Stationary convection in dilute solutions of ^3He in superfluid ^4He

(Bénard/thermohydrodynamics)

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ABSTRACT Two symmetric, convecting steady states have been observed in a novel cell of unity aspect ratio and studied over a range of temperature for two concentrations of ^3He in superfluid ^4He . An existing theory due to Parshin has been related to the conditions necessary for convection in this system, defining a Rayleigh number closely analogous to that of a classical one-component Bénard system. Values of this Rayleigh number at the onset of convection calculated from experimental data are found to have little temperature dependence, with an average value near that for a classical one-component fluid in this geometry. The Prandtl number is small and temperature dependent, with a smallest calculated value of 0.05.

After the initial recognition by Ahlers (1) of the considerable experimental advantages of using a cryogenic environment to investigate the Rayleigh–Bénard problem, several workers have reported experiments on both liquid ^4He (2–5) and liquid $^3\text{He}/^4\text{He}$ systems (6–8). In addition to the unique qualities of the helium fluids themselves, high-precision thermometry, conditions of low extraneous heat transport, and construction materials of both low specific heat and high thermal diffusivity, all necessary for the ideal experimental apparatus, are readily available at low temperatures. $^3\text{He}/^4\text{He}$ Bénard systems are particularly rich in experimental phenomena. Above the λ -transition they behave (6) like classical binary mixtures (9) with an unusually wide experimentally accessible range of parameter variation. Near the tricritical point they behave like compressible fluids (V. Steinberg, preprint), and below the λ -transition they represent the only example thus far investigated (6, 8) of a Bénard system with the novel qualities of superfluid hydrodynamics (10).

EXPERIMENTAL

An overview of this work has appeared (8) and, because a detailed description of the experimental cell will be published elsewhere, only a brief schematized description of the apparatus will be presented here. Fig. 1a *Inset* shows a scale schematic drawing of our unity aspect ratio cell, indicating how heat must be supplied from above in order to produce convection in our experimental temperature range. Also shown is one novel feature of the measurement geometry, which allows measurement of horizontal inhomogeneity in the heat flux. The upper slab was constructed of two pieces of copper pressed together but thermally isolated from one another by a thin layer of Mylar plastic. This allowed the heat flow out of each piece into the fluid to be measured independently while the temperatures of the two pieces were controlled to be equal. An annulus of fused quartz, ground to precise tolerances, separated this assembly from another polished copper slab in contact with a heat sink.

FIG. 1. (a) The Nusselt number (Nu), defined as the ratio of the effective conductance to the diffusive conductance, plotted versus the reduced temperature difference. (*Inset*) Schematized drawing of the experimental cell. (b) Power ratio, $\dot{Q}_{\text{plug}}/\dot{Q}_{\text{ring}}$, plotted versus the reduced temperature difference. O, Most stable state, identified as center-fluid-falling toroidal convection; Δ , symmetrical state with center-fluid-rising. $T = 0.825$ K; $X = 0.0024$.

Data were obtained in the form of an effective conductance, $\dot{Q}_{\text{tot}}/\Delta T$, and a power ratio, $\dot{Q}_{\text{plug}}/\dot{Q}_{\text{ring}}$ for each temperature difference at a particular temperature and concentration. Data of this type for $T = 0.825$ K and ^3He molar concentration, X , of 0.0024 are presented as an example in Fig. 1. In Fig. 1a the effective conductance has been divided by the diffusive conductance to produce a Nusselt number, Nu . Well below ΔT_c the heat flow is diffusive and neither Nu nor the power ratio depends strongly on the temperature difference. As ΔT is increased, the thermal conductance increases continuously with a slight rounding at the transition to convection. The effective conductance both above and below the transition could be fitted to straight lines whose intersection was defined as ΔT_c , the critical temperature difference for convection. Near to ΔT_c , an increase in the temperature difference brings about a decrease in the power ratio. At temperature differences such that convection is well-developed, however, the ratio of plug power to ring power is almost independent of the imposed thermal gradient, with a value of about 9–10%.

Although large step changes in the temperature difference were generally avoided, and therefore the larger temperature difference measurements were almost always preceded by smaller ones, varying the order of the measurements did not

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significantly increase their scatter. Large step changes were generally avoided because, especially in the early weeks of the experiment, they tended to produce a discontinuous transition to a time-dependent convecting state (this will be discussed elsewhere).

All of the early measurements on the time-independent convecting state were found to have power ratios below the ratio of the plug area to the area of the ring in contact with the fluid. Later on in the experiment, a power ratio was measured with a value near 14%. On the basis of our previous data and this new information, we inferred that we most often produced a toroidal velocity field with the fluid velocity in the center directed downward, parallel to the heat flow and therefore the ^4He normal fluid velocity. Occasionally the symmetrical state with upward-directed center fluid velocity was established.

Owing to the independent control of the temperatures of the plug and the ring, it was found to be possible to drive the convective state initially toward either the center-fluid-rising state or the more stable center-fluid-falling state. For example, the less-stable state could be produced reliably by deliberately setting the temperature of the plug slightly below that of the ring, with $\Delta T < \Delta T_c$, and then increasing the overall temperature difference to a value well above ΔT_c . This would produce a region, near the plug, of higher ^3He concentration and therefore lower mass density which would then be displaced radially outward, due to the gravitational instability. After center-fluid-rising steady convection had become well-established, the temperature of the plug could be restored to a value equal to that of the ring to within our typical precision. We were able to observe the center-fluid-rising steady state over a wide range of temperature and temperature difference.

In Fig. 1 the data corresponding to the center-fluid-rising state are represented as triangles. The plot of Nu as a function of $\Delta T/\Delta T_c$ shows that Nu of this less-stable state falls below that of the corresponding Nu for the center-fluid-falling state. The center-fluid-rising state was found to have a larger value of ΔT_c for all cases studied.

One other feature of the less-stable state can be seen in Fig. 1b and must be mentioned. After the convective velocity field was set up and the boundary conditions were established, the convecting state was stable over long time periods against perturbations at most temperature differences. However, if the temperature difference across the cell was gradually reduced, there always existed a temperature difference above ΔT_c where the center-fluid-rising state would decay in time to the center-fluid-falling state. For the data presented in Fig. 1 this decay occurred for $\Delta T \approx 1.065 \Delta T_c$.

Although the qualitative features of the steady convecting states were the same for all cases studied, the numerical values of some of the parameters associated with the convecting state often had strong temperature and concentration dependences. In Fig. 2 we display the measured critical temperature difference, corrected for series Kapitza resistances, plotted semilogarithmically as a function of temperature for two concentrations. The largest values of ΔT_c presented here were based on data taken at temperature differences as large as 0.04 K. This constitutes a significant fraction of the temperature. The lowest values of ΔT_c were obtained at a reduced precision because the temperature difference could be maintained constant only to within about $1\text{--}2 \times 10^{-5}$ K. For these reasons, values of ΔT_c less than about 1×10^{-3} K may be less reliable. A final consideration is that the Kapitza boundary resistance correction is largest for low temperatures and for high thermal conductivity—namely, low concentrations. For the $X = 0.0024$ data this correction was as large as 5% per boundary and, although care was taken to measure this resistance, it must be regarded as an estimate, in

FIG. 2. Critical temperature difference for convection, as defined in text, plotted semilogarithmically versus temperature, for two concentrations: O, $X = 0.0047$; Δ , $X = 0.0024$. Note that the critical temperature difference for the lower concentration decreases more rapidly with temperature. Here and elsewhere in this paper, the notation " $K \times 10^3$ " signifies that that actual value was " $x \times 10^{-3}$ K."

part due to the probable inhomogeneous nature of the Kapitza resistance (11).

The center-fluid-rising steady convecting state was always found to have a slightly larger ΔT_c than its symmetrical counterpart. In Fig. 3 we present values for the 0.24% solution of $(\Delta T_{c\uparrow} - \Delta T_{c\downarrow})/\Delta T_c$ for which data on both states were sufficient to warrant an estimate of the difference. The precision of these data extends from about 50% at high T to about 20% at low T . The displacement between $\Delta T_{c\uparrow}$ and $\Delta T_{c\downarrow}$ is probably not caused by an inhomogeneous Kapitza resistance, because experiments with quite different plug and ring temperatures had no observable effect on $\dot{Q}/\Delta T$ at a given ΔT .

Another important quantitative feature of the convective state is shown in Fig. 4. In this figure, the initial slope of the convective heat transfer increase with temperature difference is plotted, in dimensionless form, against temperature for two concentrations. As mentioned before, the aspect ratio of our cell

FIG. 3. Percentage difference between the critical temperature difference for the more-stable center-fluid-falling and the less-stable center-fluid-rising toroidal roll convective structures as a function of temperature. $X = 0.0024$. (No data are shown for the higher concentration.)

FIG. 4. The slope of the reduced effective thermal conductance curve above the transition, as defined by the change in Nu with reduced temperature difference, as a function of temperature for two concentrations: O, $X = 0.0047$; Δ , $X = 0.0024$. $t \equiv (\Delta T - \Delta T_c)/\Delta T_c$.

is very nearly 1; however, no finite aspect ratio corrections (12) have been applied to these data because the theory of convection in superfluid solutions is relatively undeveloped. The numerical values of the data on the initial slopes should be good to within 5% except perhaps at the higher temperatures and concentration at which the critical temperature differences are quite small.

Because the regimen of diffusive heat flow was usually well-defined, a reliable measurement of the thermal conductivity of these solutions could also be made. Fig. 5 displays thermal conductivity data on the two well-studied solutions as well as additional thermal conductivity measurements on a solution of ^3He at molar concentration $X = 0.0132$. Also shown for comparison are several measurements made by Ptukha (13). As with the measurements of ΔT_c , the Kapitza resistance is the

FIG. 5. Thermal conductivity of dilute solutions of ^3He in superfluid ^4He as a function of temperature. Δ , $X = 0.0024$; O, $X = 0.0047$; $\square$, $X = 0.0132$; $\times$, $X = 0.0136$ [Ptukha (13)].

major source of inaccuracies. Thus, the lower thermal conductivities are the most accurate, although the least accurate values should still have an accuracy better than 10%, with a precision considerably better than that. These data are the most reliable available for these concentrations over this temperature range for purposes of evaluating a Rayleigh number for this system.

INTERPRETATION

A one-component fluid Bénard system can be described (14, 15) by one Rayleigh number with a single boundary-condition-dependent critical Rayleigh number R_c characterizing the onset of convection, which first occurs in a stationary mode. Recent interest in one-component systems lies not only in the similarity of the onset of convection at R_c to a second-order phase transition (16) but also in the role of the nonlinear terms in the hydrodynamic equations in the production of turbulence (3) at larger Rayleigh number. In contrast, the linearized hydrodynamic equations show that the onset of convection in a classical binary mixture (17) depends on the numerical values of two independent dimensionless parameters and hence is temperature and concentration dependent, due to the competition between the magnitudes of the thermal expansion coefficient and a comparable term involving the thermal diffusion ratio (7). In addition, overstable convective states are possible (17) within certain ranges of these parameters, primarily as a result of the existence of a diffusive mass mode in addition to the diffusive thermal mode present in the homogeneous system.

In the case of the superfluid mixture (18), only one of these modes is diffusive because the other becomes a propagating second sound mode. As a consequence of there being only one diffusive mode, it was anticipated (6) that a superfluid mixture Bénard system would behave much like a one-component fluid in being described by a single Rayleigh number, again taking some critical value dependent on the boundary conditions. On the time scale of the diffusive mode, the second sound period of the fundamental cavity mode is negligibly small, so that Bénard convection should occur in the superfluid under conditions of constant pressure, p , and constant ^4He chemical potential per unit mass, μ_4 .

The conditions for the existence of the convective instability and for the existence of gravitational waves are closely related, as has been pointed out by Gitterman and Steinberg (19). Parshin (20) has examined the conditions for the existence of gravitational waves in a solution of ^3He in superfluid ^4He .

A plane gravitational wave, $\exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$, propagating through a classical one-component fluid has the dispersion relation:

$$(\omega + i\nu k^2)(\omega + i\chi k^2) - \omega_0^2 \sin^2 \theta = 0, \quad [1]$$

where ν is the kinematic viscosity, χ is the thermal diffusivity, θ is the angle between the direction of propagation and the vertical, $\omega_0^2 = \beta_T g dT/dz$, with β_T , the thermal expansion coefficient, g , the acceleration of gravity, and dT/dz , the vertical temperature gradient. For ω_0^2 sufficiently negative, at low ω , there is a solution to relation 1 which gives an exponentially growing wave. If k is estimated by the reciprocal of the plate separation, d , and if $\sin^2 \theta = 1$, then it follows from relation 1 that the condition for the growing wave is

$$R \equiv GP > 1, \quad [2]$$

where $G = -\omega_0^2 d^4/\nu^2$ and $P = \nu/\chi$ are the Grashof and Prandtl numbers for a one-component fluid. Therefore, one has

$$R = -\frac{\omega_0^2 d^4}{\nu \chi} = -\frac{g \beta_T d^4}{\nu \chi} \frac{dT}{dz}. \quad [3]$$

It is well-known (14, 15) that an exact analysis shows the condition for the onset of convection to be $R > R_c$, rather than condition 2 where the numerical value of R_c depends on the boundary conditions.

Parshin (20) studied gravitational waves in $^3\text{He}/^4\text{He}$ superfluid solutions under the assumptions

$$\vec{\nabla}p - \rho\vec{g} = 0 \quad \text{and} \quad \vec{\nabla}\mu_4 - \vec{g} = 0, \quad [4]$$

where ρ is the mass density and $\vec{g} = -g\hat{z}$, and found the same dispersion relation as relation 1. He found

$$Pr = \frac{\rho c T \eta}{\rho_n \kappa_{\text{eff}}} \left(\frac{\partial s}{\partial T} \right)_{p, \mu_4} \quad [5]$$

and

$$Gr = g \frac{\rho_n d^4}{\eta^2} \left(\frac{\partial \rho}{\partial s} \right)_{p, \mu_4} \frac{ds}{dz} \quad [6]$$

as the Prandtl and Grashof numbers pertinent to the problem, where κ_{eff} is the effective conductivity defined by Khalatnikov (10) and measured in our experiments, c is the ^3He mass concentration, η is the shear viscosity, s is the entropy per unit mass of ^3He , and ρ_n is the normal fluid density. It follows by analogy from condition 2 that

$$Ra = \frac{g \rho c T d^4}{\eta \kappa_{\text{eff}}} \left(\frac{\partial s}{\partial T} \right)_{p, \mu_4} \left(\frac{\partial \rho}{\partial s} \right)_{p, \mu_4} \frac{ds}{dz} = - \frac{g \beta_{p, \mu_4} d^4}{\nu \chi_{\text{eff}}} \frac{dT}{dz}. \quad [7]$$

In this expression, ν has replaced η/ρ and

$$\beta_{p, \mu_4} \equiv - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_{p, \mu_4} \cong \beta_T - \beta_c \frac{c}{T} \left(1 + \frac{s_{40} m_3}{k_B c} \right) \quad [8]$$

and

$$\chi_{\text{eff}} \equiv \frac{\kappa_{\text{eff}}}{\rho C_{p, \mu_4}} \cong \frac{\kappa_{\text{eff}}}{\rho} \left[C_{p, c} + \frac{k_{BC}}{m_3} \left(1 + \frac{s_{40} m_3}{k_B c} \right)^2 \right]^{-1} \quad [9]$$

where $\beta_c = -(\partial \ln \rho / \partial c)_{p, T}$, s_{40} is the entropy per unit mass of pure ^4He , m_3 is the mass of a ^3He atom, k_B is Boltzmann's constant, and C_{p, μ_4} and $C_{p, c}$ are the isobaric heat capacities per unit mass at constant μ_4 and c , respectively. The approximations indicate that the limit of dilute solutions has been taken. Steinberg (preprint) obtains the same Ra as that in Eq. 7. Fur-

ther, it can also be shown that Parshin's equations 6 and 7 can be written in a form identical to those of Chandrasekhar (14) for convection in a one-component fluid, also with the same Ra .

Parshin's assumption of constant μ_4 has reduced the analysis of the two-component superfluid mixture to that of a single-component fluid and consequently, by analogy, we might expect the onset of stationary convection to be determined by a single critical Rayleigh number. A possible objection to Parshin's approach is that, unlike with classical fluids, v_n , the normal fluid velocity, is non-zero even before the instability onset, so that the assumption of infinitesimally small v_n required to linearize the hydrodynamic equations may not be strictly correct. However, our experimental observation that ΔT_c is virtually independent of whether the stationary convective mode corresponds to center-rising or center-falling indicates that this is not serious in practice.

In Table 1 we have catalogued our experimental data and the calculated properties needed to obtain a numerical value for the critical Rayleigh number, as well as Ra_c and Pr themselves. A constant value of $\rho = 0.145 \text{ g/cm}^3$ was used throughout the table, as was a constant value $\beta_c = 0.664$ derived from Lucas and Tyler (21) by using methods appropriate to dilute solutions and values of $\alpha = (v_3/v_4 - 1)$ obtained from Ebner and Edwards (22). The characteristic length which enters the Rayleigh number was assigned the value $d = 2.07 \text{ cm}$, equal to the plate spacing in our experiment. These values should all be accurate to better than 1%.

The first column in Table 1 is the average temperature, accurately measured with a calibrated germanium thermometer. Column 2 shows the ratio of the ^3He mass density to the pure ^4He normal fluid density ρ_{n4} , tabulated to show the character of the normal fluid. The value should be as accurate as our measure of the concentration—i.e., about 5%—because ρ_{n4} was obtained from well-known data on ^4He (23).

The next two columns, labeled ΔT_c and κ_{eff} , represent smoothed experimental data obtained in this work. The errors in these quantities have already been discussed in detail and are, at worst, 3–5% for $X = 0.0047$ and twice that for the weaker solution, concentration errors notwithstanding.

Table 1. Catalog of fluid parameters pertinent to convection and tabulation of experimentally determined Prandtl and critical Rayleigh numbers

T, K	ρ_3/ρ_{n4}	$\Delta T_c,$ $\text{K} \times 10^3$	$\kappa_{\text{eff}},$ $\text{erg} \times 10^{-4}$ cm sec K	$\beta_{p, \mu_4},$ $\text{K}^{-1} \times 10^3$	$C_{p, c},$ $\text{erg} \times 10^{-5}$ g K	$s_{40} m_3,$ k_{BC}	$C_{p, \mu_4},$ $\text{erg} \times 10^{-5}$ g K	$\chi_{\text{eff}},$ cm^2 s	$\nu,$ $\text{cm}^2 \times 10^4/\text{s}$	$\rho_n/\rho,$ $\times 10^3$	Pr	$Ra_c,$ $\times 10^{-3}$
$X = 0.0024$												
0.825	1.37	13.06	5.61	-2.64	3.43	1.064	5.56	0.696	—	5.72	—	—
0.850	1.02	9.22	5.11	-2.82	4.08	1.253	6.61	0.533	—	6.17	—	—
0.875	0.777	6.12	4.65	-3.05	4.86	1.477	7.92	0.405	—	6.73	—	—
0.900	0.600	4.04	4.21	-3.32	5.78	1.744	9.54	0.304	—	7.41	—	—
0.925	0.468	2.60	3.80	-3.66	6.87	2.066	11.56	0.227	—	8.26	—	—
0.950	0.373	1.62	3.41	-4.03	8.13	2.427	13.98	0.168	—	9.24	—	—
0.975	0.298	1.08	3.06	-4.48	9.59	2.854	17.00	0.124	—	10.45	—	—
$X = 0.0047$												
0.800	3.59	3.98	3.40	-3.88	3.60	0.462	5.69	0.412	1.93	9.62	0.049	1.69
0.825	2.68	3.19	3.16	-3.97	4.13	0.543	6.46	0.337	1.88	9.95	0.056	1.74
0.850	1.99	2.43	2.95	-4.11	4.79	0.640	7.41	0.274	1.83	10.41	0.064	1.73
0.875	1.52	1.85	2.73	-4.30	5.57	0.754	8.57	0.220	1.77	10.96	0.073	1.78
0.900	1.17	1.30	2.54	-4.54	6.49	0.890	9.98	0.176	1.69	11.65	0.082	1.73
0.925	0.916	0.89	2.36	-4.84	7.56	1.055	11.69	0.139	1.62	12.49	0.093	1.67
0.950	0.730	0.66	2.19	-5.19	8.83	1.239	13.73	0.110	1.54	13.47	0.104	1.76
(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	(xii)	(xiii)

The total density used in the table was $\rho = 0.145 \text{ g/cm}^3$; $\beta_c = -(1/\rho)(\partial \rho / \partial c)_T$ was chosen constant and equal to 0.644. The characteristic dimension used in Ra_c was $d = 2.07 \text{ cm}$. For $X = 0.0024$, $c = 0.00181$; for $X = 0.0047$, $c = 0.00355$.

The fifth through eighth columns—namely, β_{p,μ_4} , $C_{p,c}$, $(s_{40}m_3/k_Bc)$, and C_{p,μ_4} —all are appropriate to the dilute solution approximation and generally represent well-known ^4He data (23). The notable exceptions are: β_c , already discussed; a small correction to β_{40} , the expansion coefficient for pure ^4He needed to approximate more closely β_T , obtained from the equation of state for low concentrations at zero pressure by Edwards *et al.* (24); and $C_{p,c}$ from the calculations of Radebaugh (25). It is expected that errors in none of the above are significant with respect to others in the problem.

χ_{eff} in column nine, is defined in Eq. 9 and can be computed from previously discussed quantities, with corresponding uncertainty.

ν , tabulated in column 10, is the ratio of the shear viscosity to the total density. We used interpolated values of the shear viscosity from data presented by Kuenhold *et al.* (26) and Staas *et al.* (27) for our higher concentration, which we believe consistent with the data to within 10%. The shear viscosity for the $X = 0.0024$ solution cannot be accurately estimated.

Column 11 displays the normal fluid mass concentration $\rho_n/\rho = [\rho_{n4} + (m^*/m_3)\rho_3]/\rho$ which has been calculated by assuming a constant inertial mass $m^* = 2.45 m_3$ (22), and is necessary to compute the Prandtl number, tabulated in column 12. These Prandtl numbers are much lower than for any other cryogenic fluid. This is important because many aspects of the Bénard problem currently being investigated have a strong dependence on the fluid Prandtl numbers—e.g., finite amplitude convection (28), over stable motion in rotating systems (29), and convection in non-Boussinesq systems (ref. 30; R. W. Walden and G. Ahlers, personal communication). However, fluids with small Prandtl numbers are rare and to our knowledge are limited to mercury ($Pr = 0.025$) whose finite electrical conductivity makes it somewhat an exception, and liquid ^4He with a minimum Pr of 0.5 except very close to the λ transition.

Finally, we have calculated from Eq. 7 and tabulated for $X = 0.0047$ in the last column a critical Rayleigh number for stationary convection. The values for Ra_c appear to have very little temperature dependence. In addition, the magnitudes of these numbers are near the magnitude for the classical one-component problem (12) in this cell, $\approx 2.2 \times 10^3$. Although several of the quantities that enter the Rayleigh number are not known accurately, and the data are far from complete, the evidence for the connection between one-component classical Bénard convection and a Bénard system using solutions of ^3He in superfluid ^4He is compelling. Thus, superfluid solutions represent an important new class of Bénard system with the substantial advantages of low temperature environment and a low Prandtl number which may be varied simply by changing the temperature.

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- Ahlers, G. (1975) *Instabilities and Phase Transitions*, ed. Riste, T. (Plenum, New York).
- Ahlers, G. (1974) *Phys. Rev. Lett.* **33**, 1185–1188.
- Ahlers, G. & Behringer, R. P. (1978) *Suppl. Prog. Theor. Phys.* **64**, 186–201.
- Libchaber, A. & Maurer, J. (1978) *J. Phys. (Paris) Lett.* **39**, L-369–L-372.
- Ahlers, G. & Walden, R. W. (1980) *Phys. Rev. Lett.* **44**, 445–448.
- Lee, G., Lucas, P., & Tyler, A. & Vavasour, E. (1978) *J. Phys. (Paris)* C-6 **39**, 178–179.
- Lee, G., Lucas, P. & Tyler, A. (1979) *Phys. Lett. A* **75**, 81–84.
- Warkentin, P. A., Hauke, H. J. & Wheatley, J. C. (1980) *Phys. Rev. Lett.* **45**, 918–921.
- Schechter, R. S., Velarde, M. G. & Platten, J. K. (1974) *Adv. Chem. Phys.* **26**, 265–301.
- Khalatnikov, I. M. (1965) *An Introduction to the Theory of Superfluidity*, translated by Hohenberg, P. C. (Benjamin, New York).
- Peterson, R. E. & Anderson, A. C. (1972) *Phys. Lett. A* **40**, 317–319.
- Charlson, G. S. & Sani, R. L., (1970) *Int. J. Heat Mass Trans.* **13**, 1479–1496.
- Ptukha, T. P. (1961) *Zh. Eksp. Teor. Fiz. Pisma Red.* **40**, 1583–1593 [translation (1961) *Sov. Phys. JETP* **13**, 1112–1119].
- Chandrasekhar, S. (1961) *Hydrodynamic and Hydromagnetic Stability* (Clarendon, Oxford).
- Gershuni, G. Z. & Zhukhovitskii, E. M. (1976) *Convective Stability of Incompressible Fluids*, Israel Program for Scientific Translation, Jerusalem (U.S. Dept. of Commerce, National Technical Information Service, Washington, DC).
- Normand, C., Pomeau, Y. & Velarde, M. G. (1977) *Rev. Mod. Phys.* **49**, 581–624.
- Gutkowicz-Krusin, D., Collins, M. A. & Ross, J. (1979) *Phys. Fluids* **22**, 1443–1450; 1451–1460.
- Halperin, B. I. & Hohenberg, P. C. (1969) *Phys. Rev.* **177**, 952–971.
- Gitterman, M. S. & Steinberg, V. A. (1972) *Fluid Dyn. Trans.* **2**, 238–243.
- Parshin, A. Ya. (1969) *Zh. Eksp. Teor. Fiz. Pisma Red.* **10**, 567–570 [translation (1969) *JETP Lett.* **10**, 362–364].
- Lucas, P. & Tyler, A. (1977) *J. Low Temp. Phys.* **22**, 281–303.
- Ebner, C. & Edwards, D. O. (1971) *Phys. Rep. C* **2**, 77–154.
- Brooks, J. S. & Donnelly, R. J. (1977) *J. Phys. Chem. Ref. Data* **6**, 51–104.
- Edwards, D. O., Ifft, E. M. & Sarwinski, R. E. (1969) *Phys. Rev.* **177**, 380–391.
- Radebaugh, R. (1967) *National Bureau of Standards Report No. 362* (U.S. Government Printing Office, Washington, DC).
- Kuenhold, K. A., Crum, D. B. & Sarwinski, R. E. (1972) *Phys. Lett. A* **41**, 13–14.
- Staas, F. A., Taconis, K. W. & Fokkens, K. (1960) *Physica* **26**, 669–686.
- Busse, F. H. (1978) *Rep. Progr. Phys.* **41**, 1929–1967.
- Clever, R. M. & Busse, F. H. (1979) *J. Fluid Mech.* **94**, 609–627.
- Busse, F. H. (1967) *J. Fluid Mech.* **30**, 625–649.