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Nonparametric Bayes Factors Based On Empirical Likelihood Ratios

Albert Vexler¹, Wei Deng, and Gregory E. Wilding

Department of Biostatistics, The State University of New York at Buffalo, Buffalo NY, 14214

Abstract

Bayes methodology provides posterior distribution functions based on parametric likelihoods adjusted for prior distributions. A distribution-free alternative to the parametric likelihood is use of empirical likelihood (EL) techniques, well known in the context of nonparametric testing of statistical hypotheses. Empirical likelihoods have been shown to exhibit many of the properties of conventional parametric likelihoods. In this article, we propose and examine Bayes factors (BF) methods that are derived via the EL ratio approach. Following Kass & Wasserman [10], we consider Bayes factors type decision rules in the context of standard statistical testing techniques. We show that the asymptotic properties of the proposed procedure are similar to the classical BF's asymptotic operating characteristics. Although we focus on hypothesis testing, the proposed approach also yields confidence interval estimators of unknown parameters. Monte Carlo simulations were conducted to evaluate the theoretical results as well as to demonstrate the power of the proposed test.

Keywords

Bayes Factor; Empirical Likelihood; Likelihood Ratio; Nonparametric Testing; Type I Error

1. Introduction

The parametric likelihood ratio is a powerful and versatile tool for hypotheses testing [e.g., 13]. Empirical likelihood methods are often proposed as data-based, nonparametric alternatives to parametric likelihood testing procedures [e.g., 12, 15, 16, 17, 19, 26]. The statistical literature has shown EL ratio tests to have asymptotic properties that are comparable with those of parametric likelihoods ratio tests. Principles of EL ratio methods are outlined in the Appendix. Traditionally, EL ratio methods consider the value of parameters under the alternative to be completely unknown. In practice, we often can expect that the parameters can only take values within a given area or range and that lower and/or upper bounds of the parameter can be assumed to be known. For example, the lower and upper bounds for mean age or the percent change in a variable can be assumed to be known for some populations of interest. To make use of this prior information, we propose to apply Bayes factors methodology.

The Bayesian approach to hypothesis testing can be conducted based on use of Bayes factors which are known powerful methods based on likelihood functions. Given that a parametric

assumption of the data distribution may be reasonably made [e.g., 4, 8, 9, 6, 1, 14, 22], Bayes factors provide the means to incorporate additional information in the assessment of a hypothesis through specification of a prior distribution, and in addition, provide a very general testing approach which does not require alternative models to be nested. Although Bayes factors support powerful hypothesis testing for parametric or non-parametric priors, the parametric likelihood function of the data is assumed to be known up to the unknown parameters.

The main objective of this article is to propose and study a new testing procedure based on empirical likelihood and Bayes factors methodologies. Lazar [11] and Schennach [21] showed, in a general context, that Bayes methods can be framed in the context of the EL as opposed to the traditional parametric likelihood when one cannot assume the distribution of observed data to be known. We derive the test-statistic for the one-sample hypothesis testing problem and provide the relevant asymptotic properties which are shown to be similar to those of BF based on parametric likelihoods [e.g., 9, 6, 14]. The proposed approach may also be used to obtain confidence intervals of unknown parameters. Monte Carlo simulations are conducted to examine the properties of the test in finite samples when using the derived asymptotic null distribution. We study the power function of the proposed tests as compared with the classical EL ratio test. Additionally, we examine the sensitivity of the proposed method in regard to the choice of prior distributions based on a scenario discussed by Sinharay & Stern [22]. We conclude with remarks in Section 4.

2. Method

Given a sample $(X_1, X_2, \dots, X_n)$ of n independent identically distributed (i.i.d.) observations, consider testing of the hypothesis $E(X_1) = \mu_0$. In this case, the well known test-statistic based on the BF is

$$BF = \frac{\int \prod_{i=1}^n f(X_i/\mu) g(\mu) d\mu}{\prod_{i=1}^n f(X_i/\mu_0)}, \quad (1)$$

where the density function $f(u|\mu)$ of X_1 is assumed to be known up to the parameter μ , $E(X_1)$ is unknown under the alternative hypothesis, and $g(\mu)$ is a prior of $\mu = E(X_1)$. the definition (1) of the BF corresponding to the point null hypothesis $E(X_1) = \mu_0$ is widely addressed in the statistical literature, e.g. Berger [2, pp. 148–150, eq (4.15)], Bernardo & Smith [3, pp. 391–392], Kass [7]. Vexler et al. [25] showed the optimality of statistical testing-strategies which utilize different modifications of the static (1).

In this article, we suggest replacing the likelihood $\prod f(X_i|\mu)$ by the EL in order to focus on the nonparametric statement of the testing problems in the Bayes context. Thus, we propose use of the test-statistic

$$BFELR = \frac{\int g(\mu) [EL(\mu)] d\mu}{EL(\mu_0)}, \quad (2)$$

where

$$EL(\mu) = \max_{p_1, \dots, p_n} \left\{ \prod_{i=1}^n p_i : \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i X_i = \mu \right\}.$$

Here, in accordance with Owen [18, p. 21], we take $EL(\mu) = 0$ by convention, if $\mu < X_{(1)} = \min(X_1, \dots, X_n)$ or $\mu > X_{(n)} = \max(X_1, \dots, X_n)$. Thus, we assume that $\int_{X_{(1)}}^{X_{(n)}} g(\mu) d\mu > 0$. See the Appendix for details related to the function $EL(\mu)$.

Following the Bayesian concept of testing [e.g., 2], one can propose that large values of $BFELR$ indicate evidence against the hypothesis that $E(X_1) = \mu_0$. Critical values of $BFELR$ corresponding to fixed significant levels are required in order to make decisions regarding the statistical hypotheses in practice. When the testing problem is parametric in nature, the asymptotic distributions of BF type test-statistics are available, e.g., Kass & Vaidyanathan [9], Gelfand & Dey [6], O'Hagan [14]. For the nonparametric setting, we make use of the next result based on an asymptotic property of the EL ratio for these purposes.

Proposition 2.1

Let $X_1, X_2, \dots, X_n$ be i.i.d. observations with finite $E|X_1|^3$. Then, under the null hypothesis $E(X_1) = \mu_0$, as $n \rightarrow \infty$,

$$2\log(BFELR) = \Lambda - 2\log(EL(\mu_0)n^n) + O_p(n^{-1/2+3\gamma}),$$

where $\gamma \in (0, 1/6)$, the $-2\log EL$ ratio, $-2\log(EL(\mu_0)n^n)$, is asymptotically distributed as χ_1^2 ,

$$\Lambda = 2\log \left\{ \int g(\mu) \exp \left(-\frac{n}{2} \frac{(\mu_0 - \mu)^2}{\sum_{i=1}^n (X_i - \bar{X})^2 / n + (\mu_0 - \mu)^2} \right) d\mu \right\}, \quad \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

and Λ can be approximated by $2\log g(\mu_0) + \log \pi + \log \left(\sum_{i=1}^n (X_i - \bar{X})^2 / n \right) - \log n$ under Laplace regular conditions as $n \rightarrow \infty$ and $\pi \simeq 3.1415$.

Note that, when the density function f at (1) is assumed to be known, it is common practice to approximate BF type statistics by integrals of approximating normal curves centered at corresponding posterior modes. These approximations can be viewed as applications of the Laplace method, e.g., DasGupta [5]. It will produce reasonable results, in the contexts of the accuracy required for testing hypotheses based on BF type statistics, as long as the related posterior is unimodal or at least dominated by a single mode, when the sample size is large enough. In this case, the useful approximation to the relevant marginal density is obtained by assuming that the posterior density, which is proportional to $\pi f(X_i/\mu) \times \text{prior}$, is highly peaked about its maximum that is the posterior mode, say, $\tilde{\mu}$. For example, the numerator at (1) can be approximated by expanding $\log(\pi f(X_i/\mu)g(\mu))$ as a quadratic about $\mu = \tilde{\mu}$ that yields asymptotically a Gaussian type form of the integral, as $n \rightarrow \infty$. To apply the approach mentioned above to evaluate the statistic (2), we need to prove that behaviors of the function $EL(\mu)$, the empirical likelihood, with respect to μ are similar to those of parametric likelihoods, for example, in a context of the maximization $\max_{\mu} EL(\mu)$. In this case, the complication is that although the function $EL(\mu)$ is well investigated when μ is asymptotically close to EX_1 , the structure of the numerator of (2) requires one to consider $EL(\mu)$ with μ -values significantly different from EX_1 . To this end, we provide the lemma below.

Lemma 2.1

Assume the $EL(\mu)$ function's argument μ belongs to the interval $(X_{(1)}, X_{(n)})$. Then $\log(EL(\mu))$ increases monotonically for $\mu < \bar{X}$ and decreases monotonically for $\mu > \bar{X}$.

The proof of Lemma 2.1 is sketched in the Appendix.

It is clear that the function $\log(EL(\mu))$ attains its maximum value at the point $\mu = \bar{X}$ and $\log(EL(\mu = \bar{X})) = -n \log(n)$ (e.g., [19]). (In this case, in the definition (2), $p_1 = \dots = p_n = 1/n$ satisfy the constraints $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^n p_i X_i = \bar{X}$ as well as these $p_1, \dots, p_n$ maximize the empirical likelihood $\prod_{i=1}^n p_i$. Lemma 2.1 is useful in the demonstration of similarity of properties related to parametric and empirical likelihoods, in various situations. We can propose Lemma 2.1 in a context of a simple nonasymptotic alternative to Lemma 1 presented in [19]. Lemma 2.1 can be applied to analyze the integrals

$$\int_{X_{(1)}}^{\bar{X}-a} g(\mu) EL(\mu) d\mu \leq EL(\bar{X}-a) \quad \text{and} \quad \int_{\bar{X}+a}^{X_{(n)}} g(\mu) EL(\mu) d\mu \leq EL(\bar{X}+a),$$

components of the numerator of the statistic $BFELR$ in (2), for positive values of a . This assists to prove Proposition 2.1 (see the Appendix for details).

Proposition 2.1 provides an asymptotic technique to control the type I error of the proposed test statistic. Corresponding schematic R codes are provided in the Appendix 5.4 (R Development Core Team [20]).

Remark 1

By virtue of the definition (2), the statistic $BFELR$ is a function of μ_0 . Utilizing Proposition 2.1, we fix the threshold C_α such that

$$Pr_{EX=\mu_0} \{2\log BFELR \leq C_\alpha\} \geq 1 - \alpha \quad \text{as} \quad n \rightarrow \infty.$$

Thus, a set of μ_0 's satisfying

$$\{\mu_0: 2\log BFELR \leq C_\alpha\}$$

provides a $(1 - \alpha) \cdot 100\%$ interval estimator of $\mu_0 = EX_1$.

Remark 2

The test-statistic $2 \log ELR$ (ELR denotes the empirical likelihood ratio) converges in distribution to χ_1^2 , under H_0 . Owen [16, pp.109–110] concluded that, under H_0 , the difference between distributions of $2 \log ELR$ and χ_1^2 depends strongly on values of the sample size n . However, one can note that $2 \log ELR$ converges in distribution to

$T = \left(\sum (X_i - \mu_0) \right)^2 / \sum (X_i - \mu_0)^2$, which converges to χ_1^2 , respectively (see, for details, Section 5.1). Thus, we can assume $2\log ELR \rightarrow \chi_1^2$ with a convergence rate that depends on a distance between the observations' distribution and a normal distribution function. In order to investigate this issue, we compared the EL ratio test-statistic with the t-test-statistic via an

extensive Monte Carlo simulation study. The results of the Monte Carlo study empirically confirmed the conclusion of this remark.

Remark 3

Note that the considered statement of testing can be easily extended to more general cases, e.g., where, for a given function $w(\cdot)$ and a fixed parameter θ , the hypothesis H_0 is $Ew(X_1) = \theta$

3. Monte Carlo study

To examine the performance of the asymptotic approximation suggested by Proposition 2.1 and power characteristics of the test-statistic (2), we conducted Monte Carlo simulations. In this section, we evaluate three types of prior distributions $g(\cdot)$ in two different scenarios based on normally and lognormally distributed data. (Here, normal and lognormal distribution functions of data-points were chosen in accordance with Remark 2; the prior distributions $Unif(0, 1)$, $N(0.7, 1)$ and $(N(1, 1) + N(-1, 1))/2$ present a "noninformative" situation as well as are commonly used, in the context of the Bayes Factor, prior functions, e.g., Sinharay & Stern [22], Berger [2].) We generated 10000 random samples of $\{X_i \sim N(\mu, 1), i = 1, \dots, n\}$ and $\{X_i = (\exp(\xi_i) - \exp(0.5)) / (\exp(2) - \exp(1))^{0.5} + \mu, \xi_i \sim N(0, 1), i = 1, \dots, n\}$ at each sample size $n = 10, 15, 25, 50, 75, 100$ and mean $\mu = \mu_0 + \delta$, where $\delta = 0, 0.5, 0.7, 1, 1.5$ is the effect size; the H_0 -mean μ_0 is zero.

Tables 1 & 2 display the Monte Carlo Type I errors of the test-statistic $2 \log(BFELR)$ comparing with those of the test based on $2 \log(ELR)$. The critical values (related to the significant levels $\alpha = 0.01, 0.05, 0.1$) of the tests were chosen utilizing the asymptotic result of Proposition 2.1 and that $2 \log(ELR)\chi_1^2$, as $n \rightarrow \infty$.

When $n = 10$, the actual Type I error of the proposed test-statistic is not well approximated by Proposition 2.1. However, when $n > 15$, the use of the result of Proposition 2.1 is seen to be adequate; whereas even when $n = 75$ and with data from the lognormal distribution, the approximation $2 \log(ELR)\chi_1^2$ is shown to be inadequate. That is, perhaps because the numerator of the EL ratio was obtained via a global maximization of EL's components (see, for details, Section 5.1), whereas the numerator of the $BFEL$ ratio (2) was a result of a local maximization of EL's components with respect to a prior $g(\cdot)$. Hence the Type I error of the $BFELR$ test is better controlled than that of the ELR test.

Table 3 presents the Monte Carlo power levels of the two testing procedures. At this rate, we applied Proposition 2.1 and the result $2 \log(ELR)\chi_1^2$, as $n \rightarrow \infty$, to fix a significant level of the tests to be $\alpha = 0.05$. Note that, following Tables 1 & 2, the actual powers (which correspond to nonasymptotic $\alpha = 0.05$'s critical values) of the test-statistic $2 \log(ELR)$ can be expected to be significantly less than those of Table 3, since $\alpha = 0.05$'s test-thresholds of $2 \log(ELR)$ are seen to be greater than 3.84. Thus, this Monte Carlo study confirms that the consideration of the Bayes Factor modification (2) of the empirical likelihood ratio (1) is reasonable.

Note that, according to Lazar [11] (see also the Oxford University technical report by S. A. Corcoran, A. C. Davison and R. H. Spady [11, p.324]), one can expect that the proposed BF empirical likelihood ratio test has the Type I error that can be more accurately controlled than that of the classical EL ratio test, whereas the power of the $BFELR$ test depends strongly on the utilized prior $g(u)$, especially when the sample size is not large. The conducted Monte Carlo simulations confirm this supposition regarding the Type I error control, showing that when $g(u)$ is chosen reasonably the $BFELR$ test is a powerful decision rule. To perform the

corresponding Monte Carlo study, we adapted a simulation scheme used by Lazar [11, Section 3] to compare the empirical likelihood based tests. Ten thousand samples of size $n = 10$ were generated from $N(0, 1)$ and $N(0.5, 1)$ distributions to evaluate the Type I error and the power of the tests for $E(X_1) = 0$. To construct the BFELR test statistic, twenty different normal priors were applied, with the five values $0, 1/n, 1/n^{1/2}, 3/n^{1/2}, 5$ of the mean and the four values $1/n, 1/n^{1/2}, 1, n$ of the variance, as depicted in Table 4.

The results presented in Table 4 can be compared with the Monte Carlo power of the 2 logELR test that was 0.4124, when the critical value 3.84 was chosen to preserve asymptotically the significance level $\alpha = 0.05$ of the ELR test. In this case, the Monte Carlo power 0.4124 of the ELR test should be considered taking into account that the actual Monte Carlo Type I error of the ELR test was observed as 0.1054. In accordance with Table 4, the Type I error of the BFELR test is controlled much better than that of the ELR test, even when we smooth the unknown alternative parameter around $EX = 0$, which is the value of the parameter under the null hypothesis. However, we must remark the proposed test becomes very conservative and powerless when the mean of the prior is chosen relatively far from the null and alternative parameters. Table 4 reports that we can increase the prior's variance to relax this issue. Intuitively, the considered BFELR test can demonstrate properties close to those of the ELR test, when the prior's variance is relatively large. Certainly, Table 4 shows the BFELR test's characteristics depend on choosing a corresponded prior. The articles [24, 25] presented discussions and the relevant literature that can assist to choose an appropriated prior to construct the BFELR test, satisfying the tester's interests.

The BF principle applied to the EL ratio test can improve the Type I error control. Owen [18, p. 33] proposed to utilize a bootstrap type procedure to obtain critical values of the ELR test. Although this procedure is computationally difficult, we conducted a brief Monte Carlo study to analyze the ELR test with the bootstrap-calibrated 95% confidence region and the proposed BFELR test with the 95% critical values obtained by virtue of Proposition 2.1. We considered samples of $\{X_i = (\exp(\xi_i) - \exp(1.5^2/2)) + \mu, \xi_i N(0, 1.5^2), i=1, \dots, n\}$, where $\mu = 0$ was set up to evaluate the Monte Carlo Type I error of the tests and $\mu = 1.5$ was used to evaluate the Monte Carlo power of the tests. The sample size n was fixed to be 25. Since the bootstrap procedure required a long computer time to be performed, we generated the data just 3000 times applying for each data 1000-resamples in the bootstrap manner. The 2 logELR test with the critical value 3.84 (corresponding to χ_1^2) and the ELR test with the bootstrap-calibrated 95% confidence region provided the Monte Carlo Type I errors and powers to be 0.2155, 0.1509 (instead of 0.05) and 0.4241, 0.1962, respectively. In this case, the BFELR tests based on the priors $Unif[-3, 3]$, $Unif[-5, 5]$ and $N(0.5, 1.5^2)$ demonstrated 0.1877, 0.1924, 0.1546 (instead of 0.05) as the Monte Carlo Type I error and 0.3757, 0.4209, 0.3491 as the Monte Carlo power. The ELR test with the bootstrap-calibrated 95% confidence region is close to being biased. Generally speaking, the considered situation motivates the modification of the EL-based tests, applying methods, e.g., proposed by Vexler et al. [23]. The bootstrap-calibrated $100(1 - \alpha)\%$ confidence regions can also be constructed for the BFELR test. In a subsequent article we plan to investigate these problems in details.

4. Concluding Remarks

The objective of this article was to propose an approach for Bayes Factors type nonparametric testing. To this end, we utilized empirical likelihood methodology. We obtained the asymptotic properties of the proposed test which were shown to be similar to those of BF tests based on parametric likelihood functions. We compared the proposed test with the empirical likelihood ratio test via the Monte Carlo study and concluded that the Bayes structure of the proposed nonparametric likelihood ratio test is associated with better Type I error than that of the classical

ELR test when in use with asymptotic approximations. That is, perhaps because the numerator of the EL ratio was obtained via a global maximization of EL's components, whereas the numerator of the BFEL ratio (2) was a result of a local maximization of EL's components with respect to a prior $g(\cdot)$. Obviously, the power of the proposed test depends on the choice of a prior $g(\cdot)$. However, we showed the BFELR test with the noninformative $g(\cdot)$ s also provides a powerful test. Note that, in practice, we often can expect that the parameters only take values within a given area or range and that lower and/or upper bounds of the parameter can be assumed to be known. For example, the lower and upper bounds for mean age or the percent change in a variable can be assumed to be known for some populations of interest. This information can improve construction of the utilized prior function. Discussions and the literature mentioned in Vexler & Wu [24] and Vexler et al. [25] can also be applied to choose a prior, preserving the efficiency of the BFELR test.

The problem of Type I error controls of ELR-type tests has been well addressed in the literature. Vexler et al. [23] investigated this issue based on finite samples that follow skewed distributions. Efficient methods to modify the empirical likelihood ratio tests were presented by Vexler et al. [23]. The proposed BF form of empirical likelihood ratios is shown to have an ability to improve the Type I error control. This approach can be merged with the ELR modifications studied in [23]. Alternatively, albeit computationally more expensive, bootstrap methods to advance the performance of the EL ratio can be considered Owen [e.g., 18, p. 33]. To attend briefly to this subject, Section 3 presented our limited simulation results, supporting the propose methodology. In a subsequent article we plan to compare the BFELR, ELR, modified ELR, modified BFELR tests, particularly utilizing bootstrap-calibrated confidence regions of the tests, which needs substantial mathematical details and extensive Monte Carlo simulations.

Vexler et al. [25] considered Equation (1)-type test statistics, when parameters under the null hypothesis are unknown. In this case, the maximum empirical likelihood technique proposed by Qin & Lawless [19] can be applied to estimate the unknown parameters appearing in the numerator of (2).

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5. Appendix

5.1. Outlined EL ratio methodology

Let $X_1, X_2, \dots, X_n$ be i.i.d. observations with $E|X_1|^3 < \infty$. The well-known empirical likelihood function has the form of

$$EL(\mu) = \max_{p_1, \dots, p_n} \left\{ \prod_{i=1}^n p_i : \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i X_i = \mu \right\}.$$

The Lagrange multiplier method yields

$$EL(\mu) = \prod_{i=1}^n \frac{1}{n + n\lambda(X_i - \mu)}, \quad (3)$$

where to satisfy the constraint $\sum_{i=1}^n p_i = 1$, λ is a root of the equation

$$\sum_{i=1}^n \frac{X_i - \mu}{1 + \lambda(X_i - \mu)} = 0.$$

When we test for $E(X_1) = \mu_0$, the empirical likelihood ratio test-statistic is

$$\begin{aligned} ELR &= \frac{\max_{p_1, \dots, p_n} \left\{ \prod_{i=1}^n p_i : \sum_{i=1}^n p_i = 1 \right\}}{\max_{p_1, \dots, p_n} \left\{ \prod_{i=1}^n p_i : \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i X_i = \mu_0 \right\}} \\ &= \prod_{i=1}^n \frac{1}{n p_i}, \quad p_i = \frac{1}{n(1 + \lambda(X_i - \mu_0))}. \end{aligned}$$

Then,

$$\log(ELR) = - \sum_{i=1}^n \log(n p_i) = - \sum_{i=1}^n \log \frac{1}{1 + \lambda(X_i - \mu_0)} = \sum_{i=1}^n \log(1 + \lambda(X_i - \mu_0)).$$

Following the proof scheme of Theorem 1 by Owen [15], the Taylor expansion method yields asymptotic notation of λ that is

$$\lambda(\mu_0) = \frac{(\bar{X} - \mu_0)}{s^2} r_0, \quad (4)$$

here,

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \mu_0)^2, \quad r_0 = \frac{\sum_{i=1}^n (X_i - \mu_0)^2}{\sum_{i=1}^n \{(X_i - \mu_0) / (1 + \eta(X_i - \mu_0))\}^2}, \quad (5)$$

$|\eta| \leq |\lambda(\mu_0)|$. Here $r_0 \xrightarrow{P} 1$ as $n \rightarrow \infty$, under the null hypothesis $EX_1 = \mu_0$. The Taylor expansion (up to a second order) of function $\log(ELR)$ around $\lambda(X_i - \mu_0) = 0$ leads to

$$\begin{aligned} 2 \log(ELR) &\approx -2 \sum_{i=1}^n \left\{ \lambda(X_i - \mu_0) - \frac{1}{2} \lambda^2(X_i - \mu_0)^2 \right\} \\ &= 2 \lambda \sum_{i=1}^n (X_i - \mu_0) - \lambda^2 \sum_{i=1}^n (X_i - \mu_0)^2. \end{aligned} \quad (6)$$

By virtue of the asymptotic result (4), under the null hypothesis $EX_1 = \mu_0$, we have

$$\begin{aligned} 2 \log(ELR) &\approx 2 \frac{\sum_{i=1}^n (X_i - \mu_0)}{\sum_{i=1}^n (X_i - \mu_0)^2} \sum_{i=1}^n (X_i - \mu_0) - \left(\frac{\sum_{i=1}^n (X_i - \mu_0)}{\sum_{i=1}^n (X_i - \mu_0)^2} \right)^2 \sum_{i=1}^n (X_i - \mu_0)^2 \\ &= \frac{\left(\sum_{i=1}^n (X_i - \mu_0) \right)^2}{\sum_{i=1}^n (X_i - \mu_0)^2} = T \chi_1^2 \end{aligned}$$

[see, for details, 15].

5.2. Proof of Lemma 2.1

We can use (3) to show that

$$\begin{aligned}\frac{\partial}{\partial \mu} \log (EL(\mu)) &= - \sum_{i=1}^n \frac{n(X_i - \mu) \frac{\partial}{\partial \mu} \lambda - n\lambda}{n + n\lambda(X_i - \mu)} \\ &= -n \left(\frac{\partial}{\partial \mu} \lambda \right) \sum_{i=1}^n (X_i - \mu) p_i + n\lambda \sum_{i=1}^n p_i = n\lambda.\end{aligned}\quad (7)$$

Consider the function

$$L(\lambda) = \frac{1}{n} \sum_{i=1}^n \frac{(X_i - \mu)}{1 + \lambda(X_i - \mu)}.$$

We are interested in a value of $\{\lambda : L(\lambda) = 0\}$. Since $\frac{\partial}{\partial \lambda} L(\lambda) \leq 0$, $L(\lambda)$ decreases with respect to λ . Assume $\bar{X} > \mu$, then $L(\lambda=0) = \bar{X} - \mu > 0$. Thus, it is clear the point $\{\lambda : L(\lambda) = 0\}$ is greater than the point $\{\lambda = 0 : L(\lambda) > 0\}$, since $L(\lambda)$ decreases and we will have sequentially the situations $L(\lambda) > 0$, $L(\lambda) = 0$, $L(\lambda) < 0$, when λ increases. This concludes that a value of $\{\lambda : L(\lambda) = 0\}$ is positive, when $\bar{X} > \mu$. In this case the equation (7) shows that

$\frac{\partial}{\partial \mu} \log (EL(\mu)) > 0$, and hence the function $\log (EL(\mu))$ increases. In a similar manner to the proof above, the situation, where $\bar{X} < \mu$ can be considered.

5.3. Proof of Proposition 2.1

We have

$$\begin{aligned}BFELR &= \frac{\int g(\mu) [EL(\mu)] d\mu}{EL(\mu_0)} \\ &= \prod_{i=1}^n [np_i(\mu_0)]^{-1} \int g(\mu) \prod_{i=1}^n \frac{1}{1 + \lambda(\mu)(X_i - \mu)} d\mu\end{aligned}\quad (8)$$

where, following Section 5.1,

$$p_i(\mu) = \frac{1}{n(1 + \lambda(\mu)(X_i - \mu))},$$

and $\lambda(\mu)$ is a root of

$$\sum_{i=1}^n \frac{X_i - \mu}{1 + \lambda(\mu)(X_i - \mu)} = 0.$$

By virtue of (8) and results of Section 5.1, the log transformed test-statistic $BFELR$ is

$$\begin{aligned} 2\log(BFELR) &= 2\sum_{i=1}^n \log(1+\lambda(\mu_0)(X_i - \mu_0)) + 2\log\left\{\int g(\mu) \exp\left(\log\left(\prod_{i=1}^n \frac{1}{1+\lambda(\mu)(X_i - \mu)}\right)\right) d\mu\right\} \\ &= 2\log(ELR) + 2\log(BELR), \end{aligned} \quad 9$$

where

$$\begin{aligned} BELR &= \int g(\mu) \exp\left(\log\left(\frac{EL(\mu)}{n^{-n}}\right)\right) d\mu \\ &= \int g(\mu) \exp\left(-\sum_{i=1}^n \log(1+\lambda(\mu)(X_i - \mu))\right) d\mu \\ &= \int_{\bar{X}-a}^{\bar{X}+a} g(\mu) \exp\left(-\sum_{i=1}^n \log(1+\lambda(\mu)(X_i - \mu))\right) d\mu + \int_{X_{(1)}}^{\bar{X}-a} g(\mu) \exp\left(\log\left(\frac{EL(\mu)}{n^{-n}}\right)\right) d\mu + \int_{\bar{X}+a}^{X_{(n)}} g(\mu) \exp\left(\log\left(\frac{EL(\mu)}{n^{-n}}\right)\right) d\mu \end{aligned}$$

Define $a = n^{-1/2+\beta}$, $0 < \beta < 1/6$ and consider the integral

$$I_1 = \int_{X_{(1)}}^{\bar{X}-a} g(\mu) \exp\left(\log\left(\frac{EL(\mu)}{n^{-n}}\right)\right) d\mu.$$

Lemma 2.1 provides the inequality

$$I_1 \leq \exp\left(-\sum_{i=1}^n \log\left(1+\lambda\left(X_i - \bar{X} + a\right)\right)\right), \quad (10)$$

where λ is a root of the equation $0=L(\lambda) \equiv \frac{1}{n} \sum_{i=1}^n \left(X_i - \bar{X} + a\right) \left(1+\lambda\left(X_i - \bar{X} + a\right)\right)^{-1}$. In a similar manner to the proof scheme mentioned in [15] on the pages 242-242, around equations (3.3)-(3.4), we can analyze, for $1/6 > \gamma > \beta$,

$$\begin{aligned} L(\pm n^{-1/2+\gamma}) &= \frac{1}{n} \sum_{i=1}^n \frac{X_i - \bar{X} + a}{1 \pm n^{-1/2+\gamma}(X_i - \bar{X} + a)} \\ &= \frac{1}{n} \sum_{i=1}^n \frac{(X_i - \bar{X} + a) \left[\left(1 \pm n^{-1/2+\gamma}(X_i - \bar{X} + a)\right) \mp n^{-1/2+\gamma}(X_i - \bar{X} + a) \right]}{1 \pm n^{-1/2+\gamma}(X_i - \bar{X} + a)} \\ &= a \mp n^{-1/2+\gamma} \sum_{i=1}^n \frac{\frac{1}{n}(X_i - \bar{X} + a)^2}{1 \pm n^{-1/2+\gamma}(X_i - \bar{X} + a)}, \\ n^{-1/2+\gamma}(X_i - \bar{X} + a) &= n^{-1/2+\gamma} O_p(n^{1/3}) = O_p(n^{-1/6+\gamma}), \quad i=1, \dots, n, \end{aligned}$$

to conclude that $n^{1/2}L(\pm n^{-1/2+\gamma}) \rightarrow \mp \infty$, i.e. $|\lambda| \leq n^{-1/2+\gamma}$ and $\lambda = O(n^{-1/2+\gamma})$, since we are interested in $\lambda : n^{1/2}L(\lambda) = 0$ and $L(\lambda)$ decreases with respect to λ (here $P\{L(n^{-1/2+\gamma}) > 0\} \rightarrow 0$ and $P\{L(-n^{-1/2+\gamma}) < 0\} \rightarrow 0$). By virtue of this result and following the proof scheme of Theorem 1 by Owen [15] outlined in Section 5.1, we can apply the Taylor expansion method to obtain asymptotic notation of the function

$$\begin{aligned} \lambda(\mu = \bar{X} - a): \quad & \sum_{i=1}^n \frac{X_i - (\bar{X} - a)}{1 + \lambda(X_i - (\bar{X} - a))} = 0, \quad \sum_{i=1}^n (X_i - (\bar{X} - a)) \\ & - \lambda \sum_{i=1}^n (X_i - (\bar{X} - a))^2 + \lambda^2 \sum_{i=1}^n \frac{(X_i - (\bar{X} - a))^3}{(1 + \eta''_i(X_i - (\bar{X} - a)))^3}, \\ & |\eta''_i| \leq |\lambda|, \quad \lambda^2 = O(n^{-1+2\gamma}), \end{aligned}$$

that is

$$\lambda = \frac{a}{n^{-1} \sum_{i=1}^n (X_i - \bar{X} + a)^2} + O(n^{-1+2\gamma}). \quad (11)$$

Similarly, we have, for $|\eta_i| \quad |\lambda| = O(n^{-1/2+\gamma})$,

$$\begin{aligned} \sum_{i=1}^n \log(1 + \lambda(X_i - \bar{X} + a)) &= \sum_{i=1}^n \left\{ \log(1) + \lambda(X_i - \bar{X} + a) - \frac{1}{2} \lambda^2 (X_i - \bar{X} + a)^2 + \frac{1}{3} \frac{\lambda^3 (X_i - \bar{X} + a)^3}{(1 + \eta_i(X_i - \bar{X} + a))^3} \right\} \\ &= \frac{na^2}{2n^{-1} \sum_{i=1}^n (X_i - \bar{X} + a)^2} + O(n^{-1/2+3\gamma}) \\ &= \frac{n^{2\beta}}{2n^{-1} \sum_{i=1}^n (X_i - \bar{X} + a)^2} + O(n^{-1/2+3\gamma}). \end{aligned}$$

Thus, the integral I_1 in (10) satisfies

$$I_1 \leq \exp \left(\frac{n^{2\beta}}{2n^{-1} \sum_{i=1}^n (X_i - \bar{X} + a)^2} + O(n^{-1/2+3\gamma}) \right). \quad (12)$$

It is clear that, one can show

$$\begin{aligned} I_2 &= \int_{\bar{X}+a}^{X_{(n)}} g(\mu) \exp \left(\log \left(\frac{EL(\mu)}{n^{-n}} \right) \right) d\mu \leq \exp \left(\log \left(\frac{EL(\bar{X} + a)}{n^{-n}} \right) \right) \\ &\leq \exp \left(- \frac{n^{2\beta}}{2n^{-1} \sum_{i=1}^n (X_i - \bar{X} + a)^2} + O(n^{-1/2+3\gamma}) \right). \end{aligned} \quad (13)$$

In accordance with (9), to analyze *BELR*, we focus on the integral

$$I = \int_{\bar{X}-a}^{\bar{X}+a} g(\mu) \exp \left(- \sum_{i=1}^n \log(1 + \lambda(X_i - \mu)) \right) d\mu. \quad (14)$$

Note that, in a similar manner to the evaluations of $EL(\mu)$ in I_1 and I_2 , applying the Taylor expansion around $\lambda = 0$ to second order for $\sum_{i=1}^n \log(1 + \lambda(X_i - \mu))$, we have, for $\bar{X} - n^{-1/2+\beta} \leq \mu \leq \bar{X} + n^{-1/2+\beta}$,

$$\sum_{i=1}^n \log(1 + \lambda(X_i - \mu)) = \sum_{i=1}^n \left\{ \log(1) + \lambda(X_i - \mu) - \frac{1}{2} \lambda^2 (X_i - \mu)^2 + \frac{1}{3} \frac{\lambda^3 (X_i - \mu)^3}{(1 + \eta'_i(X_i - \mu))^3} \right\},$$

where $|\eta'_i| \leq |\lambda| = O(n^{-1/2+\gamma})$, $1/6 > \gamma > \beta$, $X_i = O_p(n^{1/3})$ and

$$\lambda = \frac{\sum_{i=1}^n (X_i - \mu)}{\left(\sum_{i=1}^n (X_i - \mu)^2 \right)^{1/2}} + O(n^{-1+2\gamma}).$$

Thus, we conclude that

$$2 \sum_{i=1}^n \log(1 + \lambda(\mu)(X_i - \mu)) = n \frac{(\bar{X} - \mu)^2}{\sum_{i=1}^n (X_i - \bar{X})^2 / n + (\bar{X} - \mu)^2} + O(n^{-1/2+3\gamma}). \quad (15)$$

Now, by virtue of (9) and an elementary inequality, we can write

$$\begin{aligned} 2 \log(I) &\leq 2 \log(BELR) = 2 \log(I + (I_1 + I_2)) \\ &\leq 2 \log(I) + 2(I_1 + I_2)/I \leq 2 \log(I) + 2(I_1 + I_2)/I', \end{aligned}$$

where, by (15), for $\nu < \beta$,

$$\begin{aligned} I &= \int_{\bar{X}-a}^{\bar{X}+a} g(\mu) \exp \left(-\frac{n}{2} \frac{(\bar{X} - \mu)^2}{\sum_{i=1}^n (X_i - \bar{X})^2 / n + (\bar{X} - \mu)^2} + O(n^{-1/2+3\gamma}) \right) d\mu, \\ I \geq I' &= \int_{\bar{X}-n^{-1/2+\nu}}^{\bar{X}+n^{-1/2+\nu}} g(\mu) \exp \left(-\frac{n}{2} \frac{(\bar{X} - \mu)^2}{\sum_{i=1}^n (X_i - \bar{X})^2 / n + (\bar{X} - \mu)^2} + O(n^{-1/2+3\gamma}) \right) d\mu. \end{aligned}$$

in this case, the inequalities (12) and (13) imply $2 \log(BELR) = 2 \log(I) + O(n^{-1/2+3\gamma})$, as $n \rightarrow \infty$. That is, under the Laplace method regular condition [9, 6, 14], for $n \rightarrow \infty$, $2 \log(BELR)$ can be rewritten in the form of

$$\begin{aligned}
2\log(BELR) &\cong 2\log\left\{\int g(\mu)\exp\left(-\frac{n}{2}\frac{(\bar{X}-\mu)^2}{\sum_{i=1}^n(X_i-\bar{X})^2/n+(\bar{X}-\mu)^2}\right)d\mu\right\} \\
&\cong 2\log\left\{g(\bar{X})(\pi)^{1/2}\left(\frac{\sum_{i=1}^n(X_i-\bar{X})^2}{n^2}\right)^{\frac{1}{2}}\right\} \\
&= 2\log g(\bar{X}) + \log\pi + \log\left(\sum_{i=1}^n(X_i-\bar{X})^2/n\right) - \log n.
\end{aligned}$$

This and the consideration of $2\log(ELR)$ in (9) mentioned in Section 5.1 complete the proof.

5.4. Schematic R Codes

```

library(emplik)

mu0 <- 0 # Under the null hypothesis  $\mu_0 = 0$ 

n <- 25 # The sample size

prior <- function(u) 0.5 * dnorm(u, -1, 1) + 0.5 * dnorm(u, 1, 1) # The defined prior  $g(u)$ 

# Assume mc indicates a mc-th Monte Carlo repetition based on  $x[1], \dots, x[n]$ 

CLLR[mc] <- el.test(x, mu0)$ - 2LLR' # Calculation of  $-2\log(EL(\mu_0)n^n)$ 

LB <- -min(x)

UB <- -max(x) # Bounds of the integral at (2)

# Below is the function  $g(u) EL(u)n^n$  that will be integrated in (2)

C <- function(u)

p <- length(u)

R <- array(u)

for(i in 1 : p) R[i] <- -exp(0.5 * (-el.test(x, u[i])$ - 2LLR')) * prior(u[i])

return(R)

}

A <- integrate(C, LB, UB)$value # The numerator of  $BFELR$ 

LLR[mc] <- -2 * log(A) + CLLR[mc] # The value of the test statistic  $2\log(BFELR)$ 

# Below is the calculation of the test-threshold,  $C BFELR$ , defined by Proposition 2.1

h <- function(lamda) exp(-n * 0.5 * (mu0 - lamda)^2 / (var(x) + (mu0 - lamda)^2))

* prior(lamda)

```

```

C_BFELR <- -qchisq(0.95, 1) + 2 * log(integrate(h, LB, UB)[[1]])

Tresh[mc] <- -C_BFELR

print(mean(1 * (LLR >= Tresh))) # We reject  $H_0$ , if  $LLR[mc] \geq Tresh[mc]$ .

```

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Table 1
Monte Carlo type I errors of the test-statistics $2 \log ELR$ and $2 \log BFELR$ based on $N(0, 1)$ -data.

Prior g	n	$2 \log ELR$					$2 \log BFELR$				
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.1$
$Unif(0, 1)$	10	0.0398	0.1035	0.1516	0.0257	0.0673	0.1039				
	15	0.0256	0.0761	0.1308	0.0178	0.0586	0.1072				
	25	0.0162	0.0629	0.1198	0.0157	0.0572	0.1062				
	50	0.0125	0.0568	0.1055	0.0119	0.0564	0.1032				
	75	0.0093	0.0524	0.1001	0.0105	0.0513	0.1061				
$N(0.7, 1)$	10	0.0394	0.1007	0.1609	0.0312	0.0779	0.1257				
	15	0.0251	0.0733	0.1312	0.0216	0.0617	0.1075				
	25	0.0146	0.0664	0.1135	0.0125	0.0577	0.1002				
	50	0.0128	0.0542	0.1041	0.0127	0.0521	0.096				
	75	0.0105	0.0543	0.1022	0.0104	0.0537	0.0958				
$\frac{1}{2}N(1, 1) + N(-1, 1)$	10	0.0383	0.1029	0.1556	0.032	0.0845	0.1216				
	15	0.0274	0.0781	0.1309	0.0247	0.0695	0.1105				
	25	0.0182	0.0605	0.1144	0.0171	0.0559	0.1066				
	50	0.0144	0.0571	0.1177	0.0141	0.0556	0.1072				
	75	0.0105	0.0549	0.1013	0.0104	0.0531	0.0994				

Table 2

Monte Carlo type I errors of the test-statistics $2 \log ELR$ and $2 \log BFELR$ based on $\{X_i = (\exp(\xi_i) - \exp(0.5)) / (\exp(2) - \exp(1))^{0.5}, \xi \sim N(\mu, 1), i = 1, \dots, n\}$.

Prior	n	$2 \log ELR$						$2 \log BFELR$					
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
Unif(0, 1)	10	0.1152	0.1835	0.2454	0.0034	0.0287	0.0652	0.0034	0.0287	0.0652	0.0034	0.0287	0.0652
	15	0.0762	0.1469	0.2123	0.004	0.0255	0.0707	0.004	0.0255	0.0707	0.004	0.0255	0.0707
	25	0.0514	0.1198	0.1787	0.0048	0.038	0.081	0.0048	0.038	0.081	0.0048	0.038	0.081
	50	0.0304	0.0871	0.1495	0.0083	0.0447	0.0975	0.0083	0.0447	0.0975	0.0083	0.0447	0.0975
	75	0.022	0.0777	0.1395	0.0105	0.0473	0.1071	0.0105	0.0473	0.1071	0.0105	0.0473	0.1071
$N(0.7, 1)$	10	0.1176	0.1832	0.2401	0.1038	0.1583	0.2024	0.1038	0.1583	0.2024	0.1038	0.1583	0.2024
	15	0.0817	0.1526	0.2081	0.0718	0.1375	0.1897	0.0718	0.1375	0.1897	0.0718	0.1375	0.1897
	25	0.0502	0.1179	0.1809	0.0448	0.1099	0.1708	0.0448	0.1099	0.1708	0.0448	0.1099	0.1708
	50	0.0304	0.0907	0.1497	0.0291	0.0897	0.1495	0.0291	0.0897	0.1495	0.0291	0.0897	0.1495
	75	0.0239	0.0799	0.1332	0.0226	0.0797	0.134	0.0226	0.0797	0.134	0.0226	0.0797	0.134
$\frac{1}{2}N(1, 1) + N(-1, 1)$	10	0.1109	0.1839	0.2425	0.1022	0.1605	0.2046	0.1022	0.1605	0.2046	0.1022	0.1605	0.2046
	15	0.0778	0.1488	0.2092	0.0737	0.1361	0.1887	0.0737	0.1361	0.1887	0.0737	0.1361	0.1887
	25	0.0522	0.1171	0.1786	0.0496	0.112	0.1682	0.0496	0.112	0.1682	0.0496	0.112	0.1682
	50	0.0265	0.0917	0.1458	0.0262	0.0909	0.1441	0.0262	0.0909	0.1441	0.0262	0.0909	0.1441
	75	0.0225	0.0811	0.1349	0.0226	0.0815	0.1355	0.0226	0.0815	0.1355	0.0226	0.0815	0.1355

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Table 3

The Monte Carlo powers of the test-statistics $2 \log BFELR$ and $2 \log ELR$ at $\alpha = 0.05$.

Prior	n	δ	$X \sim N(\delta, 1)$		$X \sim Lognormal(\delta, 1)$	
			$2 \log ELR$	$2 \log BFELR$	$2 \log ELR$	$2 \log BFELR$
Unif(0, 1)	10	0.5	0.4203	0.4809	0.8654	0.8926
	10	1	0.8842	0.9134	0.9998	0.9884
	10	1.5	0.996	0.9857	1	0.984
	15	0.5	0.5235	0.7352	0.9653	0.9979
	15	1	0.9636	0.9893	1	0.9977
	15	1.5	0.999	0.9999	1	0.9998
	25	0.5	0.6962	0.8532	0.9980	0.9984
	25	1	0.9984	0.9995	1	0.9996
	25	1.5	1	1	1	0.9999
	50	0.5	0.9424	0.9814	1	1
N(0.7, 1)	50	1	1	1	1	1
	75	0.5	0.9878	0.9969	1	1
	10	0.5	0.4257	0.4227	0.8563	0.8702
	10	0.7	0.6416	0.6878	0.9967	1
	10	1	0.8864	0.8957	0.9997	0.9999
	10	1.5	0.9952	0.9951	1	0.9992
	15	0.5	0.5117	0.5780	0.9623	0.9700
	15	0.7	0.7755	0.8373	0.9993	0.9989
	15	1	0.9667	0.9895	0.9999	0.9999
	15	1.5	0.9998	0.9998	1	1
	25	0.5	0.7128	0.8205	0.9982	0.9991
	25	0.7	0.9288	0.9875	0.9999	0.9997
	25	1	0.9974	0.9982	1	0.9999
	25	1.5	1	1	1	0.9999
	50	0.5	0.9375	0.9648	1	1
	50	0.7	0.9974	0.9992	1	1
	75	0.5	0.9924	0.9997	1	1

			$X \sim N(\delta, 1)$				$X \sim \text{Lognormal}(\delta, 1)$			
Prior	n	δ	$2 \log ELR$		$2 \log BFELR$		$2 \log ELR$		$2 \log BFELR$	
$\frac{1}{2}N(1, 1) + N(-1, 1)$	10	0.5	0.4215		0.3998		0.8574		0.7998	
	10	0.7	0.6439		0.6984		0.9977		0.9979	
	10	1	0.8911		0.9534		1		0.9999	
	10	1.5	0.9957		0.9956		1		1	
	15	0.5	0.5246		0.5195		0.9631		0.9597	
	15	0.7	0.7806		0.8391		0.9998		0.9999	
	15	1	0.9614		0.9996		1		1	
	15	1.5	0.9997		0.9997		1		1	
	25	0.5	0.7024		0.6999		0.9973		0.9979	
	25	0.7	0.9309		0.9945		1		1	
	50	0.5	0.9367		0.9388		1		1	
	50	0.7	0.9986		1		1		1	
	75	0.5	0.9901		0.9998		1		1	

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Table 4
The Monte Carlo type I errors and powers of the test-statistics 2 log *BFELR*, when the asymptotically expected $\alpha = 0.05$

Prior	$X \sim N(0, 1)$	$X \sim N(0.5, 1)$	Prior	$X \sim N(0, 1)$	$X \sim N(0.5, 1)$
$N(0,0.1)$	0.0514	0.2466	$N(0,1)$	0.0674	0.3560
$N(0.5,0.1)$	0.0655	0.4938	$N(0.5,1)$	0.0653	0.4099
$N(0.6,0.1)$	0.0667	0.5355	$N(0.6,1)$	0.0698	0.4233
$N(0.82,0.1)$	0.0729	0.5738	$N(0.82,1)$	0.0759	0.4301
$N(1.46,0.1)$	0.0607	0.4906	$N(1.46,1)$	0.0711	0.4585
$N(5,0.1)$	0.0025	0.0289	$N(5, 1)$	0.0032	0.0849
$N(0,0.32)$	0.0604	0.3239	$N(0,10)$	0.0774	0.3578
$N(0.5,0.32)$	0.0677	0.4259	$N(0.5,10)$	0.0801	0.3705
$N(0.6,0.32)$	0.0689	0.4601	$N(0.6,10)$	0.0768	0.3836
$N(0.82,0.32)$	0.0711	0.4836	$N(0.82,10)$	0.0748	0.3775
$N(1.46,0.32)$	0.0680	0.5244	$N(1.46,10)$	0.0703	0.3827
$N(5,0.32)$	0.0021	0.0214	$N(5,10)$	0.0712	0.3853