Heat capacity and entropy of a GaAs quantum dot with Gaussian confinement

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(Received 9 July 2012; accepted 25 September 2012; published online 22 October 2012)

The heat capacity and entropy effects in a GaAs quantum dot with Gaussian confinement are calculated in the presence of a magnetic field and its interaction with the electron spin using the canonical ensemble approach. It is shown that the heat capacity shows a Schottky-like anomaly at a low temperature, while it approaches a saturation value $2k_B$ as the temperature increases. As a function of the magnetic field, the heat capacity shows a maximum and then reduces to zero. Also the width of the maximum becomes wider with temperature. It is also shown that the heat capacity remains constant up to a certain value of the confinement length beyond which it displays a monotonic increase. However as a function of the confinement strength, though the heat capacity initially shows a significant drop, it remains constant thereafter. At low temperatures like $T = 10$ and 20 K, the entropy is found to decrease with increasing magnetic field, but at higher temperatures, it remains almost independent of the magnetic field. At high temperatures, entropy shows a monotonic increase with temperature, but at a sufficiently low temperature as the magnetic field decreases, the entropy is found to develop a shoulder which becomes more and more pronounced with decreasing magnetic field. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4759350>]

I. INTRODUCTION

The subject of quantum dots has emerged as a very fascinating area in condensed matter physics in recent times.^{1–9} The reason is primarily two-fold. First and foremost, the quantum dots have a great potential for applications in micro-electronic devices such as quantum dot lasers, solar cells, single electron transistors and quantum computers. Second and not less importantly, quantum dots can be considered as a tiny laboratory where the predictions of quantum mechanics can be tested comprehensively. It is essential to know the nature of the confining potential to formulate any theory of quantum dots. Early experiments indicated that this potential is harmonic in nature. However, some recent experiments have suggested that the confining potential is rather anharmonic and has a finite depth. Adamowsky *et al.*¹⁰ have proposed an attractive Gaussian potential which seems to be a better model for the confining potential. We would like to mention in passing that the Gaussian potential has been solved approximately for a single-particle problem by several authors.^{11–18}

There have been several investigations in the past few years on the heat capacity and entropy of quantum dots and other related low-dimensional systems in the presence of a magnetic field.^{19–27} However, most of these investigations have used either a square well model (SQM) or a parabolic potential model (PPM) for the confining potential. Nguyen and Peeters¹⁹ have investigated the thermodynamic quantities such as the heat capacity, the magnetization and susceptibility in the presence of a single magnetic ion and a

perpendicular magnetic field taking into account the electron-electron correlations in a many-electron quantum dot. They have observed that the cusps in the energy levels show up as peaks in the heat capacity and the susceptibility. Müller and Koonin²⁰ have calculated the heat capacity associated with the rotational degree of freedom of deformed many-body states. Oh *et al.*²¹ have studied the heat capacity of quantum wires and dots and have found that the calculated heat capacities at low temperatures show an oscillating behavior. De Groote *et al.*²² have calculated the magnetization, magnetic susceptibility and heat capacity of helium-like confined quantum dots and obtained an additional structure in the heat capacity. Maksym and Chakraborty²³ have calculated the electronic heat capacity taking into account the electron-electron interaction. In all these cases, the confining potential has been taken as parabolic. To our knowledge, no investigation has so far been made on the temperature-dependent heat capacity and entropy in a gaussian quantum dot in the presence of the spin-Zeeman interaction. The purpose of this present paper is to make an attempt in this direction.

II. MODEL AND FORMULATION

The Hamiltonian of a system of an electron moving in a two-dimensional (2D) confining potential V in the presence of an external magnetic field $\mathbf{B}$ may be written as

$$H = \frac{1}{2m^*} \left(\mathbf{p} + \frac{e}{c} \mathbf{A} \right)^2 + V(\boldsymbol{\rho}), \quad (1)$$

where $\boldsymbol{\rho}$ refers to the position vector of an electron in two dimensions, $\mathbf{p}$ is the corresponding momentum operator, m^*

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is the electron effective mass and $\mathbf{A}$ is the vector potential corresponding to the magnetic field $\mathbf{B}$ which has been applied in the z direction. Choosing the gauge of $\mathbf{A}$ as $\mathbf{A} = B(-y/2, x/2, 0)$ such that $\mathbf{A}$ is divergence-less and including the spin-Zeeman term, we can write Eq. (1) as

$$H = -\frac{\hbar^2}{2m^*} \nabla_\rho^2 + V(\rho) + \frac{1}{8} m^* \omega_c^2 \rho^2 + \frac{1}{2} \hbar \omega_c (\hat{L}_z + g^* \hat{S}_z), \quad (2)$$

where L_z is the z -component of the angular momentum of the electron, ω_c is the bare cyclotron frequency given by $\omega_c = eB/m^*$ (units have been chosen in which $(c = 1)$), B being the applied magnetic field and $V(\rho)$ is the confining potential which we take as: $V(\rho) = -V_0 \exp(-\rho^2/2R^2)$, where V_0 denotes the depth of the potential and R gives a measure of the range of the potential and thus represents the effective confinement length or the size of the quantum dot.

We assume that the deviation of the Gaussian confining potential from the parabolic potential is small enough so that it can be treated as a parabolic potential plus a perturbation. This is a reasonable assumption for small r and since in a quantum dot, r will be generally small, it can be considered as a fairly good approximation. So we rewrite the Hamiltonian (1) as

$H = H_0 + H_1$, where

$$H_0 = -\frac{\hbar^2}{2m^*} \nabla_\rho^2 + \frac{1}{2} m^* \tilde{\omega}^2 \rho^2 + \frac{1}{2} \hbar \omega_c (\hat{L}_z + g^* \hat{S}_z) - V_0, \quad (3)$$

$$H_1 = -\lambda \left[\frac{1}{2} m^* \omega_h^2 \rho^2 - V_0 \exp(-\rho^2/2R^2) \right], \quad (4)$$

where $\tilde{\omega}^2 = \omega_h^2 + \omega_c^2/4$, $\omega_h^2 = V_0/m^*R^2$, and $\lambda=0$ for a PPM and $\lambda=1$ for a Gaussian potential model (GPM). We assume that the sole effect of H_1 is to renormalize the frequency $\tilde{\omega}$. So we treat H_1 by the first order perturbation theory at the mean field level. More specifically, we write H_1 as

$$H_1 = \lambda \left[\frac{V_0}{\langle \rho^2 \rangle} - \frac{1}{2} m^* \omega_h^2 - \frac{V_0}{\langle \rho^2 \rangle} \langle \exp(-\rho^2/2R^2) \rangle \right] \rho^2, \quad (5)$$

where $\langle \rho^2 \rangle$ is the expectation value of ρ^2 with respect to ground state wave function of the harmonic oscillator of frequency $\tilde{\omega}$. The GPM problem now reduces to an effective PPM problem described by Hamiltonian^{28,29}

$$H = -\frac{\hbar^2}{2m^*} \nabla_\rho^2 + \frac{1}{2} m^* \omega^2 \rho^2 + \frac{1}{2} \hbar \omega_c (\hat{L}_z + g^* \hat{S}_z) - V_0, \quad (6)$$

where $\omega = [(1 - \lambda)\omega_h^2 + \omega_c^2/4 + 2\lambda V_0(\tilde{\omega}/(\hbar + 2m\tilde{\omega}R^2))]^{1/2}$ and g^* is effective Lande-g factor (which is equal to -0.44 for GaAs). The effective problem now satisfies the Schrödinger equation: $H\Psi_{nls}(\rho, \theta, \sigma) = E_{nls}\Psi_{nls}(\rho, \theta, \sigma)$, where $\Psi_{nls}(\rho, \theta, \sigma)$ is the wave function given by the Fock-Darwin states: $\Psi_{nls}(\rho, \theta, \sigma) = (\alpha/\sqrt{\pi})(n!/(n+|l|)!)^{1/2} (\alpha\rho)^{|l|} L_n^{|l|}(\alpha^2\rho^2) \exp(-\frac{1}{2}\alpha^2\rho^2 + il\theta) \chi_s(\sigma)$, where $\alpha = (m^*\omega/\hbar)^{1/2}$, n is the radial quantum number, $l = (0, \pm 1, \pm 2, \dots)$ is the azimuthal angular momentum quantum number, $L_n^{|l|}$ is

the associated Laguerre polynomial, $\chi_s(\sigma)$ is the eigenstate of the spin operator S_z with eigenvalue $s = \pm 1/2$ and E_{nls} is given by

$$E_{nls} = (2n + |l| + 1)\hbar\omega + \frac{1}{2}\hbar\omega_c(l + g^*s) - V_0. \quad (7)$$

In the present work, we are interested in the temperature dependence of the heat capacity and entropy. We shall however neglect the electron-electron and the spin-orbit interactions. The partition function for the present system can be exactly calculated and is given by³⁰

$$Z(T, B) = \frac{1}{2} \cosh\left(\frac{1}{4} g^* \beta \hbar \omega_c\right) \operatorname{csch}\left\{\frac{\beta \hbar}{2} \left(\omega - \frac{\omega_c}{2}\right)\right\} \times \operatorname{csch}\left\{\frac{\beta \hbar}{2} \left(\omega + \frac{\omega_c}{2}\right)\right\}, \quad (8)$$

where $\beta = 1/k_B T$. The partition function can be used to obtain the average energy E , heat capacity (C) and the entropy (S) as

$$\begin{aligned} \langle E \rangle = \frac{\hbar}{2} & \left[\left(\omega - \frac{\omega_c}{2}\right) \coth\left\{\frac{\beta \hbar}{2} \left(\omega - \frac{\omega_c}{2}\right)\right\} \right. \\ & + \left(\omega + \frac{\omega_c}{2}\right) \coth\left\{\frac{\beta \hbar}{2} \left(\omega + \frac{\omega_c}{2}\right)\right\} \Big] \\ & - \frac{1}{4} g^* \hbar \omega_c \tanh\left(\frac{1}{4} g^* \beta \hbar \omega_c\right), \end{aligned} \quad (9)$$

$$\begin{aligned} C = \frac{\hbar^2 k_B}{4\beta^2} & \left[\left(\frac{\omega_c}{2} - \omega\right)^2 \operatorname{csch}^2\left\{\frac{\beta \hbar}{2} \left(\omega - \frac{\omega_c}{2}\right)\right\} \right. \\ & + \left(\frac{\omega_c}{2} + \omega\right)^2 \operatorname{csch}^2\left\{\frac{\beta \hbar}{2} \left(\omega + \frac{\omega_c}{2}\right)\right\} \Big] \\ & + \frac{k_B g^{*2}}{16\beta^2} (\hbar \omega_c)^2 \operatorname{sech}^2\left(\frac{1}{4} g^* \beta \hbar \omega_c\right), \end{aligned} \quad (10)$$

$$\begin{aligned} S = \frac{1}{2} k_B \ln & \left[\cosh\left(\frac{1}{4} g^* \beta \hbar \omega_c\right) \operatorname{csch}\left\{\frac{\beta \hbar}{2} \left(\omega - \frac{\omega_c}{2}\right)\right\} \right. \\ & \times \operatorname{csch}\left\{\frac{\beta \hbar}{2} \left(\omega + \frac{\omega_c}{2}\right)\right\} \Big] \\ & + \frac{\beta \hbar}{2} k_B \left[\left(\omega + \frac{\omega_c}{2}\right) \coth\left(\frac{\beta \hbar}{2} \left(\omega + \frac{\omega_c}{2}\right)\right) \right. \\ & + \left. \left(\omega - \frac{\omega_c}{2}\right) \coth\left(\frac{\beta \hbar}{2} \left(\omega - \frac{\omega_c}{2}\right)\right) \right] \\ & - \frac{\beta \hbar}{4} k_B g^* \omega_c \tanh\left(\frac{1}{4} g^* \beta \hbar \omega_c\right), \end{aligned} \quad (11)$$

We have studied the behaviour of the heat capacity and entropy as a function of temperature (T), magnetic field (B), confinement length (R) and the depth of the confining potential (V_0) numerically.

III. NUMERICAL RESULTS AND DISCUSSION

In Fig. 1, we plot the thermal heat capacity (C) per electron in a GaAs QD as a function of temperature T incorporating the effect of the electron spin. Unless specifically

FIG. 1. Heat capacity (C/k_B) vs temperature (T) of a GaAs QD for $B = 0.5, 1, 5$ and 10 T.

mentioned otherwise, it should be understood that the results being reported are for GPM. We observe that at very low temperatures, the heat capacity shows a peak structure. As the magnetic field is increased, the peak shifts to the right and become broader in width. In fact, at $B = 10$ T, the peak transforms into a shoulder. Naturally at higher magnetic fields, one would expect a monotonic increase in the heat capacity with temperature. However, as the temperature keeps increasing further, the heat capacity finally saturates. The saturation value the heat capacity approaches at a high temperature is found to be about $2k_B$ in conformity with result reported in Ref. 20. The origin of the peak structure observed in C is not quite clear, though it is reminiscent of the well-known Schottky anomaly observed in the heat capacity of two-level systems at low temperatures. We also observe that the average thermodynamic energy initially increases very slowly with temperature (though not shown in the figure) but above a certain temperature, it increases rather rapidly. In Fig. 2, we plot the heat capacity of a GaAs QD as a function of the applied magnetic field for three values of the temperature. The figure shows that the heat capacity peaks at a certain value of the magnetic field and then reduces to zero, more quickly at a low temperature. The critical value of the magnetic field B at which the heat capacity becomes maximum increases as the temperature increases. Furthermore, the peaks also become broader with increasing

FIG. 2. Heat capacity (C/k_B) vs magnetic field B of a GaAs QD for $T = 2, 10$ and 20 K.

temperature. This scenario can be explained in a simple way. With the given parameter values of the GaAs QD, one would have certain discrete energy spectrum. The surrounding environment will be able to excite the system thermodynamically most efficiently only at a certain critical temperature. Or put differently, at a given temperature, the system will absorb maximum energy from the environment only if the applied magnetic field has a critical value. At other magnetic field values, absorption of energy will be much less leading to a sharp fall in the specific heat. The argument sounds credible because at high temperature, the peak in the heat capacity shifts towards right and becomes broader.

In Fig. 3, we plot the heat capacity as a function of the effective Lande g factor (g^*) for $B = 5$ T and $T = 30$ K. The first thing to note is that the heat capacity C is symmetric in the Lande g factor, g^* . Second, the heat capacity initially increases with g^* , reaches a maximum and then starts decreasing with increasing g^* and finally above a certain value of g^* ($|g^*| \sim 5$), it becomes almost independent of the Lande g factor g^* . We have also plotted the results for the parabolic potential model. It appears that the parabolic model underestimates the heat capacity. The saturation of the heat capacity for large g^* may be ascribed to the fact that in the presence of a large magnetic field B , beyond a certain value of g^* electron spin moments will be saturated and the Zeeman contribution to the average energy becomes independent of temperature and consequently, g^* will not have any effect on the heat capacity. In fact one can see from Eq. (9) that in the presence of a strong magnetic field, the average energy for large g^* approaches $-\frac{1}{4}g^*\hbar\omega_c$ and therefore the Zeeman term does not contribute to the heat capacity at large g^* and then the heat capacity becomes independent of g^* . In Fig. 4, we plot the heat capacity as a function of the quantum dot size, R for temperature, $T = 30$ K and $B = 5$ T. One can observe that in the presence of spin, the heat capacity remains constant at small R up to a certain value of R beyond which the heat capacity increases monotonically as R increases (solid line). Interestingly, without the spin-Zeeman term, the heat capacity is zero in the region where it is flat in the presence of spin (dashed line). The zero value of the heat capacity below a certain critical dot size in the absence of spin may be because of the occurrence of large energy spacings at low R . The

FIG. 3. C/k_B vs. g^* for a QD with GPM (solid line) and PPM (dashed line).

FIG. 4. C/k_B vs. R for a GaAs QD.

behavior of the heat capacity beyond R essentially displays the extensive property of the heat capacity g^* , since it scales linearly with the size R . The flat part of the heat capacity in the presence of spin arises due to R -independent contribution of the spin-Zeeman interaction.

In Fig. 5, we plot the behavior of heat capacity as a function of the depth (V_0) of the confinement potential. Fig. 5(a) shows the results for high values of V_0 and Fig. 5(b) displays the behavior for low values of V_0 . One can observe an interesting crossing behavior at low values of V_0 . It is, however, more important to note that at low values of V_0 , the heat

FIG. 5. C/k_B vs. V_0 of a GaAs QD in presence of spin for high (a) and low (b) values of V_0 . FIG. 6. C/k_B vs. m^*/m_0 of a Gaussian QD.

capacity undergoes a substantial reduction in its magnitude as V_0 increases and then remains essentially constant on further increase in the value of V_0 . This sharp fall in the value of heat capacity may have the genesis in the large energy separations arising for the low-lying excited states due to the increase in V_0 . Since the excitation energies for the low-lying excited states become large, the thermal energy available in the ambient environment as indicated Fig. 5 cannot excite the system leading to a very low heat capacity. This explanation looks plausible in view of the observation that the lower the temperature, the sharper the fall in the heat capacity.

In Fig. 6, we plot the heat capacity as a function of effective mass at $B = 1$ T and $T = 1, 50$ and 100 K. We find the behavior quite interesting. At high temperatures like 50 K and 100 K, the heat capacity increases very rapidly with m^* at small values of m^* and then tends to saturate as m^* becomes a little larger. The saturated value of the heat capacity appears to be about $2k_B$. However, at a very low temperature like 1 K, the behavior is qualitatively different. In this case, the heat capacity increases with m^* initially, attains a maximum and then decreases to zero. The explanation is again pretty straight-forward. The effective mass m^* in a way increases the cyclotron frequency and the Zeeman term, leading to larger excitation energies for the low-lying excited states. So at larger temperatures like $T = 50$ K and 100 K, the required excitation energies are available and one obtains a sizable heat capacity. However, beyond a certain value of the parameters, the potential cannot sustain any more excited states, eventually heat capacity saturates. However, at very low temperature the situation is quite different. For instance, at very small m^* value, the system may be able to absorb heat and show a small heat capacity, but as m^* increases beyond a critical value, the surroundings with 1 K may not be able to provide the sufficient energy to excite the system and consequently, the heat capacity would reduce to zero.

Another important thermodynamics quantity we have studied is the entropy. We have calculated the entropy S/k_B as a function of magnetic field B using the same Gaussian confining potential at temperatures $T = 10$ K, 20 K, 50 K and 100 K. The results are shown in Fig. 7. At $T = 50$ K and 100 K, the entropy does not change much with B . But at relatively lower temperatures like $T = 10$ K and 20 K, as the

FIG. 7. S/k_B vs. B of a GaAs QD.

magnetic field increases, entropy decreases. One possible explanation for the observed behavior could be that the order brought in the system by the magnetic field might be counterbalanced by the kinetic energy due to confinement together with the thermodynamic disorder at higher temperatures. At low temperatures, however, the magnetic energy wins over, and the entropy decreases with increasing magnetic field. Of course the entropy increases with temperature at a fixed magnetic field, as is generally expected. In Fig. 8, we plot the entropy explicitly as a function of temperature (T) for $B = 1, 5$ and $10 T$. As expected, the entropy increases with temperature at a fixed value of the magnetic field. However, the behavior is qualitatively different at lower temperatures as compared to that at relatively higher temperatures. For example, at higher temperature, entropy increases monotonically with temperature for all the values of the magnetic field considered and becomes essentially independent of the magnetic field. However, at low temperature, the behavior is very different and is crucially dependent on the magnetic field. It is clear that as the magnetic field decreases, the entropy develops a shoulder. The shoulder becomes more and more distinct as the magnetic field decreases. At $B = 1 T$, as the temperature is increased to a value slightly greater than zero (though still being pretty close to zero), the entropy increases quite rapidly to a certain value. Then in a certain temperature window, entropy shows a rather marginal increase in its value and finally merges with the essentially magnetic-field-independent monotonically increasing entropy. This behavior is consistent with the heat

FIG. 8. S/k_B vs. T for a GaAs QD. FIG. 9. C/k_B as a function of T and B for a GaAs QD.

capacity behavior shown at low temperature at $B = 1 T$ in Fig. 1. In Fig. 9, we present the contour diagram for the heat capacity as a function of magnetic field (B) and temperature (T) for $R = 10$ nm and $V_0 = 36.7$ meV for a GaAs quantum dot with Gaussian confinement.

IV. CONCLUSION

We have investigated the temperature-dependent heat capacity and the corresponding entropy of a GaAs quantum dot with Gaussian confinement in the presence of the full Zeeman term using the canonical ensemble approach. We have shown the variation of the heat capacity and entropy as a function of different parameters. As a function of temperature, the heat capacity shows at a very low temperature, a sharp peak structure reminiscent of the Schottky anomaly. As a function of the magnetic field, the heat capacity is found to peak at a certain value of the magnetic field which depends on the temperature, and then reduces to zero, more quickly at a low temperature. The critical value of the magnetic field B at which the heat capacity becomes maximum increases as the temperature increases. We have shown that as a function of the Lande g factor, the heat capacity initially increases with g^* , reaches a maximum and then starts decreasing with increasing g^* and finally above a certain value of g^* ($|g^*| \sim 5$), it becomes almost independent g^* . As a function of the dot size R , the heat capacity remains constant up to a certain value of R at small R beyond which the heat capacity increases monotonically as R increases. We find that initially the heat capacity decreases significantly with increasing confinement strength V_0 , and thereafter remains essentially constant on further increase in V_0 . As a function of the electron effective mass m^* , at temperatures like 50 and 100 K, the heat capacity increases very rapidly with m^* at small m^* and then tends to saturate as m^* becomes a little larger. The saturated value of the heat capacity appears to be about $2k_B$. However, at a very low temperature like 1 K, we observe that initially the heat capacity increases with m^* , attains a maximum and then decreases to zero. It may be pointed out that the behavior of the average energy for an electron gas in general depends on the density of states. However in the case of quantum dots, the energy

levels are not quasi-continuous but discrete and therefore the average energy can be said to depend on the distribution of energy levels. Consequently, the heat capacity should depend both on the energy level distribution and the temperature dependence of the occupation probability of the states.

As we have already mentioned, we have also presented some interesting behavior for the entropy. At $T = 50$ and 100 K, the entropy does not seem to change much with the magnetic field B , but at relatively lower temperatures like $T = 10$ and 20 K, entropy decreases as B increases. As a function of temperature, entropy increases monotonically with temperature at high temperatures for the magnetic fields considered in this work and becomes essentially independent of the magnetic field. However at low temperature, the behavior is very different and is crucially dependent on the magnetic field. It is clear that as the magnetic field decreases, the entropy develops a shoulder which becomes more and more prominent as the magnetic field decreases. We have attempted to provide explanations for almost all the behavior of the heat capacity and entropy we have presented in this work.

¹M. A. Kastner, *Rev. Mod. Phys.* **64**, 849 (1992).

²N. F. Johnson, *J. Phys.: Condens. Matter* **7**, 965 (1995).

³R. Ashoori, *Nature* **379**, 413 (1996).

⁴U. Woggon, *Optical Properties of Semiconductor Quantum Dots* (Springer, 1997).

⁵L. Jacak, P. Hawrylak, and A. Wójs, *Quantum Dots* (Springer, 1998).

⁶D. Bimberg, M. Grundmann, and N. N. Ledentsov, *Quantum Dot Heterostructures* (Wiley, 1999).

⁷S. Mukhopadhyay and A. Chatterjee, *Acta Phys. Pol. B* **32**, 473 (2001).

⁸L. P. Kouwenhoven, D. Austing, and S. Tarucha, *Rep. Prog. Phys.* **64**, 701 (2001).

⁹J. Kainz, S. Mikhailov, A. Wensauer, and U. Rössler, *Physica E* **12**, 888 (2002).

¹⁰J. Adamowski, M. Sobkowicz, B. Szafran, and S. Bednarek, *Phys. Rev. B* **62**, 4234 (2000).

¹¹N. Bessis, G. Bessis, and B. Joulakian, *J. Phys. A*, **15**, 3679 (1982).

¹²C. S. Lai, *J. Phys. A*, **16**, L181 (1983).

¹³R. E. Crandall, *J. Phys. A*, **16**, L395 (1983).

¹⁴M. Cohen, *J. Phys. A*, **17**, L101 (1984).

¹⁵A. Chatterjee, *J. Phys. A*, **18**, 2403 (1985).

¹⁶A. Chatterjee, *Phys. Rep.* **186**, 249 (1990).

¹⁷S. S. Gomez and R. H. Romero, *Cent. Eur. J. Phys.* **7**, 12 (2009).

¹⁸B. Boyacioglu, M. Saglam, and A. Chatterjee, *J. Phys.:Condens. Matter.* **19**, 456217 (2007).

¹⁹N. T. T. Nguyen and F. M. Peeters, *Phys. Rev. B* **78**, 045321 (2008).

²⁰H. M. Müller and S. E. Koonin, *Phys. Rev. B* **54**, 14532 (1996).

²¹J. C. Oh, K. J. Chang, G. Ihm, and S. J. Lee, *J. Korean Phys. Soc.* **28**, 132 (1995).

²²J. J. S. De Groote, J. Hornos, and A. Chaplik, *Phys. Rev. B* **46**, 12773 (1992).

²³P. A. Maksym and T. Chakraborty, *Phys. Rev. Lett.* **65**, 108 (1990).

²⁴J. H. Oh, K. J. Chang, G. Ihm, and S. J. Lee, *Phys. Rev. B* **50**, 15397 (1994).

²⁵S. J. Lee, H. Chang Jeon, T. Won Kang, and S. Souma, *Physica E* **40**, 2198 (2008).

²⁶E. Gornik, R. Lassnig, G. Strasser, H. L. Störmer, A. C. Gossard, and W. Wiegmann, *Phys. Rev. Lett.* **54**, 1820 (1985).

²⁷O. Voskoboynikov, O. Bauga, C. P. Lee, and O. Tretyak, *J. Appl. Phys.* **94**, 5891(2003).

²⁸S. Mukhopadhyaya, B. Boyacioglu, M. Saglam, and A. Chatterjee, *Physica E* **40**, 2776 (2008).

²⁹B. Boyacioglu and A. Chatterjee, *Int. J. Mod. Phys. B*, **26**, 1250018 (2012).

³⁰L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Int. Lib., 1975).