RESEARCH

Pose determination of a cylindrical (dental) implant in three-dimensions from a single two-dimensional radiograph

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Objectives: The aim was to develop an analytical algorithm capable of determining localization and orientation of a cylindrical (dental) implant in three-dimensional (3D) space from a single radiographic projection.

Methods: An algorithm based on analytical geometry is introduced, exploiting the geometrical information inherent in the 2D radiographic shadow of an opaque cylindrical implant (RCC) and recovering the 3D co-ordinates of the RCC's main axis within a 3D Cartesian co-ordinate system. Prerequisites for the method are a known source-to-receptor distance at a known locus within the flat image receptor.

Results: Accuracy, assessed from a small feasibility experiment in atypical dental radiographic geometry, revealed mean absolute errors for the critical depth co-ordinate ranging between 0.5 mm and 5.39 mm. This translates to a relative depth error ranging from 0.19% to 2.12%.

Conclusions: Experimental results indicate that the method introduced is capable of providing geometrical information important for a variety of applications. Accuracy has to be enhanced by means of automated image analysis and processing methods.

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Introduction

The loss of depth owing to the two-dimensional (2D) reproduction of a three-dimensional (3D) scene is a major limitation in projection radiography. In general, no depth information can be gained from a single-view 2D radiograph, because many points along each X-ray are mapped on to only one point in the image plane (many-to-one-mapping). The 3D position of a metallic reference sphere, however, can be determined exactly from its elliptical distortion in combination with the location of its shadow.¹ This information can be used directly to compute the effective projection geometry,^{2,3} which is essential information for understanding the image formation, for example, for 3D reconstruction from projections.² It is important to note that the approach detailed by Schulze and d'Hoedt is not a simple pose estimation by using the rule of proportion, *i.e.* to obtain

an estimation of the object-to-receptor distance by dividing the known object dimension by the projected one, and multiplying the result with the source-to-object distance. This simple estimation would only return valuable results if, and only if, the object is positioned parallel to the receptor plane. Rather, Schulze and d'Hoedt use all information inherent in the projected object shadow to compute an exact pose (position plus orientation) of the reference object. The basic idea for the work reported here is that a dental implant is also a reference body of known dimensions. Owing to its reduced symmetry when compared with a spherical object, however, a cylinder produces a much more complex radiographic shadow if its pose is arbitrary within the projection geometry. Dental implants commonly have, at least in part, the shape of a right circular cylinder (RCC), mathematically defined as a solid of circular cross-section in which the centres of the circles all lie on a single line. Hence, finding a method using the *a priori* knowledge of the implant size for exact determination of its 3D spatial position during radiographic

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exposure would be of general interest. It is important to note that although specified for the dental implant scenario here, the method has a more general application to all medical or even industrial radiographic evaluations in which regular cylinders are involved. Mathematically, a known pose of the RCC reduces the degrees of freedom (df) for a rigid object from six to only one. This is shown in this article. Assuming a point source, a general radiographic projection geometry involves nine image-relevant df, six of which are covered by the rigid object.² Using a holding device eliminates the three df between source and receptor plane, leaving the six degrees of the object. Obviously, exact pose determination of the object solves for these remaining df. This information may be used for various applications, *e.g.* localization of the implant inside the human body or motion analysis in the case of medical radiography. Here, cylindrical parts of endoprostheses deeply submerged in the human body could be exactly localized with respect to the surrounding tissue, and a possible (unintended) movement with respect to the tissue could be followed. By means of radiostereometric analysis,⁴ an even higher level of localization accuracy may be obtained. Also, the knowledge obtained from the method introduced here may be used to reconstruct 3D information from two or more 2D radiographic projections.^{5,6} Although research has been performed to use implants for co-registration of CT and digital 2D radiographs⁷ or to determine the 3D position of geometrically known sparse objects in an iterative optimization procedure from a single projection,⁸ the author is not aware of any work published using an analytical approach based on exploitation of the object geometry to infer the exact 3D position of a cylinder of known dimensions.

The key problem to solve here is to locate the main axis of a given RCC of known diameter in a 3D co-ordinate system from a single radiographic projection. The objective of this research can be stated as follows: given a moderately constrained projection geometry which is practically relevant (*i.e.* in holding device-based intraoral or in C-arm-based medical radiography), determines the 3D position of distinct points located on the main axis of a cylindrical implant using *a priori* knowledge of its diameter. This article will (1) introduce a mathematical solution based on analytic geometry and (2) present some experimental data validating the algorithm.

Materials and methods

Algorithm

The author makes two assumptions regarding the projection geometry, both of which are fulfilled in the holding device-based intraoral radiographic technique; first, that the shortest distance, $\overline{FO'} = |z_F|$, between the focal spot, F , and the image receptor is known and, second, that the flat receptor is centred relative to the source in such a way that the locus (origin $O'(0,0,0)$) of

the ray (central X-ray) incident perpendicular onto the receptor is defined. All points located within the $x-y$ receptor plane are indicated by a prime mark. Let the central X-ray be collinear with the z -axis in a 3D Cartesian co-ordinate system. We seek to recover the co-ordinates of the centre points $N(x_N, y_N, z_N)$ and $M(x_M, y_M, z_M)$ of the RCC end caps (Figures 1 and 2).

The image outline boundary of an RCC cast from a single point source, ignoring scatter, is necessarily produced by X-rays tangential to it. The shadow's shape will be rectangular or trapezoidal with two lines defining its lateral sides. The sum of all tangents to the lateral straight cylinder sides form one plane (I, II) at each side, with the RCC being encapsulated by them at a mathematically defined position, as this article will show (Figure 1). Each plane is defined by three points: the upper ($P'(x_P, y_P, z_P)$, $Q'(x_Q, y_Q, z_Q)$) and lower ($R'(x_R, y_R, z_R)$, $S'(x_S, y_S, z_S)$) tangent end points at both the lateral sides of the RCC image boundary and the source point $F(0,0,z_F)$, common to both planes. The equations of the planes I and II in 3D are given by:

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 \text{ and} \\ A_2x + B_2y + C_2z + D_2 &= 0 \end{aligned} \quad (1)$$

Plane I and II intersect in a line, L_I , through F , which is necessarily parallel to the long axis of the RCC (for proof see the Appendix). From Equation (1) we obtain L_I as follows:

$$x(A_1 - A_2) + y(B_1 - B_2) + z(C_1 - C_2) + D_1 - D_2 = 0 \quad (2)$$

The coefficients $A_1, A_2, \dots, D_1, D_2$ are computed from the determinant:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \quad (3)$$

where the subscript 1, 2, 3 denotes the three plane-defining points F, P', R' or F, Q', S' , respectively.

Since L_I is parallel to the main axis L_{II} of the RCC, its direction vector, v_I , determines the angulation of the cylinder relative to the receptor. The co-ordinates of v_I are given by:

$$x_{v_I} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}; y_{v_I} = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}; \text{ and } z_{v_I} = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \quad (4)$$

If L_I is not parallel to the receptor ($x-y$) plane, the equation for L_{II} can be transformed to:

$$x = kz + a \text{ and } x = hz + b \quad (5)$$

The specific case of parallelism simplifies the algorithm considerably and will be discussed in a separate paragraph at the end of this section.

Figure 1 Drawing of the projection geometry including all cylinder landmark points. For the sake of clarity, the image of the cylinder and the respective image points are not included here (see Figure 2 for image landmark points). The x - and y -axis of the co-ordinate system represent the horizontal and vertical axis, respectively, of the flat image receptor, the z -axis is collinear with the central X-ray. Planes I and II represent the sum of all tangents to both lateral cylinder sides emanating from the focal spot F and intersecting in a line, L_I , at angle φ . L_I is necessarily parallel to the line L_{II} through the main cylinder axis. The distance $|KM|$ between L_I and the line through the RCC main axis, L_{II} , is calculated from φ and the known RCC radius r . As L_I and L_{II} are parallel, they span a plane III bisecting φ which intersects with the $x-z$ plane in a line through $\overline{FF'}$, on which $G(x_G, 0, z_G)$ represents the point of intersection of L_{II} and $\overline{FF'}$. x_G and y_G are calculated from $|KM|$ and the angle ζ , which can be derived from the equations of L_I and $\overline{FF'}$.

The not parallel case

From Equation (5) and the equation defining a line through a given point ($F(0,0,z_F)$) parallel to v_I :

$$\frac{x-x_1}{x_{VI}} = \frac{y-y_1}{y_{VI}} = \frac{z-z_1}{z_{VI}} \quad (6)$$

we obtain:

$$x = \frac{x_{VI}}{z_{VI}}z + \left(-z_F \frac{x_{VI}}{z_{VI}}\right) \text{ and } y = \frac{y_{VI}}{z_{VI}}z + \left(-z_F \frac{y_{VI}}{z_{VI}}\right) \quad (7)$$

Since we seek to find the spatial position of L_{II} , we have to determine the shortest distance d between L_I and L_{II} given by:

$$d = \frac{r}{\cos \frac{\varphi}{2}} \quad (8)$$

with r representing the radius of the RCC and φ the angle between plane I and II, calculated from

$$\varphi = \arccos \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{(A_1^2 + B_1^2 + C_1^2)(A_2^2 + B_2^2 + C_2^2)}} \quad (9)$$

If we shift L_I by a vector with the norm $|d|$ to L_{II} , we obtain the co-ordinates we are interested in, *i.e.* the position of the long RCC axis. Since L_I and L_{II} are parallel, they define a plane III bisecting φ as follows:

$$x(A_1 + A_2) + y(B_1 + B_2) + z(C_1 + C_2) + D_1 + D_2 = 0 \quad (10)$$

To find the intersection point $G(x_G, 0, z_G)$ of L_{II} with the $x-z$ -plane (Figure 1), we first have to find the equation for the line through $\overline{FF'}$ resulting from the intersection of plane III and the $x-z$ plane. Solving equation (10) for x , setting $y=0$ yields

$$x_{F'} = -\frac{D_1}{(B_1 + B_2)} - \frac{D_2}{(B_1 + B_2)} \quad (11)$$

where

Figure 2 Image landmark points in the experimental projection radiograph, in which for explanatory reasons, the x - y -receptor coordinate system with origin $O'(0,0,0)$ has been included. P' , Q' and R' , S' represent the points to be identified for evaluation, since they are the images of the upper and lower end points of the tangents formed by all X-rays being tangent to the lateral sides of the RCC. The lines defined by $P'R'$ and $Q'S'$ clearly converge towards the top of the image, indicating an angulated position of the RCC relative to the x - y -detector plane

$$x = -\frac{x_{F'}}{z_F} z + x_{F'} \quad (12)$$

We can now calculate the direction vector $v_{FF'}$ from

$$z_{VFF'} = \cos \alpha_{FF'} = \frac{-z_F}{FF'} \text{ and } x_{VFF'} = \cos \beta_{FF'} = \frac{x_{F'}}{FF'} \quad (13)$$

Note that $y_{VFF'} = 0$, since this line is lying within the x - z -plane. To find $G(x_G, 0, z_G)$, we have to calculate the norm of the segment $\overline{FG}$ given by

$$|\overline{FG}| = \frac{d}{\sin(180^\circ - \varsigma)} \quad (14)$$

with

$$\varsigma = \arccos \frac{(z_{VFT} z_{VFF'} + x_{VFT} x_{VFF'})}{\sqrt{(z_{VFT}^2 + x_{VFT}^2)(z_{VFF'}^2 + x_{VFF'}^2)}} \quad (15)$$

$\overline{FT}$, the segment between the source point and the point of intersection, $T(x_T, y_T, 0)$, between L_I and the receptor (x - y -) plane, is computed from solving Equation (7) for x_T and y_T and inserting the co-ordinates of F and T in

$$\overline{FT} = \sqrt{x_F^2 + y_T^2 + z_T^2} \quad (16)$$

The co-ordinates $x_{VFT}, y_{VFT}, z_{VFT}$ of the direction vector v_{FT} are calculated according to Equation (13).

$$G(x_G, 0, z_G) \text{ is calculated from } z_G = z_F - \sqrt{\overline{FG}^2 - x_G^2} \quad (17)$$

$$\text{and } x_G = \frac{\overline{FG} x_{F'}}{\overline{FF'}} \quad (18)$$

$$\text{with } \overline{FF'} = \sqrt{z_F^2 + x_{F'}^2} \quad (19)$$

Next, we have to compute the points of interest, $N(x_N, y_N, z_N)$ and $M(x_M, y_M, z_M)$, as defined by the intersection of L_{II} with the lines connecting F with the images of N and M , i.e. $N'(x_{N'}, y_{N'}, 0)$ and $M'(x_{M'}, y_{M'}, 0)$. Hereby, $M' \in \overline{P'Q'}$ and $N' \in \overline{R'S'}$. $N'(x_{N'}, y_{N'}, 0)$ is calculated from the co-ordinates of the end points and those of the direction vector $v_{R'S'}$ as follows:

$$x_{N'} = x_{R'} - x_{VR'S'} p_{R'S'} \text{ and } y_{N'} = y_{R'} - y_{VR'S'} p_{R'S'} \quad (20)$$

with

$$x_{VR'S'} = x_{S'} - x_{R'} \text{ and } y_{VR'S'} = y_{S'} - y_{R'} \quad (21)$$

and

$$p_{R'S'} = \frac{x_{R'}(B_1 + B_2) + x_{R'}(C_1 + C_2) + D_1 + D_2}{(B_1 + B_2)x_{VR'S'} + (C_1 + C_2)y_{VR'S'}} \quad (22)$$

$M'(x_{M'}, y_{M'}, 0)$ is computed analogously. Since the line through $\overline{FN'}$ is given by

$$x = z \left(-\frac{x_{N'}}{z_F} \right) + x_{N'} \text{ and } y = z \left(-\frac{y_{N'}}{z_F} \right) + y_{N'} \quad (23)$$

we finally find the z -co-ordinate of N from

$$z_N = \frac{x_{N'} - \left(-\left(\frac{x_{VI} z_G}{z_{VI}} + x_G \right) \right)}{\left(\frac{x_{VI}}{z_{VI}} \right) - \left(-\frac{x_{N'}}{z_F} \right)} \quad (24)$$

and z_M analogously. Finally, x - and y -co-ordinates of M and N are calculated from Equation (7).

The parallel case

If Equation (5) is not defined, parallelism between the RCC main axis L_{II} and the receptor ($x-y$) plane is necessarily the case, yielding $\overline{KM} = \overline{FG}$. Calculating $z_M = z_N$ is straightforward:

$$z_G = z_M = z_N = \frac{(x_{F'} - x_G)z_F}{x_{F'}} \quad (25)$$

with x_G obtained from Equation (18).

x - and y -co-ordinates are computed from:

$$x_N = \frac{x_{N'}(z_F - z_N)}{z_F} \quad (26)$$

and

$$y_N = \frac{y_{N'}(z_F - z_N)}{z_F} \quad (27)$$

with x_M, y_M calculated analogously.

Experimental evaluation

On an optical bench (source-to-receptor distance 255.0 mm) complying with the requirements specified above, a steel RCC (diameter 6.00 mm, length 15.55 mm) was exposed at two different angulations (1: 12° both vertical and horizontal tilt; 2: -25° vertical and 10° horizontal tilt) on a dental charge-coupled device (CCD) sensor (Full Size, Sirona Dental Systems, Bensheim, Germany; physical pixel size: 19.5 $\mu\text{m} \times 19.5 \mu\text{m}$). Both angulations were exposed with and without scattering equivalent (4 wax plates of 1.5 mm thickness each) at two exposure times (0.08 s, 0.12 s) and two computed pixel sizes (19.5 μm ; 39.0 μm), yielding a total of 12 images. Exported as 8 bit uncompressed bitmap files, the co-ordinates of the four landmark points, P', R', Q' and S' , were visually identified by one observer (RS). By means of a software-implemented, mouse-driven measurement tool of image-editing software (Adobe Photoshop 7.0, Adobe Software, Mountain View, CA), their co-ordinates were manually assessed in triplicate. x -, y - and z -co-ordinates of M and N were computed using the algorithm implemented in spreadsheet software (Excel 2000, Microsoft Corporation, Redmond, CA). Truth was assessed to the nearest 0.5 mm by means of a calliper. Accuracy was computed as absolute differences

between true and calculated co-ordinates over the entire set of assessments and images. Precision was calculated from intraindividual differences between assessments on each individual image, and averaged over all images.

Results

The true z -(depth) co-ordinate of the upper landmark point, M , was 42.5 mm in the first and 15.1 mm in the second configuration. For the lower end point N , it was 39.0 mm in the former and 21.5 mm in the latter angulation. Depth co-ordinates yielded least accuracy, with an average error of 1.7 mm for the lower point, N , and 2.6 mm for the upper end point, M (Table 1). The vast majority of the differences were positive, indicating a clear trend towards underestimation of depth (Figure 3). Absolute error in accuracy was dependent on the actual exposure setting, with the largest values (5.4 mm) found for 1:1 binning and 0.12 s exposure time in combination with the scattering equivalent (Figure 4). Precision ranged between 0.00 mm and 1.55 mm for the critical depth co-ordinate (Table 1).

Theoretical error estimation

Assuming that the implant diameter is known at sufficient accuracy (≤ 0.1 mm), the following input parameter will affect method accuracy: the source-to-receptor distance z_F and the co-ordinates of the shadow end points P', R', Q' and S' . Since z_F essentially operates as multiplier in the determinant (Equation (1)) as well as in subsequent equations of the algorithm, errors in z_F result in depth errors of similar magnitude. Essentially, they induce scaling errors in the projection geometry. Small errors in the definition of the origin $O'(0,0,0)$ will also be of limited effect, as they result only in a small change of the landmark-point co-ordinates, the relation, of which, to one another yet remains stable.

The critical z -co-ordinate is determined by the distance d between the intersecting line, L_I , of the tangential planes I and II and the computed main implant axis, L_{II} , (Equation (8)). This step is directly dependent on the angle φ between plane I and II (Equation (9)), which is computed from the coefficients A_1 , to D_2 derived from the input co-ordinates. Hence, even small errors in assessing the latter will have a major influence on φ and the resulting z -co-ordinate. Without loss of generality, we may consider the parallel case, in

Table 1 Mean absolute accuracy ($\pm$ standard deviation) and mean precision ($\pm$ standard deviation) for x -, y -, z -co-ordinates averaged over all images and assessments. All values are given in millimetres

Error	z_N^*	x_N^*	y_N^*	$z_M^\dagger$	$x_M^\dagger$	$y_M^\dagger$
Accuracy	1.50 (± 1.36)	0.72 (± 0.04)	0.06 (± 0.20)	2.57 (± 1.38)	0.19 (± 0.10)	0.24 (± 0.15)
Range accuracy	-1.45; 3.57	0.67; 0.81	-0.20; 0.35	0.50; 5.39	0.04; 0.37	-0.03; 0.55
Precision	0.55 (± 0.52)	0.01 (± 0.01)	0.02 (± 0.03)	0.63 (± 0.60)	0.02 (± 0.02)	0.03 (± 0.02)
Range precision	0.13; 1.40	0.00; 0.02	0.00; 0.04	0.00; 1.55	0.00; 0.05	0.00; 0.10

*Lower end point on main cylinder axis

†Upper end point on main cylinder axis

Figure 3 Box plots representing accuracy as expressed as difference between true and calculated co-ordinates for the upper (*M*) and lower (*N*) RCC main axis end point. Within each box, the median is represented by a bold horizontal line, and the whiskers define the three-fold interquartile distance. While accuracy for *x*- and *y*-coordinates was generally good, the boxes of the critical *z*-co-ordinates indicated a clear trend towards underestimation of true distance of *M* and *N* from the image receptor

which the central X-ray passes through L_{II} , which is located at a depth $z_G = 30.0$ mm ($z_F = 250.0$ mm) and orientated parallel to the *y*-axis. Here, an error of 1 pixel (± 0.039 mm) at either side will result in a depth error of ± 3.35 mm for an RCC with $r = 2.5$ mm. If $r = 1.5$ mm, the depth error will even be as large as ± 5.14 mm. It will increase with increasing pixel size and

Figure 4 Accuracy (difference: truth-calculated) with respect to different exposure configurations (1–6): 1:1:1 binning, 0.8 s, without scatter equivalent; 2:2:2 binning, 0.8 s, without scatter equivalent; 3:1:1 binning, 0.8 s, with 6 mm scatter equivalent; 4:2:2 binning, 0.8 s, with 6 mm scatter equivalent; 5:1:1 binning, 1.2 s, with 6 mm scatter equivalent; 6:2:2 binning, 1.2 s, with 6 mm scatter equivalent

decreasing RCC diameter. A larger source-to-receptor distance will decrease the ratio between shadow and pixel size, thereby also increasing the absolute depth error.

Discussion

Dental implants, or at least parts thereof, commonly have the shape of an RCC, the dimensions of which are accurately known *a priori*. It is generally accepted to use the implants' radiographic shadow as a reference to determine local magnification,^{9,10} although this method is very sensitive to alignment errors.^{10–12} Again, it is important to notice that our analytical approach, at least in theory, for error-free radiographs is independent on the true object pose, *i.e.* the angulation and position of the RCC relative to the receptor plane. It is not a simple guess by using the RCC's magnification to roughly estimate its distance from the detector; rather, it nails down the exact location of it plus its angulation. This is done by exploiting the entire geometrical information inherent in the RCC's radiographic shadow in such a way that the spatial position plus orientation of the RCC is derived from it. Mathematically, this is only feasible within a moderately constrained environment, such as applied in holding-device-based intraoral radiography, in which, because of the construction of the device, the central X-ray intersects the detector plane at its centre. Holding devices also allow for easy determination of the source-to-receptor distance, for example, when a scale is attached to them. Consequently, in this well-established radiographic technique, the df are limited to six possible object movements (three translational and three rotational). The author's approach provides two distinct points (*M*, *N*) in space located on the RCC's main axis. They account for five df: three translational and two rotational. One degree remains unknown: the rotation about the main axis. One additional reference point located not collinearly with this axis and identifiable in each 2D view would be sufficient to solve for this rotation.

Information obtained from the algorithm facilitates *a posteriori* calculation of the projection geometry, accurate assessment of local magnification, localization of the implant within the body, motion analysis from a series of 2D images or 3D reconstruction from two or more views. Applications of the method are not restricted to dental radiography, since cylinders are also common shapes in other medical implants (for example, stents, screw-shaped implants, etc). This means that, for instance, quantitative evaluation of a radiograph would be much easier, as lengths measured could be much more accurately corrected for distortion and magnification. By computing the RCC's position in two follow-up radiographs, a relative motion between the radiographs can be estimated. This information would be helpful if the radiograph is evaluated, for example, for scientific purposes; however, it may be relevant, primarily, in

medical (endoprosthesis) or industrial (cylindrical interior object parts) applications. In cases for which the imaging geometry has to be fixed *a priori*, our method may be used to compute it *a posteriori*. Radiostereometric analysis utilizes reference bodies to infer relative motion between two or more instances of projections to a very high degree of accuracy.⁴ An RCC located at a distinct position relative to one projection would clearly facilitate this process. Another interesting application of the method is to use the information of the projection geometry to back-project image information into some volume, *i.e.* for 3D reconstruction from two or more images.² An accurately known imaging geometry is the fundamental prerequisite for the back-projection process used in 3D reconstruction techniques. It has been shown that, after determination of the effective imaging geometry, back-projection techniques to acquire 3D information on the object are straightforward.^{2,5}

Geometric unsharpness and noise obviously deteriorate the level of accuracy of the algorithm. Three main factors causing geometric unsharpness have to be considered: (i) focal spot size, (ii) scatter and (iii) the discrete data-sampling process. Also, structures attached to or superimposed over the RCC image render the stable and accurate identification of its boundary points or lines a challenging task. On the other hand, the author is very confident that fully automatic detection of the required landmarks is possible at a very high level of accuracy. The key problem to solve will be the correct landmark point definition, *i.e.* correct detection of the shadow boundary cast by the RCC. As evident from the error estimation, even small errors of only ± 1 pixel may result in a considerable error in depth. For the typical spatial relations as applied in the study, it can be concluded that, as to be expected, the manual evaluation is very inaccurate. A visual/manual detection accuracy of ± 1 to 2 pixels was observed for each landmark point. Reproducibility, *i.e.* precision, was also low, accounting

for up to 1.6 mm error of a maximum accuracy error of 5.4 mm. Accuracy in assessing the true co-ordinates, however, was also limited to ± 0.5 mm at maximum. Most of the computed depth co-ordinates were short of the true distance from the receptor, corresponding to an underestimation of the angle φ between the tangential planes encapsulating the RCC. In other words, the shadow's width was also underestimated. Obviously, boundary pixels belonging to the implant's shadow were assigned to the background because of the non-linear decrease of grey values towards the RCC boundary. The penumbra caused by the focal spot size also adds to this error. It is quite obvious that automated image analysis methods are required to enhance accuracy. It has been demonstrated that, by means of sophisticated image analysis and processing methods adapted to the specific task, a very high level of accuracy in the 3D localization of a reference body from a single projection can be reached.² The most crucial task is to accurately identify the tangent lines spanned by $\overline{P'R'}$ and $\overline{Q'S'}$, respectively, which will be initiated by automated segmentation of the RCC image, for example, by an edge detection algorithm (see, for example, Canny¹³). In the presence of noise (and anatomical structures) a line-based Hough transform^{14,15} will accurately detect the required lines. The author is currently developing prototype software implementing these methods.

In conclusion, the method locates a cylinder in 3D space accurately from a single radiographic projection by means of analytical geometry. This position information is useful for a variety of applications. Errors inherent in current digital radiographic imaging technology deteriorate the level of accuracy of the method, particularly if the image evaluation process is performed manually. Future developments and automated image feature recognition software, however, should help to overcome a great part of these shortcomings.

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Appendix

The fundamental assumption of the algorithm is that the tangential planes I and II intersect in a line, L_I , orientated parallel to the RCC's main axis, L_{II} . Under this assumption, any given tangent point W will remain a tangent point, regardless of the actual position of the focal spot, F , as long $F \in L_I$ and $L_I \parallel L_{II}$.

Proof

Let W_1, W_1' be tangent points to both sides of the RCC obtained from a source position F_1 , where the long implant axis L_{II} is normal to the plane defined by W_1, W_1', F_1 , i.e. the cross-section cut by this plane through the cylinder is a true circle (Figure 5a,b). We define K_1 as the Cartesian co-ordinate system coplanar with W_1, W_1', F_1 , its z -axis being collinear with the line

through F_1 and the centre M_1 (= origin of K_1) of the circular cross-section with the RCC radius r (Figure 5b). W_2 and W_2' are the corresponding tangent points for any arbitrary position of F_2 on a line L_I parallel to L_{II} at an arbitrary distance d (Figure 5c), where the cross-section through the RCC is necessarily an ellipse. The latter is proved by the fact that the cross-section is a conic section. It follows that $\{M_1, M_2\} \in L_{II}$.

We simply have to prove, that $W_1 = W_2$ or $W_1' = W_2'$.

A tangent to an ellipse at a given point $W(x_0, z_0)$ is given by:

$$\frac{xx_0}{a^2} + \frac{zz_0}{b^2} = 1 \Rightarrow \frac{x_0}{a^2}x + \frac{z_0}{b^2}z - 1 = 0 \quad (A1)$$

or

$$A^2a^2 + B^2b^2 + C^2 = 0 \quad (A2)$$

b

c

Figure 5 Drawing introducing all elements necessary for the proof of parallelism between the line, L_I , obtained from the intersection of the tangential planes I and II and the line through the RCC main axis, L_{II} . (a) The view from the side, with the source position F_1 defined as that particular source position, in which the central X-ray through F_1M_1 intersects L_{II} perpendicularly at the point $M_1 \in L_{II}$, resulting in a circular RCC cross-section of the radius, r , within the plane defined by F_1, M_1, W . We have to prove, that the resulting tangent point W will also be a tangent point within the respective plane F, M, W (with $M \in L_{II}$) for any arbitrary position of F on a line L_I parallel to L_{II} at an arbitrary distance d . Apart from source position F_1 , all resulting RCC cross-sections will be an ellipse with the short radius, r , and the long radius, a . (b) The two-dimensional co-ordinate system, K_1 , resulting from the source position F_1 , with the z -axis collinear with the central X-ray through F_1M_1 . (c) The co-ordinate system, K_2 , which is obtained from source position F_2 , again with the z -axis being aligned with the central X-ray

with $a=r$ denoting the long and b the short diameter of the ellipse, respectively.

By substituting the parameters A, B, C from Equation (A 1), in (A 2) we obtain

$$\begin{aligned} \frac{x_0^2}{r^4} r^2 + \frac{z_0^2}{b^4} b^2 + 1 &= 0 \Rightarrow \\ \frac{x_0^2}{r^2} + \frac{z_0^2}{b^2} + 1 &= 0 \end{aligned} \quad (\text{A3})$$

In K_1 , W_1 can be expressed by its co-ordinates x_{W_1}, z_{W_1} (Figure 5b), where

$$z_{W_1} = \frac{r^2}{d} \quad (\text{A4})$$

and x_{W_1} , represents the altitude in the right triangle $\Delta F_1 M_1 W_1$ given by:

$$x_{W_1} = \sqrt{dz_{W_1} - z_{W_1}^2} = \sqrt{r^2 - \frac{r^4}{d^2}} \quad (\text{A5})$$

Thus, if $W_1 = W_2$, the following equation must be correct:

$$z_{W_2} = \frac{z_{W_1}}{\cos \phi} = \frac{r^2}{d \cos \phi} \quad (\text{A6})$$

Since $a=r$ (Figure 5a) is given by

$$a = \frac{r}{\cos \phi}$$

we can substitute z_0 in Equation (A3) by $z_{W_1} = z_{W_2}$ and x_0 by x_{W_2} and obtain

$$\frac{r^4 (\cos \phi)^2}{d^2 (\cos \phi)^2 r^2} + \frac{r^2 - \frac{r^4}{d^2}}{r^2} + 1 = 0 \quad (\text{A7})$$

and after simplification

$$\frac{r^2}{d^2} + 1 = \frac{r^2}{d^2} + 1 \quad (\text{A8})$$

This is obviously correct, hence $W_1 = W_2$.