SC-GRAPPA: Self-constraint noniterative GRAPPA reconstruction with closed-form solution

Yu Ding

Davis Heart and Lung Research Institute, The Ohio State University, Columbus, Ohio 43210

Hui Xue^{a)}

Siemens Corporate Research, Princeton, New Jersey 08540

Rizwan Ahmad and Samuel T. Ting

Davis Heart and Lung Research Institute, The Ohio State University, Columbus, Ohio 43210

Orlando P. Simonetti

Davis Heart and Lung Research Institute, The Ohio State University, Columbus, Ohio 43210; Department of Internal Medicine, The Ohio State University, Columbus, Ohio 43210; Department of Biomedical Engineering, The Ohio State University, Columbus, Ohio 43210; and Department of Radiology, The Ohio State University, Columbus, Ohio 43210

(Received 15 March 2012; revised 4 November 2012; accepted for publication 5 November 2012; published 3 December 2012)

Purpose: Parallel MRI (pMRI) reconstruction techniques are commonly used to reduce scan time by undersampling the k-space data. GRAPPA, a k-space based pMRI technique, is widely used clinically because of its robustness. In GRAPPA, the missing k-space data are estimated by solving a set of linear equations; however, this set of equations does not take advantage of the correlations within the missing k-space data. All k-space data in a neighborhood acquired from a phased-array coil are correlated. The correlation can be estimated easily as a self-constraint condition, and formulated as an extra set of linear equations to improve the performance of GRAPPA. The authors propose a modified k-space based pMRI technique called self-constraint GRAPPA (SC-GRAPPA) which combines the linear equations of GRAPPA with these extra equations to solve for the missing k-space data. Since SC-GRAPPA utilizes a least-squares solution of the linear equations, it has a closed-form solution that does not require an iterative solver.

Methods: The SC-GRAPPA equation was derived by incorporating GRAPPA as a prior estimate. SC-GRAPPA was tested in a uniform phantom and two normal volunteers. MR real-time cardiac cine images with acceleration rate 5 and 6 were reconstructed using GRAPPA and SC-GRAPPA.

Results: SC-GRAPPA showed a significantly lower artifact level, and a greater than 10% overall signal-to-noise ratio (SNR) gain over GRAPPA, with more significant SNR gain observed in low-SNR regions of the images.

Conclusions: SC-GRAPPA offers improved pMRI reconstruction, and is expected to benefit clinical imaging applications in the future. © 2012 American Association of Physicists in Medicine. [<http://dx.doi.org/10.1118/1.4768162>]

Key words: MRI, parallel imaging, GRAPPA, self-constraint, real-time MRI

I. INTRODUCTION

Parallel MRI (pMRI) reconstruction techniques reduce scan time by undersampling k-space; this directly improves temporal resolution in real-time cine imaging.^{1,2} GRAPPA, a k-space based pMRI technique, is widely used clinically for MRI because of its robustness.¹ It estimates the missing k-space points by solving a set of linear equations; however, this method does not take advantage of the correlations within the missing k-space data. In reality, all k-space data within a neighborhood are correlated. These correlations can be estimated easily, and be regarded as self-constraint conditions which are not considered in GRAPPA. While it has been established that incorporating these self-constraints in parallel reconstruction greatly improves the image quality,^{3–5} this has only been previously demonstrated using iterative solutions. We propose a new formulation to incorporate this correlation

into the GRAPPA reconstruction algorithm, without requiring an iterative solver. This strategy is expected to improve the image quality by reducing noise as a consequence of the increased number of linear equations.

In the past 5 yr, linear constraints with iterative solvers have been used to improve the performance of GRAPPA reconstruction. Zhao and Hu proposed to estimate a self-constraint kernel that utilizes the reconstructed lines to interpolate the acquired lines [iterative GRAPPA (iGRAPPA)].⁵ Lustig and Pauly proposed the iterative self-consistent parallel imaging reconstruction (SPIRiT) technique, which exploits the correlations in k-space by applying a full neighborhood kernel.³ Zhang *et al.* proposed the parallel reconstruction using null operations (PRUNO) technique, which uses the null-space method to take advantage of k-space correlations.⁴

However, these methods do not have closed-form solutions, and the corresponding reconstruction problem can only

be solved using iterative solvers. There are some ubiquitous difficulties associated with iterative methods. First, defining an appropriate stopping criterion is problematic; second, convergence may not be 100% guaranteed; and third, these methods are computationally demanding and real-time reconstruction may be infeasible.

Here, we propose another approach to self-constraint GRAPPA (SC-GRAPPA) reconstruction that provides a non-iterative, closed-form solution. The constraints are formulated as a set of homogeneous linear equations. The SC-GRAPPA reconstruction solves the combination of traditional GRAPPA equations and the extra set of equations created by the self-constraint condition. Using the simple least-squares principle, a closed-form solution is derived without introducing any free parameters or regularization terms. Similar to GRAPPA, the SC-GRAPPA method can be applied using a k-space sliding window and therefore is a direct extension of GRAPPA with self-constraints.

In this paper, we first briefly review the prior art, i.e., GRAPPA, iGRAPPA, SPIRiT, and PRUNO methods, and then derive the SC-GRAPPA approach; second, we demonstrate the performance of SC-GRAPPA in phantom and volunteer studies; and finally, we discuss the results in the context of other methods.

II. THEORY

II.A. A brief review of GRAPPA reconstruction

The GRAPPA technique is a k-space channel-by-channel reconstruction method. The unacquired k-space data of each channel are estimated by interpolating the acquired k-space data of all channels within the neighborhood. Therefore, GRAPPA can be performed using convolution in k-space or can be computed efficiently in the image domain using pixel-wise multiplication.⁶ The convolution kernel can be estimated from a set of fully sampled k-space lines, called automatic calibration signal (ACS) lines. In comparison to SENSE, the GRAPPA method uses a relatively small convolution kernel which is estimated using linear regression instead of computed directly from first principles.

It is well-accepted that GRAPPA is more robust than the SENSE technique;⁷ GRAPPA can tolerate a relatively large motion mismatch between the ACS lines and the down-sampled k-space lines without generating severe artifacts. While GRAPPA reconstruction is widely used in real-time cardiac imaging, it has one shortcoming: GRAPPA has lower signal-to-noise ratio (SNR) than SENSE reconstruction. It has been observed that SENSE provides the optimal SNR and shows slightly better image quality than GRAPPA.⁸

GRAPPA reconstruction uses the acquired k-space data as the only input, and the output is an estimate of the missing k-space data. For an arbitrary k-space neighborhood, the GRAPPA technique can be represented by the following equation:

$$\mathbf{k}_{\text{unacq}} = \mathbf{g} \otimes \mathbf{k}_{\text{acq}}, \quad (1a)$$

where $\mathbf{k}_{\text{unacq}}$ and $\mathbf{k}_{\text{acq}}$ are vectorized unacquired and acquired k-space data, respectively, $\mathbf{g}$ is the GRAPPA kernel, and $\otimes$ represents convolution. Since the convolution operation can be represented as a matrix multiplication, Eq. (1) can be written as a matrix equation:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ G_{21} & -I \end{bmatrix} \begin{bmatrix} \mathbf{k}_{\text{acq}} \\ \mathbf{k}_{\text{unacq}} \end{bmatrix}, \quad (1b)$$

where G_{21} is the GRAPPA kernel convolution matrix and I is the identity matrix. Each element of Eq. (1b) is a block.

II.B. k-space self-constraint condition

All k-space data have local correlations.^{3,4} These correlations can be represented by a set of extra equations that are not included in standard GRAPPA, Eq. (1). The new method proposed in this article incorporates these equations to generate a new (larger) set of equations. Image reconstruction is performed by finding the optimal solution to the new set of equations. Three recently published papers proposed different approaches to assemble the equations which originate from the local k-space correlations. In Secs. II.B and II.C, we briefly review all three techniques and then describe our new approach to incorporate the self-constraint condition with GRAPPA.

iGRAPPA (Ref. 5) is a method that uses missing data to interpolate the acquired data as a self-constraint condition that can be summarized using the following matrix equations:

$$\begin{bmatrix} \mathbf{k}_{\text{acq}} \\ \mathbf{k}_{\text{unacq}} \end{bmatrix} = \begin{bmatrix} 0 & G_{12} \\ G_{21} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{k}_{\text{acq}} \\ \mathbf{k}_{\text{unacq}} \end{bmatrix}, \quad (2)$$

where G_{21} is the GRAPPA kernel convolution matrix [identical to Eq. (1b)] and the G_{12} is the matrix that represents the self-constraint condition.

SPIRiT (Ref. 3) utilizes a full-neighborhood kernel to interpolate the unacquired k-space data. Mathematically, SPIRiT can also be written as a self-constraint condition where each sample in k-space, acquired or unacquired, can be represented by a weighted sum of neighboring acquired and unacquired samples, yielding

$$\begin{bmatrix} \mathbf{k}_{\text{acq}} \\ \mathbf{k}_{\text{unacq}} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \mathbf{k}_{\text{acq}} \\ \mathbf{k}_{\text{unacq}} \end{bmatrix} \quad (3a)$$

or

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} S_{11} - I & S_{12} \\ S_{21} & S_{22} - I \end{bmatrix} \begin{bmatrix} \mathbf{k}_{\text{acq}} \\ \mathbf{k}_{\text{unacq}} \end{bmatrix}. \quad (3b)$$

The matrix S is the SPIRiT convolution kernel. Equations (1) and (2) can be considered as special cases of Eq. (3). The matrices G and S in Eqs. (2) and (3), respectively, can both be estimated using linear regression of the ACS lines, similar to the GRAPPA kernel estimation procedure.

Another method that utilizes k-space correlations is PRUNO.⁴ In PRUNO formulation, every k-space neighborhood can be projected onto a null space by the same projection matrix N :

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \mathbf{k}_{\text{acq}} \\ \mathbf{k}_{\text{unacq}} \end{bmatrix}. \quad (4)$$

Instead of the linear regression method, PRUNO uses singular value decomposition (SVD) to estimate the projection matrix N .⁴

The similarity of Eqs. (2)–(4) indicates how all three methods take advantage of the local k-space correlations among acquired and unacquired samples that GRAPPA does not. Both SPIRiT and PRUNO use full k-space kernels, and iGRAPPA uses kernels that either span the acquired or the unacquired k-space lines. In other words, SPIRiT and PRUNO are able to exploit more generic correlations than iGRAPPA. However, both SPIRiT and PRUNO utilize iterative solvers for image reconstruction.

There are two major differences, however, between SPIRiT and PRUNO. First, two different mathematical tools are used to solve these equations, i.e., linear regression for SPIRiT vs SVD for PRUNO. Second, the number of equations is different. The number of rows of Eq. (4) is higher than that of Eq. (3); PRUNO results in more equations because it utilizes the null-space kernels to interpolate k-space. The number of null-space kernels can be considerably larger than the number of channels.⁴ In SC-GRAPPA, we use GRAPPA reconstruction as a prior and use PRUNO equations as the self-constraint condition.

II.C. Self-constraint GRAPPA reconstruction

In this study, we propose a new linear reconstruction method, SC-GRAPPA, to incorporate self-constraint conditions into the GRAPPA reconstruction. Unlike iGRAPPA/SPIRiT/PRUNO, the new method has a closed-form solution. There are two mathematical methods that can be used to formulate this problem; the first is a convex optimization with linear constraints, and the second is the least squares (LS) problem with prior estimation. It can be proven that the second approach is more generic. Please refer to the Appendix for more details.

The linear equations of SC-GRAPPA are the following:

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{w}, \quad (5a)$$

$$0 = \mathbf{N}\mathbf{k} + \mathbf{v}. \quad (5b)$$

$\mathbf{k}_0$ in Eq. (5a) represents the GRAPPA reconstruction, used as the prior estimation. Both $\mathbf{k}$ and $\mathbf{k}_0$ are vectorized local k-space data. Equation (5b) is the self-constraint condition from PRUNO. The definition of matrix $\mathbf{N}$ is the same as Eq. (4). Both $\mathbf{w}$, $\mathbf{v}$ are random noise terms with covariance matrices $\mathbf{Q}$ and $\mathbf{R}$, respectively (details in the Appendix). According to Eqs. (A5)–(A8), the LS solution of Eq. (5) is

$$\mathbf{k} = \mathbf{k}_0 - \mathbf{R}\mathbf{N}^T(\mathbf{Q} + \mathbf{N}\mathbf{R}\mathbf{N}^T)^{-1}\mathbf{N}\mathbf{k}_0. \quad (6)$$

There is no well-established method to estimate the covariance matrices $\mathbf{Q}$ and $\mathbf{R}$. For simplicity, we set each of them as the identity matrix in this study.

Equation (6) can be implemented as a convolution in k-space, and can be calculated in every local k-space neighborhood using a sliding window. Since each k-space data point can be assessed multiple times when a sliding window is applied, a simple average is used to reach the final result. For example, if a 5×5 sliding window is selected and an 8-channel

phased-array coil is used, then the input $\mathbf{k}_0$ in Eq. (6) is a 1D vector with dimension 1×200 and the output $\mathbf{k}$ is a 1D vector of the same size. If the full k-space dimension is 128×128 , then the k-space data can be vectorized as 16384 1D vectors, each with dimension 1×200 . Because each k-space point appears in exactly 25 vectors, and each may have a different value, the final result will be the average of these 25 values.

The computation of the SC-GRAPPA reconstruction has the following four steps:

1. Estimate the GRAPPA kernel from ACS lines.
2. Estimate the null projection matrix $\mathbf{N}$ from ACS lines.
3. Perform GRAPPA reconstruction to calculate $\mathbf{k}_0$.
4. Calculate k-space of SC-GRAPPA using Eq. (6), and apply channel combination using either the sum-of-squares or B1-weighted summation.

III. METHODS

The study was approved by our institution's Human Subjects Committee and all subjects gave written informed consent to participate. All phantom and human images were acquired on a 1.5 T MRI system (MAGNETOM Avanto, Siemens Healthcare Inc., Erlangen, Germany). A 32-channel cardiac array coil (Quality Electrodynamics, Mayfield Village, OH) was used for data acquisition. The SC-GRAPPA method was implemented and all images were processed using MATLAB® 2011a (MathWorks, Natick, Massachusetts) running on a computer with Intel® Xeon® E5620 2.4 GHz CPU, 40 GB system RAM. Utilizing Eq. (6), the SC-GRAPPA reconstruction can be computed in k-space using a sliding window. The convolution operation in Eq. (6) was implemented in image space as a pixel-wise multiplication to avoid the time-consuming k-space convolution and thereby increase the speed of reconstruction significantly.⁶ This image space implementation is a direct application of the convolution theorem. The conjugate gradient (CG)-PRUNO algorithm⁴ without regularization terms was also implemented in MATLAB using a 7×7 kernel and was applied in the phantom study to provide a comparison with another self-constraint reconstruction method.

III.A. Phantom study

Seventeen time frames were acquired from the coronal orientation slice of two spherical water bottles using a steady-state free precession (SSFP) real-time cine sequence with no parallel acceleration. Detailed parameters for the SSFP sequence were: flip angle = 70° , TE = 1.15 ms, TR = 2.28 ms, pixel bandwidth = 1502 Hz/pixel, matrix size = 128×120 , FOV = 400×375 mm², and slice thickness = 5.0 mm.

The first frame of the image series was used for the ACS lines for GRAPPA, CG-PRUNO, and SC-GRAPPA. The remaining 16 fully sampled frames were uniformly down-sampled and then reconstructed using GRAPPA, CG-PRUNO, and SC-GRAPPA. All reconstructions were tested for every possible k-space trajectory, e.g., for $R = 6$, all six possible uniform down-sampled k-space trajectories were reconstructed. The kernel size used in GRAPPA reconstruction

was 4×5 ; the kernel size used in SC-GRAPPA reconstruction was 7×7 ; and the sliding window sizes used in null-space selection were 7×7 , 9×9 , and 11×11 . Using the corresponding fully sampled k-space as the gold standard, the root-mean-square-error (RMSE) of the reconstructed k-space was evaluated as a metric to measure the fidelity of both reconstructions. Since there is no well-established method to find the optimal dimension of the null-space, we plotted the RMSE vs null-space dimension selection to study it qualitatively. Both GRAPPA and SC-GRAPPA reconstructions and the corresponding RMSE measurements were repeated using the conventional GRAPPA acquisition pattern. Forty ACS lines in the center of k-space were fully sampled, and the rest of the k-space was uniformly down-sampled (please note that this strategy reduces the effective acceleration rate).

III.B. In vivo real-time cardiac MR cine images

SC-GRAPPA was tested in multiple cardiac views in two healthy volunteers. MR real-time cardiac cine images with uniformly down-sampled temporally interleaved k-space were reconstructed using both GRAPPA and SC-GRAPPA. We acquired two SSFP real-time cine series using TGRAPPA with parallel acceleration rates $R = 5$ and $R = 6$, in vertical and horizontal long-axis, and one short-axis views. Imaging parameters (for $R = 5/6$) were $192 \times 95/96$ matrix reconstructed from $192 \times 15/16$ acquired matrix, 6 mm thick slice, flip angle = 48° , TE/TR = 1.0/2.56 ms, pixel bandwidth = 1447 Hz/pixel, FOV = 380×285 mm². A total of 256 frames were acquired per image series to support statistical analysis. The ACS lines were generated using *average-all-frames*.⁹ A 7×7 sliding window size was used in the volunteer study. The null-space selection was set to be the same as the number of channels, i.e., only 32 modes belong to signal-space, and all other modes belong to null-space.

After reconstruction, the signal was estimated as the temporal mean. The noise level of images was evaluated using the MP-law method based on the random matrix theory.¹⁰ The Karhunen-Loeve transform was applied along the temporal direction, and the noise-only eigenimages were identified by assuming they follow a specific distribution function, i.e., the Marcenko-Pastur law. Once identified, the noise level can be accurately estimated from the noise-only eigenimages.¹⁰ The image artifact level was estimated using a spatial cross-correlation based method.¹¹ A template including 1% of the highest signal pixels was generated for each frame, typically comprising the brightest subcutaneous fat of the chest wall. An artifact score was defined as the cross-correlation coefficient evaluated between the template and the corresponding frame at phase encoding FOV/R (where R is the acceleration rate).¹¹

IV. RESULTS

IV.A. Phantom study

Figure 1 shows the RMSE vs iterations of CG-PRUNO in a phantom with acceleration rate $R = 6$. For CG-PRUNO reconstruction, the RMSE is consistently worse than GRAPPA

FIG. 1. The RMSE vs iterations of CG-PRUNO in a uniform phantom with acceleration rate $R = 6$. Results for two of the six possible uniformly down-sampled k-space trajectories are shown. The convergence and the RMSE improvement of CG-PRUNO depend strongly on the k-space trajectory. The SC-GRAPPA reconstruction is consistently better than the GRAPPA reconstruction.

reconstruction, regardless of the null-space selection. In addition, large fluctuation was observed among different k-space trajectories ($>20\%$). However, the SC-GRAPPA reconstruction shows a consistent RMSE improvement over GRAPPA reconstruction. The relative RMSE fluctuation for GRAPPA reconstruction in all six trajectories is less than 1.1%, and for SC-GRAPPA is less than 1.2% for all null-space selections. The relative RMSE fluctuation is even smaller at lower acceleration rate. The reconstruction times for GRAPPA and SC-GRAPPA are 0.19 and 2.0 s, respectively. The reconstruction time for CG-PRUNO varies from 2.0 to 15 s, depending on the k-space trajectory.

Figure 2 shows the GRAPPA and SC-GRAPPA reconstructions of data acquired in the phantom and the difference image. The acceleration rate $R = 6$, the sliding window size was 11×11 , and the null-space selection was 48. Note that the noise reduction in the air region was more significant than in regions of the image with high signal.

Figure 3(a) shows the SC-GRAPPA RMSE vs null-space selection at accelerate rate $R = 6$. When the null-space was properly selected, the RMSE of SC-GRAPPA was smaller than GRAPPA. Figure 3(b) shows the scaled SC-GRAPPA RMSE vs acceleration rate. The RMSE of GRAPPA reconstruction was set to be 100% for all acceleration rates. SC-GRAPPA shows higher RMSE reduction at high acceleration. Figures 3(c) and 3(d) show the results for the conventional GRAPPA acquisition pattern. The RMSE improvement was less sensitive to null-space selection.

The same results were observed for all three sliding window sizes; 7×7 , 9×9 , and 11×11 . The maximum RMSE reduction was consistently higher when the sliding window was larger. The optimal PRUNO mode selection also increased with the sliding window size.

IV.B. In vivo real-time cardiac MR cine images

Figure 4 shows the GRAPPA reconstruction, the SC-GRAPPA reconstruction, and 10 times the difference

FIG. 2. Example images acquired in a uniform water phantom. The SC-GRAPPA reconstructed image (top left), the corresponding GRAPPA reconstructed image (top right), the difference image of GRAPPA reconstruction and fully sampled k-space (bottom right), and the difference image of GRAPPA reconstruction (bottom left) demonstrate a decreased noise level in the SC-GRAPPA reconstruction. The image window width and level setting are identical for all three images. A low window level is used to provide sufficient dynamic range in the low signal region to visualize the noise.

image in a typical 4-chamber view slice with acceleration rate $R = 6$. The difference image shows that the SNR improvement of SC-GRAPPA was not spatially homogeneous. Low SNR regions of the image showed the highest noise reduction. In the lung region marked in Fig. 4, the noise variance was reduced by 54%, which is more than the effect of averaging two acquisitions. In high SNR regions, e.g., in the chest wall, the SC-GRAPPA reconstruction and the GRAPPA reconstruction were nearly identical. The difference image showed spatially variant noise.

Figure 5 shows a typical plot of the artifact score of GRAPPA reconstruction vs the artifact score of SC-GRAPPA reconstruction in the short-axis view with acceleration rate $R = 6$. The SC-GRAPPA reconstruction showed significantly lower artifact level (p -value < 0.001). At both acceleration rates ($R = 5$ and $R = 6$), the Student t -test showed that the SC-GRAPPA reconstruction had a lower artifacts level than the GRAPPA reconstruction (p -value < 0.001).

Table I shows the global SNR gain of SC-GRAPPA over GRAPPA. The SNR improvements are statistically significant ($p < 0.001$) for both acceleration rates $R = 5$ and $R = 6$.

V. DISCUSSION

We proposed a new k-space pMRI reconstruction technique called SC-GRAPPA. It combines the traditional GRAPPA reconstruction and the recently described k-space

TABLE I. The SNR gain of SC-GRAPPA compared to GRAPPA reconstruction in the volunteer images.

		Slice 1 (%)	Slice 2 (%)	Slice 3 (%)	Mean (%)
Volunteer 1	Rate 5	12.6	10.9	21.9	15.13 ($p < 0.001$)
	Rate 6	9.0	3.3	18.0	10.10 ($p < 0.001$)
Volunteer 2	Rate 5	9.8	14.6	18.9	14.43 ($p < 0.001$)
	Rate 6	14.2	9.4	5.0	9.53 ($p < 0.001$)

self-constraint condition, which originates from the correlations in k-space data. The present study indicates that the self-constraint condition can help improve the GRAPPA reconstruction by combining the self-constraint condition and GRAPPA reconstruction. The new method has a closed-form solution, thereby avoiding the need for an iterative solver. Compared to iterative self-constraint reconstruction methods,³⁻⁵ SC-GRAPPA has two generic advantages: lower computation load and higher robustness. There are no *ad hoc* assumptions about the image content. The self-constraint condition can be derived from the standard ACS lines used by GRAPPA and, therefore, the method can be utilized as a viable alternative to GRAPPA. Both the phantom study and the volunteer study demonstrate that the SC-GRAPPA reconstruction improves the image SNR without introducing observable artifacts.

The SC-GRAPPA algorithm has an interesting property: higher noise suppression effect is observed in image regions with lower SNR. In the high SNR regions, the signal content usually satisfies the self-constraint condition automatically, and applying the additional self-constraint condition may not lead to significant improvement. In the low SNR regions, the random noise does not satisfy the self-constraint condition, and applying the additional self-constraint condition shows stronger noise reduction effect. This property may help improve the local image SNR in regions where signal penetration is problematic.

There is a generic trade-off between the artifact level and the SNR. Typically, any improvement in SNR comes with a corresponding increase in artifact level. We did observe that more aggressive selection of the null-space dimension led to higher aliasing/artifacts but lower random noise in the phantom study. Our *in vivo* experiments demonstrated that SC-GRAPPA improves both SNR and artifact over GRAPPA when the null-space selection was simply set to the number of phased-array coil channels. Intuitively, optimal selection of the sliding window size and the null-space dimension should improve the performance of SC-GRAPPA. Unfortunately, there are no well-established methods to optimally select either parameter. The same problem also exists in the SPIRiT and PRUNO techniques.^{3,4} Therefore, the major weakness of the proposed method is that it depends on the sliding window size and null-space dimension of the PRUNO method. Fortunately, the reported phantom studies revealed that SC-GRAPPA is not sensitive to either parameter. In fact, SC-GRAPPA showed significant improvement over GRAPPA for all three sliding window sizes: 7×7 , 9×9 ,

FIG. 3. The RMSE of phantom images with PRUNO sliding window size = 7×7 , 9×9 , and 11×11 utilized in the SC-GRAPPA algorithm. (a) and (b): the k-space is unformed down-sampled, and the ACS lines are acquired separately. (c) and (d): a conventional GRAPPA acquisition pattern is used, i.e., fully sampled 40 lines in the center of k-space are utilized as ACS lines, and the rest of k-space is uniformly down-sampled. (a) The RMSE vs the null-space selection with acceleration rate $R = 6$ (the x-axis indicates the number of dimensions excluded from null-space selection). Even with a very conservative null-space cutoff, SC-GRAPPA still shows significant RMSE improvement compared to GRAPPA. (b) The maximum relative RMSE gain of SC-GRAPPA compared to GRAPPA. The RMSE improvement is higher at high acceleration rate and larger sliding window. (c) The RMSE vs the null-space selection with acceleration rate $R = 4$ (outer k-space only). (d) The maximum relative RMSE gain of SC-GRAPPA compared to GRAPPA.

and 11×11 . In addition, the RMSE vs the selection of the null-space dimension curve is rather flat. This indicates that even a suboptimal null-space dimension selection in SC-GRAPPA would still have lower RMSE and higher SNR than GRAPPA. Therefore, we may adopt a relatively conservative

null-space selection and a relatively large sliding window, and guarantee an improvement over GRAPPA reconstruction.

The performance of SC-GRAPPA can be further improved by incorporating additional data to fine tune the parameters.

FIG. 4. Example images acquired in one normal volunteer. The SC-GRAPPA reconstructed image (left), the corresponding GRAPPA reconstructed image (center), and the difference image magnified by 10 times (right) demonstrate that the random noise reduction is more significant in low-SNR regions of the image.

FIG. 5. A typical aliasing score for an image series with acceleration rate $R = 6$ reconstructed using GRAPPA and SC-GRAPPA. The SC-GRAPPA reconstruction shows a consistently lower artifact score (p -value < 0.001). The dashed line indicates the identity relation.

For example, if two sets of ACS lines are available, the cross-validation method can be used to find the optimal sliding window size and the modes of the null-space selection; one set of ACS lines can be used for calibration and the other set of ACS lines can be used for reconstruction validation. The optimal parameters can be determined by finding the minimal residual.⁵ As a practical approach, several prescan low-resolution images with variant slice location/orientation may also be used to further optimize the parameters.

In our prototype SC-GRAPPA implementation, the reconstruction was performed either in k -space, or in image space. Our experiments showed that the computational time was relatively low by implementing the k -space convolution as an image space point-wise multiplication.

In summary, we propose a new framework to combine the self-constraint condition with the GRAPPA technique to improve the reconstructed image quality. We must clarify that the inclusion of self-constraint condition to improve pMRI reconstruction is not new and has been proposed previously. Our contributions, however, reside in developing and validating a closed-form solution. This proposed framework combines GRAPPA with other linear constraints given in Eq. (5b).

VI. CONCLUSION

We have described a new channel-by-channel k -space image reconstruction method by incorporating the self-constraint condition into the standard GRAPPA reconstruction method. SC-GRAPPA framework, in general, has more equations than GRAPPA, but, more importantly, it has a closed-form solution. No additional assumptions or information are needed beyond those required by GRAPPA; SC-GRAPPA can be utilized in any scenario where GRAPPA can be applied. Our experiments showed that the technique has overall higher SNR than GRAPPA in real-time dynamic cardiac cine imaging, with higher gain in regions of low SNR. SC-GRAPPA offers improved SNR over GRAPPA and should provide advantages in situations where high acceleration rates are needed.

ACKNOWLEDGMENTS

This research has been supported by the National Heart, Lung, and Blood Institute; Grant No. R01HL102450. The authors thank Dr. Randall Kroeker (Siemens Healthcare, Erlangen, Germany) and Dr. Peter Kellman (National Institutes of Health, MD, USA) for their help with the MATLAB implementation of the GRAPPA reconstruction algorithm.

APPENDIX: SELF-CONSTRAINT GRAPPA

From Eq. (5), there are two intuitive ways to formulate the self-constraint condition: it can either be formulated as a problem of convex optimization with linear constraints or as a least squares problem with prior estimation. Next, we will manifest both formulations and reveal the relation between them.

The corresponding convex optimization problem with linear constraint

$$\min_x \frac{1}{2} \|k - k_0\|^2 \quad \text{s.t.} \quad Nk = b, \quad (\text{A1})$$

where k , k_0 , b , and N are vectors and matrix in the complex domain. N^H represents the Hermitian transpose of matrix N . Equation (A1) can be solved using the Lagrange method, generating the following linear equations:

$$\begin{bmatrix} I & -N^H \\ -N & 0 \end{bmatrix} \begin{bmatrix} k \\ \lambda \end{bmatrix} = \begin{bmatrix} k_0 \\ -b \end{bmatrix} \quad (\text{A2})$$

with solution

$$\begin{bmatrix} k \\ \lambda \end{bmatrix} = \begin{bmatrix} (I - N^H(NN^H)^{-1}N)k_0 + N^H(NN^H)^{-1}b \\ -(NN^H)^{-1}Nk_0 + (NN^H)^{-1}b \end{bmatrix}, \quad (\text{A3})$$

which can be written as

$$k = k_0 + N^H(NN^H)^{-1}(b - Nk_0). \quad (\text{A4})$$

The alternative formulation is the least squares problem with prior estimation:

$$\begin{aligned} k_0 &= k + w \\ b &= Nk + v, \end{aligned} \quad (\text{A5})$$

where w and v are random noise terms, representing the uncertainties in both equations. In matrix form

$$\begin{bmatrix} b \\ k_0 \end{bmatrix} = \begin{bmatrix} N \\ I \end{bmatrix} k + \begin{bmatrix} v \\ w \end{bmatrix}. \quad (\text{A6})$$

Let us assume that the noise covariance matrix can be written as

$$P = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}. \quad (\text{A7})$$

The best linear unbiased estimation (BLUE) is

$$\begin{aligned} k &= (N^H Q^{-1} N + R^{-1})^{-1} (N^H Q^{-1} b + R^{-1} k_0) \\ &= k_0 + R N^H (Q + N R N^H)^{-1} (b - N k_0). \end{aligned} \quad (\text{A8})$$

Now we will show that the second formulation can be converted back to the first formulation under proper assumptions. If we introduce two preconditions:

$$(I + N N^H)^{-1} = (N N^H)^{-1} ((N N^H)^{-1} + I)^{-1} \approx (N N^H)^{-1}, \quad (\text{A9a})$$

$$Q = R = I, \quad (\text{A9b})$$

then Eq. (A8) can be rewritten as

$$k = k_0 + N^H (N N^H)^{-1} (b - N k_0), \quad (\text{A10})$$

which is exactly the same as Eq. (A4). Obviously, Eq. (A8) is a more general form of Eq. (A4).

^{a)} Author to whom correspondence should be addressed. Electronic mail: hui-xue@siemens.com; Telephone: +1 (609) 734-6560 and +1 (609) 712-3398; Fax: +1 (609) 734-6565.

- ¹M. A. Griswold, P. M. Jakob, R. M. Heidemann, M. Nittka, V. Jellus, J. Wang, B. Kiefer, and A. Haase, "Generalized autocalibrating partially parallel acquisitions (GRAPPA)," *Magn. Reson. Med.* **47**, 1202–1210 (2002).
- ²K. P. Pruessmann, M. Weiger, M. B. Scheidegger, and P. Boesiger, "SENSE: Sensitivity encoding for fast MRI," *Magn. Reson. Med.* **42**, 952–962 (1999).
- ³M. Lustig and J. M. Pauly, "SPIRiT: Iterative self-consistent parallel imaging reconstruction from arbitrary k-space," *Magn. Reson. Med.* **64**, 457–471 (2010).
- ⁴J. Zhang, C. Liu, and M. E. Moseley, "Parallel reconstruction using null operations," *Magn. Reson. Med.* **66**, 1241–1253 (2011).
- ⁵T. Zhao and X. Hu, "Iterative GRAPPA (iGRAPPA) for improved parallel imaging reconstruction," *Magn. Reson. Med.* **59**, 903–907 (2008).
- ⁶H. Saybasili, P. Kellman, M. A. Griswold, J. A. Derbyshire, and M. A. Guttman, "HTGRAPPA: Real-time B1-weighted image domain TGRAPPA reconstruction," *Magn. Reson. Med.* **61**, 1425–1433 (2009).
- ⁷S. Skare, R. D. Newbould, D. B. Clayton, G. W. Albers, S. Nagle, and R. Bammer, "Clinical multishot DW-EPI through parallel imaging with considerations of susceptibility, motion, and noise," *Magn. Reson. Med.* **57**, 881–890 (2007).
- ⁸M. Blaimer, F. Breuer, M. Mueller, R. M. Heidemann, M. A. Griswold, and P. M. Jakob, "SMASH, SENSE, PILS, GRAPPA: How to choose the optimal method," *Top. Magn. Reson. Imaging* **15**, 223–236 (2004).
- ⁹F. A. Breuer, P. Kellman, M. A. Griswold, and P. M. Jakob, "Dynamic autocalibrated parallel imaging using temporal GRAPPA (TGRAPPA)," *Magn. Reson. Med.* **53**, 981–985 (2005).
- ¹⁰Y. Ding, Y. C. Chung, and O. P. Simonetti, "A method to assess spatially variant noise in dynamic MR image series," *Magn. Reson. Med.* **63**, 782–789 (2010).
- ¹¹Y. Ding, Y. C. Chung, M. Jekic, and O. P. Simonetti, "A new approach to autocalibrated dynamic parallel imaging based on the Karhunen-Loeve transform: KL-TSENSE and KL-TGRAPPA," *Magn. Reson. Med.* **65**, 1786–1792 (2011).