection.

Figure 6(a) shows mean noise amplification factors in static and dynamic regions, and Fig. 6(b) the condition number of the forward matrix, both as a function of the dynamic FOV fraction. These values were obtained with a constant number of 128 phase-encodings, for 2, 5, 10, 16, and 20 frames, using the Stairwell algorithm for data selection.

III.B. Optimization by exhaustive search

Table I summarizes results of experiments for optimization of (k, t) -space subset selection through exhaustive search. A total of 527 984 141 matrix inversions were called in the process. The table lists for all 64 analyzed cases the acceleration factor R_{Noquist} , the number of possible subset selections, the number of optimal solutions found, optimal values of the three SNR criteria described above, and corresponding values for the Stairwell algorithm. The three optimization criteria yielded the same optimal solutions in all cases. In 46 of the 64 (71.9%) cases (100% of cine-compliant cases), the Stairwell algorithm yielded optimal solutions. The mean optimization criteria values were $R_{\text{cond_MEAN}} = 3.8706$, $\Phi_{\text{mean}D_MEAN} = 1.6200$, and a $\Phi_{\text{max}D_MEAN} = 1.6363$. The overall mean deviation of the Stairwell algorithm results from optimum was small: 0.35% (for R_{cond}), 0.10% ($\Phi_{\text{mean}D}$), and 0.66% ($\Phi_{\text{max}D}$) from the optimal combinations obtained through exhaustive search. For no case did the deviation from optimum exceed 3.2% (R_{cond}), 1.0% ($\Phi_{\text{mean}D}$), and 4.9% ($\Phi_{\text{max}D}$).

To illustrate these numbers with some example results, Fig. 7 shows the k, t -space selection for all six optimal combinations, obtained out of 245 025 exhaustive choices, for the case $(S, D, T) = (8, 4, 2)$. These six solutions are all equivalent under permutation of phases and/or cycling of phase encodings, and include the pattern generated by the Stairwell algorithm pattern. Figures 3(b)–3(d) show k, t -space combinations for further example cases $(S, D, T) = (2, 2, 2)$, $(3, 3, 3)$, and $(4, 4, 4)$ to illustrate the range of feasible solutions. For each case we show one or more ill-conditioned combinations, one or more well-conditioned solutions, and one optimal combination which is in all three cases a pattern generated by our algorithm. These observations permit us to extrapolate the Stair-

well algorithm to larger image sizes to which an exhaustive search is not feasible to study the performance of our proposed algorithm for those cases.

III.C. Variational changes to the “Stairwell” pattern

In tests to challenge optimality of the algorithm at $(S, D, T) = (128, 128, 16)$: 256 phase-encodings and 16 phases at 50% static FOV, the algorithm itself scored $R_{\text{cond}} = 12.6489$, $\Phi_{\text{mean}D} = 1.6956$, and a $\Phi_{\text{max}D} = 1.6956$. In 10 000 tries each of random modification of 1, 2, 5, 10, and 20 phase-encodings per phase from the original Stairwell pattern, none of the resulting new patterns provided better results than the original one. As observed in Fig. 5, the upper bounds for a 16 phases acquisition are $\Phi_{\text{mean}D} = 1.6956$ and $R_{\text{cond}} = 12.6489$, exactly the same results obtained using our proposed “Stairwell” algorithm for the case $(S, D, T) = (128, 128, 16)$.

Experiments at $(S, D, T) = (96, 48, 24)$, i.e., 144 phase-encodings and 24 phases with 67% static FOV, yielded very similar results. The Stairwell pattern scored $R_{\text{cond}} = 23.4683$, $\Phi_{\text{mean}D} = 2.0412$, and a $\Phi_{\text{max}D} = 2.0412$. Again, in 10 000 tries with 1, 2, 5, 10, and 20 random changes in phase view selections per phase from the algorithm result, better SNR properties were not ever observed. When compared with the upper bounds (Fig. 5), where $\Phi_{\text{mean}D} = 1.7078$ and $R_{\text{cond}} = 18.3302$ for any 24 phases acquisition, the “Stairwell” algorithm provides a discrepancy of 19.5% for the $\Phi_{\text{mean}D}$ and 28% for the R_{cond} for the 24 phase acquisition, being the best results obtained from any of the thousands tested selections.

IV. DISCUSSION

Our results consistently confirm optimal or near-optimal performance of the “Stairwell” algorithm. These results establish the robustness of the method with respect to SNR characteristics. Observations from our optimization experiments allow us to formulate recommendations for dimensioning Noquist image acquisition parameters which guarantee stable inversion. Moreover, these results allow prediction of the anticipated SNR properties of the reconstruction for given image

FIG. 5. (a) Optimal mean noise amplification factor for the dynamic region; (b) optimal conditioning number of the reconstruction matrix as a function of the number of cardiac phases; and (c) scatter diagram of mean dynamic SNR as a function of acceleration for a wide variety of experimental parameters (number of phases, image sizes). $1/R_{\text{Noquist}}$ is linearly proportional with the scan time, and $1/\Phi_{\text{meanD}}$ is linearly proportional with the SNR in the dynamic FOV. All reconstructions were performed using the Stairwell algorithm. The $y = \sqrt{x}$ function is provided as a general theoretical reference for SNR as a function of imaging time.

dimensions (S, D, T), relative to SNR in a conventional full-grid acquisition.

Figure 6(a) shows stable and predictable SNR characteristics of Noquist as a function of the dynamic FOV fraction for a range of temporal resolutions T for a fixed image size of 128 phase encodings. Local minima in noise amplification show the optimal solutions, illustrating the benefit of symmetry in image size parameters as obtained with the conditions described above. The data points in the plot include both “true cine cases,” with exactly equal numbers of phase encodings in each phase, and slightly asymmetric grids for images sizes where in some phases one more phase encoding is ac-

FIG. 6. These plots show the noise amplification factor of the static and the dynamic region (a) and the conditioning number with respect to inversion of the reconstruction matrix (b) as a function of the % of dynamic region size over the total FOV. The plots represent, in descending order, a 128 phase-encoding Noquist reconstructions with 2, 5, 10, 16 and 20 phases. All acquisitions were made using the Stairwell algorithm. Some local optimum can be observed in these plots near the 25%, 50%, and 75% dynamic/FOV points.

quired than in others. It is interesting to observe that the mean noise amplification factor for the dynamic region converges to a specific value > 1 as the dynamic FOV fraction approaches unity. This value increases monotonically with the number of phases to an apparent asymptotic value near 1.75.

The optimization work performed in this paper allows us to formulate recommendations for dimensioning Noquist image acquisition parameters that guarantee stable inversion. In the following paragraphs we formulate: (1) necessary conditions for a nonsingular forward matrix; (2) recommendations to improve the conditioning of the matrix and reduce the noise amplification factor; (3) permutations in the data selection, to which the condition of the forward matrix is insensitive; and (4) we state key features of an algorithm which can, given any approximate Noquist problem dimensions, provide exact dimensions for the closest optimal solutions, including the associated data selection pattern.

Necessary conditions for a nonsingular matrix were confirmed by all experiments for optimization by exhaustive search.

1. All k -space grid locations should be sampled for at least one time sample.

FIG. 7. All six optimal combinations from a total of 245 025 possible cine combinations for the $(S,D,T) = (8,4,2)$ case. It can be seen that all six combinations are the same, just cycling in k or t directions. R_{cond} , Φ_{meanD} , and Φ_{maxD} represent the condition number with respect to inversion and the mean and maximum noise amplification factor of the dynamic region respectively. The three optimization parameters provide the same values for all six combinations.

2. No two phases should have the same selected subset of phase encodings.

SNR- and condition-invariant permutations in (k,t) pattern:

1. permutation of t -axis samples (“columns” in Figs. 3, 4, 7, and 8);
2. cyclic translation along k -axis;
3. mirror symmetry in the k -axis.

These symmetry conditions explain the multiplicity of optimal patterns reported in Table I. For example, the $(S,D,T) = (3,3,3)$ case has 6×3 squares in the (k,t) grid, and the Stairwell pattern gives optimal results. There are $3! = 6$ possible t -axis permutations. Either the even or the odd k -values are sampled persistently, yielding $6 \times 2 = 12$ different permutations before duplicates by further k -cycling or mirroring. For the example $(S,D,T) = (8,4,2)$ of Fig. 7 there are $2! = 2$ t -permutations. Every third k -value is sampled persistently, giving three nonduplicate k -cyclings for a total of $2 \times 3 = 6$ optimal patterns.

The exhaustive-search experiments reveal SNR optimality of the Stairwell pattern in 46 of the 64 visited problem sizes, and near-optimal results for the algorithm in all other cases.

IV.A. SNR-optimal data dimensions

The principal observations from our data, underlying the algorithm design for optimal stairwell design, are the following:

1. Cine-compliant static FOV sizes, i.e., with exactly equal numbers of phase encodings for each phase, must have complete flights, thus S must be an integer multiple κ of T .
2. Optimal SNR is observed [see Fig. 8(a) and 8(e)] with the Stairwell algorithm when the pattern of floors is regularly spaced:

- a. (for $D \leq S$) Each flight must be partitioned equally into equal segments by single floors, including a mandatory “ground floor.”
- b. (for $D > S$) Every step (a single “static” phase encoding view) of each of the flights is separated from the next by the same number of floors. This implies $D = mS$ with integer m .

We have been able, by use of prime factorizations of κ and T , to formulate an algorithm that enumerates exact optimal combinations (S,D,T) that possess flat SNR characteristics in the dynamic FOV, for given approximate image size and dynamic FOV fraction. This algorithm (the “Optimal Stairwell Algorithm”) is presented in the Appendix. In Table I the right-hand column shows a “+” for dimensions that comply with this algorithm, and “−” for image sizes which do not comply. There are a few cases with optimal SNR in the dynamic region which do not comply with the algorithm for designing optimal dimensions. In some of these cases we observed differences between the mean and maximum noise amplification factors in the *static* FOV region, indicating nonuniform static-region SNR—not considered as optimization criterion in our experiments. Importantly, we found optimal and flat (dynamic and static) SNR characteristics for *all* image dimensions that comply with the algorithm of the Appendix.

Optimality of the Stairwell algorithm under the symmetry conditions detailed in the Appendix is further supported by the results of our experiments to randomly test variational changes to the pattern for two realistic problem sizes. The symmetry conditions for hypothesized optimality are satisfied in both $(S,D,T) = (128,128,16)$, i.e., 256 phase encodings, 16 phases, and 50% dynamic FOV, and $(S,D,T) = (96,48,24)$, i.e., 144 phase encodings, 24 phases, and 33% dynamic FOV. Changes were applied to the pattern by switching in the pattern for each frame 1–20 randomly selected phase encoding views for different, also randomly selected, grid points. A total of 50 000 tries each were evaluated for each set of image dimensions. No instances were observed for either where changes resulted in better SNR.

IV.B. Near-optimal image dimensions

In Table I we further observe that in the 28.1% (18 of 64 cases) of cases where the Stairwell algorithm’s data selection did not score best, the SNR penalty rarely exceeds 1%. Although at larger image sizes this number increases some, data with reasonably high SNR allow flexibility in dynamic FOV fraction beyond the symmetry requirements of the Optimal Stairwell algorithm: we have anecdotally observed at practical image dimensions ($\sim 256 \times 256$) modest additional SNR penalties for violation of the associated symmetry constraints [typically less than 10% over optimal mean dynamic SNR factors shown in Fig. 5(a)].

Figure 8 illustrates in further detail how the SNR characteristics in a Noquist reconstruction are affected by violation of any of the symmetry rules stated above. Data selection was in all cases performed by the Stairwell algorithm. We observe optimal SNR factors Φ and constant static- and

FIG. 8. Distribution of the noise amplification factor through the FOV where the pixels are consecutively numbered, first through the static FOV, then through the phases of the dynamic FOV (four phases in all cases). Next to the noise amplification factor through the FOV the selection pattern for each case is also shown. (a) $(S,D,T) = (8,8,4)$ case, (b) $(7,9,4)$ case, (c) $(12,4,4)$ case, (d) $(9,9,4)$ case, (e) $(4,12,4)$ case, and (f) $(3,13,4)$ case.

dynamic-region SNR [Figs. 8(a) and 8(e)] for symmetric patterns. Whenever the number of (“static”) k -samples is not equal for all phases [Figs. 8(b), 8(d), and 8(f)] we see a corresponding asymmetry across the phases in noise levels. Finally, in cases where the spacing of persistently acquired (“dynamic”) k -samples is not uniform across the FOV [Figs. 8(b), 8(c), and 8(f)] we observe that the noise is variable across the dynamic and/or static FOV regions. Similar observations have also been made for larger image sizes. Quantitative aspects of these effects appear to be modest in many situations but are more difficult to predict, and vary with the problem dimensions by complex and not readily predictable patterns. For reliably stable SNR characteristics we therefore recommend use of the “optimal stairwell” algorithm, detailed in the Appendix.

V. CONCLUSIONS

A favorable subset of k -space phase-encoding views for Noquist reduced-data cine MR imaging is important, since this selection influences the SNR of the reconstruction. In this

work we have demonstrated SNR-optimality of the “Stairwell” selection algorithm for small image dimensions, and performed additional experiments which all support hypothesized optimality of the algorithm for any image dimensions that satisfy certain symmetry constraints. Furthermore, we have presented overall SNR characteristics associated with use of the Noquist method by this algorithm for practical clinical image dimensions. Our observations on small-scale image models showed in all cases investigated, which are all possible image size combinations under the constraints to have full “cine compliant” symmetry and computationally feasibility, the Stairwell solutions to be the only optimal solutions, with zero exceptions to the observed results. For not-completely compliant problem dimensions our results suggest that additional noise follows predictable and well-behaved patterns.

In conclusion:

- The Noquist method may be used with stable, near-optimal inversion properties for practically any image dimensions, and with optimal inversion/SNR properties under moderate size parameter constraints.

- Closest constrained parameter combinations for optimal SNR may be found for any temporal and spatial resolution through an empirical algorithm, designed to satisfy observed symmetry requirements for such optimality.
- The overall cost in SNR of accelerated imaging by Noquist is in the same range as the square-root law for SNR behavior with respect to imaging time.

Optimization of Noquist phase encoding view selection would be different when parallel imaging and Noquist were combined and would depend on coil characteristics. We note that the issue of coil-dependent view selection in parallel imaging is an important but almost totally unexplored question, and it has been outside the scope of this paper. At the current state of the art, a sensible approach is to seek for data selection methods that are stable and compatible with both accelerator methods. The new insights offered by the present paper address stability and optimization for Noquist alone. The methods and results of the this work may very well be important in future investigations into optimization of data sampling parallel imaging.

Not yet investigated in this research were the noise properties of the Noquist method with respect to nonrandom noise in the data, which may cause violation of the main constraint of the method that part of the FOV does not change over time. Such properties will be under investigation in a continuation of the present work.

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APPENDIX: “OPTIMAL STAIRWELL ALGORITHM” FOR DESIGN OF IMAGE DIMENSIONS WITH OPTIMAL SNR CHARACTERISTICS

The “Stairwell” algorithm:

- For each independent static FOV pixel (i.e., before any interpolation) there will be a “static” phase encoding view, acquired in only one cardiac phase of the sequence.
- A run of S (static) phase encoding views, distributed across the T cardiac phases, will be called a “flight.”
- For each dynamic FOV pixel there will be a “dynamic” phase encoding view, acquired for each phase of the sequence. These are the “floors,” spaced at regular intervals in the stairwell.

The principal observations from our data, underlying the algorithm design for optimal stairwell algorithm design, are the following:

1. Stairwell patterns with constant and optimal SNR in dynamic regions must be “cine-compliant,” i.e., must have exactly equal numbers of phase encodings for each phase, therefore must have complete “flights.” This implies that $S = \kappa T$ with integer κ .
2. For constant SNR in both the dynamic and static regions the pattern of the floors must be identical for each flight, i.e., $D = n\kappa$ with positive integer n .
3. Furthermore, the pattern of floors must be regularly spaced:
 - a. (for $D \leq S$) Each flight must be partitioned equally into equal segments by single floors, including a mandatory “ground floor.”
 - b. (for $D > S$) Every step (a single “static” phase encoding view) of each of the flights is separated from the next by the same number of floors. This implies $D = mS$ with integer m .

1. The optimal Stairwell algorithm

Primary timing-related MRI scan parameters such as TR, views per segment/echo train length, and heartbeat dictate the maximum number of independent phases in a cine scan. Therefore, in order to design Noquist cine image dimensions with optimal SNR response and flat SNR characteristics across the dynamic FOV, we start with a given number of phases T , desired approximate resolution (number of phase encodings) N , and a desired static FOV fraction φ . Determination of Noquist image dimensions follows a two-step process:

1. Select an integer “number of flights” κ to yield a number of static pixels $S = \kappa T$ approximately equal to the desired fraction φ of N :

$$\kappa \approx \varphi N / T. \quad (\text{A1})$$

2. Select a matching number of dynamic pixels $D \approx N - S$, as dictated by conditions (2) and (3):
 - a. (For $\kappa < D < S$) perform a prime factorization of T :

$$T = \prod_{\{i\}} T_i, \quad (\text{A2})$$

where similarly $\{i\}$ denotes the set of prime factors T_i of T . If again divisors T^* of T defined are accordingly:

$$T^* = \prod_{\{j\} \subset \{i\}} T_j, \quad (\text{A3})$$

the possible choices for D are in this case $D = \kappa T^*$.

- b. (For $D > S$) possible choices of D are $D = mS$ for any integer m .

By computing and ordering possible choices for D , the corresponding value φ^* closest to the original target φ is selected as the desired solution. For safe practice it is recommended to further constrain φ^* to only consider values less than φ , to avoid forcing a reconstruction with partially dynamic regions which are reconstructed as static. Since for given S the choices in D are limited, the complete selection process may require multiple tries of S (one or more T -multiples above or below first choice) if with the initial choice no combination can be found close to the target parameters. Additional or alternative convenience rules may be added to the algorithm as desired.

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