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An Implicit Elastic Theory for Lung Parenchyma^{☆,☆☆}

Alan D. Freed and

Department of Mechanical Engineering, Saginaw Valley State University, 202 Pioneer Hall, 7400 Bay Road, University Center, MI 48710, USA

Daniel R. Einstein

Systems Toxicology, Pacific Northwest National Laboratory, 1007 Rogers Street Northwest, Olympia, WA 98502, USA

Abstract

The airways and parenchyma of lung experience large deformations during normal respiration. Spatially accurate predictions of airflow patterns and aerosol transport therefore require respiration to be modeled as a fluid-structure interaction problem. Such computational models in turn require constitutive models for the parenchyma that are both accurate and efficient. Herein, an implicit theory of elasticity is derived from thermodynamics to meet this need, leading to a generic template for strain-energy that is shown to be an exact analogue for the well-known Fung model that is the root of modern constitutive theory of tissues. To support this theory, we also propose a novel definition of Lagrangian strain rate. Unlike the classic definition of Lagrangian strain rate, this new definition is separable into volumetric and deviatoric terms, a separation that is both mathematically and physically justified. Within this framework, a novel material model capable of describing the elastic contribution of the nonlinear response of parenchyma is constructed and characterized against published data.

Keywords

biomechanics; bulk response; compressible solids; constitutive equations; deviatoric strain; elastic moduli; finite deformation; shear response; soft solids; strain-energy function; tangent moduli; thermodynamics

1. Introduction

The soft tissues of the parenchyma, which comprise the majority of lung tissue, are considered to include the alveoli, the alveolar ducts, the respiratory bronchioles, and the capillary fields that surround them, but they are dominated in terms of volume by the alveoli themselves [1]. Reviews on the structure and function of lung have been written by Faffe

[☆]Dedicated to Prof. Kumbakonam R. Rajagopal.

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Corresponding author (ADF): adfreed@svsu.edu.
daniel.einstein@pnnl.gov

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and Zin [2], Fredberg and Kamm [3], Fung [4, 5], Mead [6], Otis [7, 8], Stamenovi [9], Suki and Bates [10], Suki et al. [11], Weibel and Gil [12], and West [1].

Literature on the modeling of lung parenchyma is quite substantial and goes back at least a century and a half [7]. The well established non-linearity of the pressure-volume response of lung has made work in this field challenging [13]. These issues have sometimes lead to the impression that the parenchyma, as a continuum, is anisotropic [14], which it is not [15]. Experimental characterization has also been challenging. Due to the spongy nature of the parenchyma, excision—always a source of artefact—is especially confounding. At the same time, *in situ* measurement of tissue strains and, especially, site-specific pressures are also difficult to attain without causing undue risk for the patient.

Broadly speaking, modeling efforts have been divided into two categories: 1) continuum models in which material behavior is represented by a single relationship between stress and strain; and 2) structural models in which the shape, arrangement, and properties of the alveolar extracellular constituents are modeled and whose mean stress-strain response emerges as an ensemble average. Within the context of creating a finite element model of an entire lung, continuum constitutive equations for the parenchyma (to be solved at each Gauss point within a lung model) are analytic and, therefore, are apt to be ‘more efficient’. Structural models for the parenchyma are usually constructed as finite element models themselves out of which an homogenized parenchymal response is extracted (i.e., the material model to be solved at each Gauss point within a lung model is itself a finite element model) and, therefore, are likely to be ‘less efficient’ in this context. A continuum model for the elastic contribution to parenchymal response is developed in this paper.

The first continuum theory to use finite-strain elasticity to describe lung response was based on the stretching of a balloon [16] even though, physiologically, this model had been known to be incorrect for some time [17]. Nevertheless, it was intuitive, it was solvable; in addition it provided insight and, as such, it had value during the early days of studying lung mechanics. Later, investigators proposed hyperelastic material models for describing the passive response of parenchyma, e.g., Frankus and Lee [18], Fung [19], Karakaplan et al. [20], Lee and Frankus [21], Stamenovi and Wilson [22], and Vawter [23]. These are models where stress $\mathbf{S}$ is described as a function of strain $\mathbf{E}$, i.e., $\mathbf{S} = \mathcal{F}(\mathbf{E})$. In contrast, Freed and Einstein [15] recently proposed a hypo-elastic model for parenchyma where stress rate is a homogeneous function in strain rate, viz., $d\mathbf{S} = \mathcal{F}(\mathbf{S}) d\mathbf{E}$.¹

We will now further that mathematical development by deriving a novel, implicit, elastic theory applicable for parenchyma wherein $d\mathbf{S} = \mathcal{F}(\mathbf{S}, \mathbf{E}) d\mathbf{E}$. The capability of this model in capturing the essence of parenchymal F response is improved over that of our prior hypo-elastic model. This will be shown by parametrizing an implicit model for parenchyma against published data and validating model predictions against independent data. Our main contributions are: 1) the development of an implicit framework that is shown to be an exact analogue to the well-known one-dimensional Fung model [25]; 2) the introduction of a novel formulation for Lagrangian strain rate whose decomposition into volumetric and deviatoric parts is both mathematically and physically justified; 3) the novel solution of an experimentally tractable boundary value problem that is shown to yield the elastic (tangent) modulus; and 4) the development of a simple yet predictive constitutive model for parenchyma based on our new framework.

¹The authors’ hypo-elastic model for parenchyma [15] was designed to be compatible with an integrated approach to fluid-structure interaction (FSI) recently introduced by Greenshields and Weller [24], thereby making it straightforward to implement into code.

2. Theory

The implicit theory of elasticity put forward in this paper is a special case of a more general structure developed by Rajagopal and Srinivasa [26, 27] that, in turn, was developed from an idea formulated by Rajagopal [28]. From thermodynamics, Rajagopal and Srinivasa [26, 27] determined that an implicit elastic solid will have the following mathematical structure

$$\mathbb{A}:\mathrm{d}\mathbf{S}+\mathbb{B}:\mathrm{d}\mathbf{E}=0 \quad (1)$$

wherein $\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{I})$ is the Green strain tensor and $\mathbf{S}=\det(\mathbf{F})\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$ is the second Piola-Kirchhoff stress tensor, with $\mathbf{F}$ being the deformation gradient, $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ being the Green deformation, and $\mathbf{T}$ being the Cauchy stress, while $\mathbb{A}$ and $\mathbb{B}$ are fourth-order tensors with appropriate symmetries.

The implicit theory of elasticity has several features that the classic explicit theory of elasticity, e.g. hyperelasticity, does not. Of relevance to the field of biomechanics is that implicit elastic solids can possess limiting states, either in terms of strain or in terms of stress. Soft biological tissues often exhibit a limiting state in strain. Clinically, the limiting state of lung is expressed in terms of transpulmonary pressure but it may be similarly conceived of in terms of a limiting volume. Implicit elastic models possess moduli that readily convert between their compliance and stiffness constructions, thereby making it possible to construct accurate and efficient algorithms to handle these limiting states, at least in principle.

It is well known that the parenchyma is viscoelastic [16] with likely causes being the tissue itself [29], the surfactant film [9, 30], and alveolar recruitment [31-33]. Here we seek to understand the underlying nonlinear elastic contribution to the overall parenchymal response so that a proper viscoelastic theory for parenchyma can be constructed in the future.

From mass conservation comes an expression for mass continuity [34]

$$\Delta=\frac{\mathrm{d}V}{\mathrm{d}V_0}=\det \mathbf{F} \quad \text{so that} \quad \mathrm{d}\ln \Delta=\mathrm{tr}(\mathbf{F}^{-1}\mathrm{d}\mathbf{F}) \quad (2)$$

where Δ is the dilation², $\ln \Delta$ is the dilatation, and $\mathrm{d}V$ is the current volume of an elemental volume whose initial volume in its reference state is $\mathrm{d}V_0$. Dilation Δ is a volumetric stretch, while dilatation $\ln \Delta$ is a volumetric strain.³ In lung mechanics, dilation Δ can be experimentally measured through imaging by quantifying $\mathrm{d}V/\mathrm{d}V_0$ with $\mathrm{d}V_0$ being the pixel volume.

For purely mechanical theories of continua where temperature is held fixed, which is a reasonable approximation for most soft mammalian tissues, the Clausius-Planck inequality governing continuous media states [34]

$$\mathfrak{D}=\mathrm{tr}(\mathbf{S}\mathrm{d}\mathbf{E})-\mathrm{d}\mathfrak{U} \quad \text{where} \quad \mathfrak{D} \geq 0 \quad (3)$$

with the inequality being a consequence of the second law of thermodynamics. State function $\mathfrak{D}$ represents the rate of dissipation occurring at a continuum point, while state function $\mathfrak{U}$ represents the internal energy at that location.

²It is common in the mechanics literature to see the dilation Δ denoted as J since it is the Jacobian determinant [35].

³The concept of a logarithmic strain was formulated and developed by Hencky [36] from an idea that he [37] credits to Ludwig.

Definition 1. A material response is *elastic* if $\mathfrak{D}=0$ whenever $d\mathbf{E} = \mathbf{0}$.

2.1. Implicit Elastic Solid

The internal energy $\mathfrak{U}$ of an isothermal elastic solid has the physical interpretation of being an elastic strain-energy function, conventionally denoted as $W=\mathfrak{U}$.

Definition 2. An *implicit elastic solid* obeys Defn. 1 with $W=\mathfrak{W}(\mathbf{E}, \mathbf{S})$.

Considering an implicit elastic solid, while assuming $\mathfrak{W}$ to be a continuous and sufficiently differentiable function of state, an expansion of the differential $d\mathfrak{W}$ by the chain rule allows one to write

$$\mathbf{S}:d\mathbf{E} - \frac{d\mathfrak{W}(\mathbf{E}, \mathbf{S})}{d\mathbf{E}}:d\mathbf{E} - \frac{\partial\mathfrak{W}(\mathbf{E}, \mathbf{S})}{\partial\mathbf{S}}:d\mathbf{S}=0. \quad (4)$$

Now, taking the derivative of this expression with respect to stress from the left yields:⁴

Theorem 1. *The implicit elastic solid of Defn. 2 obeys*

$$\left(\mathbf{I} \odot \mathbf{I} - \frac{\partial^2\mathfrak{W}(\mathbf{E}, \mathbf{S})}{\partial\mathbf{S}\partial\mathbf{E}} \right):d\mathbf{E} - \frac{\partial^2\mathfrak{W}(\mathbf{E}, \mathbf{S})}{\partial\mathbf{S}\partial\mathbf{S}}:d\mathbf{S}=0 \quad (5)$$

and, therefore, possesses the mathematical structure of a Rajagopal and Srinivasa [26, 27] elastic solid, as stated in Eq. (1).

The inner dyadic product $\odot$ within the fourth-order identity tensor $\mathbf{I} \odot \mathbf{I}$ creates a mixed

tensor field with components $\frac{1}{2}(\delta_i^K \delta_j^L + \delta_i^L \delta_j^K)$ where δ_j^I is the Kronecker delta function with the superscript being the row index and the subscript being the column index.

Considering that $\det(\partial^2\mathfrak{W}/\partial\mathbf{S}\partial\mathbf{S}) \neq 0$ allows one to impose:

Definition 3. A *Rajagopal elastic solid* is any model derived from

$$d\mathbf{S}=\mathbb{M}:d\mathbf{E}=\left(\frac{\partial\mathfrak{W}(\mathbf{E}, \mathbf{S})}{\partial\mathbf{S}\partial\mathbf{S}}\right)^{-1}:\left(\mathbf{I} \odot \mathbf{I} - \frac{\partial\mathfrak{W}(\mathbf{E}, \mathbf{S})}{\partial\mathbf{S}\partial\mathbf{E}}\right):d\mathbf{E} \quad (6)$$

where $\mathbb{M}$ is a fourth-order tensor commonly known as the tangent modulus.

This becomes $dS^{IJ} = M^{IJKL}dE_{KL}$ in component notation. The class of all Rajagopal elastic solids is a subclass to the class of all implicit elastic solids. What distinguishes the tangent modulus of a Rajagopal elastic solid from that of a hyperelastic solid [35, pp. 355-357] is how $\mathbb{M}$ is constructed from its respective elastic strain-energy function; specifically, in hyperelasticity $\mathbb{M}=d^2\mathfrak{W}(\mathbf{E})/d\mathbf{E}d\mathbf{E}$, which should be contrasted with the formula in Defn. 3.

Because stress rate is a homogeneous function of strain rate in a Rajagopal elastic solid, it necessarily follows that:

⁴In the authors' prior paper [15] they constructed a hypo-elastic theory by taking the derivative of Eq. (4) with respect to strain $\mathbf{E}$ from the left.

Corollary 1. For a given initial stress, the final state of stress in a Rajagopal elastic solid depends only on the paths by which the material points reach the final state and not upon the rate at which they traverse these paths.

The proof of this corollary is the same as the proof to a like theorem by Noll [38, pg. 35] that applies to hypo-elastic solids. Although expressed as a rate theory, this corollary states that the response of an implicit elastic solid does not depend upon time. This corollary also implies that each path traveled through state space can have its own elasticity. Alternatively, different paths traversed through state space that reach a common final state may require different amounts of work to accomplish the feat, as observed in the experimental data of Criscione et al. [39, 40].

The three-dimensional theory of a Rajagopal elastic solid has a mathematical structure reminiscent of the one-dimensional phenomenological law proposed by Fung [25, Eq. 22]

$$dT = E (1 + \beta T/E) d\lambda \quad (7)$$

where T is the traction, λ is the stretch, E is Young's modulus, and β is Fung's parameter which is dimensionless. Contrasting Eqs. (6 & 7) suggests:

Definition 4. A *Fung elastic solid* is any Rajagopal elastic solid (Defn. 3) whose mixed derivative $\partial^2 \mathbb{W} / \partial S \partial E$ is first order in stress.

As a consequence, a Fung elastic solid will have a coupled contribution in its overall strain-energy function that is quadratic in stress and linear in strain.

Definition 5. A strain dependence arising from the compliance $\partial^2 \mathbb{W} / \partial S \partial S$ caused by a coupling between stress and strain within a strain-energy function $W = \mathbb{W}(E, S)$ is classified as a *Rajagopal effect*.

Thermodynamics requires a Fung elastic solid to exhibit a Rajagopal effect.

2.2. Dilatoric-Deviatoric Splits

There are two ways that one can construct a finite, deviatoric, strain-rate tensor. The approach used in this paper is presented below, while the alternate approach and its outcomes are sketched out in Appendix A.

A differential change of the deformation gradient $d\mathbf{F}$ can be split into dilatoric and deviatoric parts by defining

$$d\bar{\mathbf{F}} = d\mathbf{F} - \frac{1}{3} d \ln(\Delta) \mathbf{F} \quad \text{where} \quad \text{tr}(\mathbf{F}^{-1} d\bar{\mathbf{F}}) = 0 \quad (8)$$

because $\text{tr}(\mathbf{F}^{-1} d\bar{\mathbf{F}}) = \text{tr}(\mathbf{F}^{-1} d\mathbf{F}) - d \ln \Delta = 0$ from Eq. (2); consequently, $d\bar{\mathbf{F}}$ is a deviatoric measure for a rate of change in the deformation gradient. Recall that $\ln \Delta = \ln(\det \mathbf{F})$ from physics and $d \ln(\det \mathbf{F}) = \text{tr}(\mathbf{F}^{-1} d\mathbf{F})$ from linear algebra.

From the definition for the Green deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ comes $d\mathbf{C} = d\mathbf{F}^T \mathbf{F} + \mathbf{F}^T d\mathbf{F}$ from which one can advance a definition for a deviatoric rate of the Green deformation as

$$d\bar{\mathbf{C}} = d\bar{\mathbf{F}}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot d\bar{\mathbf{F}} = d\mathbf{C} - \frac{2}{3} d \ln(\Delta) \mathbf{C} \quad \text{where} \quad \text{tr}(\mathbf{C}^{-1} d\bar{\mathbf{C}}) = 0 \quad (9)$$

with the second line following from a substitution of Eq. (8) into the first line whose trace yields $\text{tr}\left(C^{-1}d\bar{C}\right)=\text{tr}\left(C^{-1}dC\right)-2d\ln\Delta=\text{tr}\left(F^{-T}dF^T\right)+\text{tr}\left(F^{-1}dF\right)-2d\ln\Delta=0$ from Eq. (2); consequently, $d\bar{C}$ is a deviatoric measure for the rate of deformation.

In accordance with the above definitions one can immediately write

$$d\bar{E}=dE-\frac{1}{3}d\ln(\Delta)C \quad \text{where} \quad \text{tr}\left(C^{-1}d\bar{E}\right)=0 \quad (10)$$

because from $E=\frac{1}{2}(C-I)$ comes $dE=\frac{1}{2}dC$ with the deviatoric strain rate $d\bar{E}=\frac{1}{2}d\bar{C}$ following by analogy. This deviatoric strain rate differs from the one derived from the alternative approach put forward in Appendix A.

The challenge with adopting the above approach is quantifying $\bar{E}$. Suppose that this field is

described by $\bar{E}=E-\frac{1}{3}\ln(\Delta)C+X$ where X is to be determined. Differentiating this formula

leads to $d\bar{E}=dE-\frac{1}{3}d\ln(\Delta)C-\frac{1}{3}\ln(\Delta)dC+dX$ that when contrasted with Eq. (10)

implies that $dX=\frac{1}{3}\ln(\Delta)dC$ whose integration allows us to write

$$\bar{E}=E-\frac{1}{6}\ln(\det C)C+\frac{1}{6}\int_0^t \ln(\det C(t'))dC(t') \quad (11)$$

where it is assumed that $C(t=0)=I$. It follows that $\bar{E}$ is *not* deviatoric because

$\text{tr}\left(C^{-1}\bar{E}\right)\neq 0$, even though $d\bar{E}$ is deviatoric.⁵ Here we use the fact that $\Delta=\det F=\sqrt{\det C}$.

The second Piola-Kirchoff stress S splits into an additive decomposition between dilatoric and deviatoric components in that⁶

$$S=\varphi C^{-1}+\bar{S} \quad \text{where} \quad \varphi=\frac{1}{3}\text{tr}(SC)=\frac{1}{3}\Delta \text{tr}(T) \quad (12)$$

wherein φ is a hydrostatic pressure and $\bar{S}$ is a deviatoric stress. $\bar{S}$ is deviatoric because it is traceless, i.e., $\text{tr}(\bar{S}C)=0$. Stresses S and $\bar{S}$ and pressure φ all have units of force per unit undeformed area.

2.3. Constitutive Equations

With both strain rate and stress decomposing into separate dilatoric and deviatoric parts according to Eqs. (10 & 12) one arrives at:

Corollary 2. For compressible materials, a Rajagopal elastic solid (Defn. 3) described by $dS=M:dE$ has a *tangent modulus with additive constituents*

⁵This is also true of the alternative formulation for deviatoric strain rate presented in Appendix A.

⁶Pressure is usually defined in the mechanics literature as an Eulerian field via the formula $p=-\frac{1}{3}\text{tr}(T)$. Here it is defined as a Lagrangian field and with opposite sign. This was done so that our definition would agree with its clinical interpretation in lung physiology.

$$\mathbb{M} = \mathbb{V} + \mathbb{D} \quad (13)$$

whose volumetric tangent modulus is given by

$$\mathbb{V} = \frac{d\varphi}{d \ln \Delta} \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2\varphi \mathbf{C}^{-1} \odot \mathbf{C}^{-1} \quad (14)$$

with

$$\frac{d\varphi}{d \ln \Delta} = \left(\frac{\partial^2 \mathbb{W}_b(\ln \Delta, \varphi)}{\partial \varphi^2} \right)^{-1} \left(1 - \frac{\partial^2 \mathbb{W}_b(\ln \Delta, \varphi)}{\partial \varphi \partial \ln \Delta} \right), \quad (15)$$

and whose deviatoric tangent modulus is given by

$$\mathbb{D} = \frac{d\bar{\mathbb{S}}}{d\bar{\mathbb{E}}} : \left(\mathbf{I} \odot \mathbf{I} - \frac{1}{3} \mathbf{C} \otimes \mathbf{C}^{-1} \right) \quad (16)$$

with

$$\frac{d\bar{\mathbb{S}}}{d\bar{\mathbb{E}}} = \left(\frac{\partial^2 \mathbb{W}_s(\bar{\mathbb{E}}, \bar{\mathbb{S}})}{\partial \bar{\mathbb{S}} \partial \bar{\mathbb{S}}} \right)^{-1} : \left(\mathbf{I} \odot \mathbf{I} - \frac{\partial^2 \mathbb{W}_s(\bar{\mathbb{E}}, \bar{\mathbb{S}})}{\partial \bar{\mathbb{S}} \partial \bar{\mathbb{S}}} \right)^{-1} \quad (17)$$

wherein $\mathbb{W}_b = \mathbb{W}_b(\ln \Delta, \varphi)$ and $\mathbb{W}_s = \mathbb{W}_s(\bar{\mathbb{E}}, \bar{\mathbb{S}})$ are the independent, bulk and shear strain-energy functions, respectively.

Proof of this corollary is given in Appendix B.

Remark 1. The Lagrangian hydrostatic pressure p is often defined as

$$p = -\frac{1}{3} \text{tr}(\mathbf{S}\mathbf{C}) = -\frac{1}{3} \Delta \text{tr}(\mathbf{T}) \quad \text{so that} \quad \mathbf{S} = -p \mathbf{C}^{-1} + \bar{\mathbb{S}}. \quad (18)$$

Consequently, Eqs. (14 & 15) of Corollary 2 become

$$\mathbb{V} = 2p \mathbf{C}^{-1} \odot \mathbf{C}^{-1} - \frac{dp}{d \ln \Delta} \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \quad (19)$$

and

$$\frac{dp}{d \ln \Delta} = - \left(\frac{\partial^2 \mathbb{W}_b(\ln \Delta, p)}{\partial p^2} \right)^{-1} \left(1 + \frac{\partial^2 \mathbb{W}_b(\ln \Delta, p)}{\partial p \partial \ln \Delta} \right) \quad (20)$$

while the remaining formulæ of Corollary 2 remain intact.

3. Regarding the Reference State

Strain is a two-state field. Strain is a relative measure of the stretch taking place in a current configuration with respect to some reference configuration. Strain is normalized to be zero valued whenever the current configuration coincides with its reference configuration.

In the mechanics of inanimate objects, selecting a reference state for strain is straightforward. A reference state is typically chosen to associate with a stress-free state. Such reference states can be realized in inanimate objects.

In the mechanics of animate beings, and in parts thereof, selecting a meaningful reference state for strain is not straightforward at all. In living entities, stress-free states often have no physiologic interpretation, because constituents of living things seldom know states without stress; typically, they are not realizable short of death. For example, cells are born, live, adapt, and die while residing within a stressed environment. Furthermore, the cells that will replace them will likely be born into a different environment than their progenitors were born into, as their host lives, grows, and ages. The mechanics of these and other issues pertinent to living organisms have been discussed at length by Humphrey and Rajagopal [41].

It is common practice for clinicians to normalize the volume of air within a lung against its total lung capacity (TLC). The transpulmonary pressure needed to inflate a lung to TLC is typically 25-30 cm H₂O in humans. From a mechanics perspective TLC is not a reference state, but to a clinician it is. One's choice of selecting a reference configuration for strain will likely effect the affect of the Rajagopal effect, when present.

4. Models

A model arises whenever a strain-energy function is specified or, as is the case with Corollary 2, whenever a pair of bulk and shear strain-energy functions are specified. A model becomes viable only after it has been exercised and shown to correlate with reality, and after its limitations have been established and are understood.

Physics provides the framework to derive a theory. Experiments provide the framework to create a model. Human intellect provides the judgment.

4.1. Hookean Solid

To illustrate, consider the elastic strain energies of a Hookean solid

$$\mathfrak{W}_b = \frac{1}{2\kappa} p^2 \quad \text{and} \quad \mathfrak{W}_s = \frac{1}{4\mu} \text{tr}(\bar{\mathbf{S}} \mathbf{C} \bar{\mathbf{S}} \mathbf{C}) \quad (21)$$

where κ and μ are the respective bulk and shear moduli. Interpreting pressure in a classical sense, an introduction of these energies into the constitutive structure of Remark 1 and Corollary 2 leads to

$$dp = -\kappa d \ln \Delta \quad \text{and} \quad d\bar{\mathbf{S}} = 2\mu \mathbf{C}^{-1} \odot \mathbf{C}^{-1} : d\mathbf{E} \quad (22)$$

that, after an integration, can be combined with Eq. (18) to establish the total state of stress $\mathbf{S}$. This material's tangent moduli are

$$\mathbb{D} = 2\mu \left(\mathbf{C}^{-1} \odot \mathbf{C}^{-1} - \frac{1}{3} \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \right) \quad (23)$$

and

$$\mathbb{V} = \kappa \left(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2 \ln(\Delta) \mathbf{C}^{-1} \odot \mathbf{C}^{-1} \right) \quad (24)$$

where pressure integrates as $p = -\kappa \ln \Delta$. This is a Hookean solid written in rate form that uses tensor fields appropriate for finite deformation analysis.

4.2. Fungian Solid

In most soft biological tissues, distortions caused by shearing dominate the deformation because $\kappa \gg \mu$, thereby allowing an assumption of material incompressibility to be imposed. That is not the case for the parenchyma of lung. The human lung typically dilates 10-12% during tidal breathing, up to 57-60% during a deep sigh, and up to 76-80% during heavy breathing, according to the data of Table 1. Therefore, any realistic modeling of lung will require the use of finite deformation theory. Consequently, Corollary 2 is an appropriate point of departure for the development of an elastic continuum model for parenchyma.

4.2.1. Dilatoric Response—The inflation experiment discussed in Appendix C mimics normal lung behavior as a first approximation. It is used to establish the bulk tangent modulus $\bar{\kappa}$ defined by $V dP/dV$ [43] where P is the inflation pressure imposed on the lung with V being its associated volume, consequently

$$\bar{\kappa} = V \frac{dP}{dV} \equiv \Delta \frac{d\varphi}{d\Delta} = \frac{d\varphi}{d \ln \Delta} \quad (25)$$

so there is a simple scalar relationship between theory and this experiment.

4.2.2. Deviatoric Response—Considering the shear response to be an isotropic function of its two independent state variables $\bar{\mathbf{E}}$ and $\bar{\mathbf{S}}$, the Cayley-Hamilton theorem allows us to replace these two tensorial arguments with ten scalar arguments [44]

$$\begin{aligned} I_1 &= \text{tr}(\bar{\mathbf{C}}^{-1} \bar{\mathbf{E}}) & I_2 &= \text{tr}(\bar{\mathbf{C}}^{-1} \bar{\mathbf{E}} \bar{\mathbf{C}}^{-1} \bar{\mathbf{E}}) \\ I_3 &= \text{tr}(\bar{\mathbf{C}}^{-1} \bar{\mathbf{E}} \bar{\mathbf{C}}^{-1} \bar{\mathbf{E}} \bar{\mathbf{C}}^{-1} \bar{\mathbf{E}}) & I_4 &= \text{tr}(\bar{\mathbf{S}} \bar{\mathbf{C}}) \\ I_5 &= \text{tr}(\bar{\mathbf{S}} \bar{\mathbf{C}} \bar{\mathbf{S}} \bar{\mathbf{C}}) & I_6 &= \text{tr}(\bar{\mathbf{S}} \bar{\mathbf{C}} \bar{\mathbf{S}} \bar{\mathbf{C}} \bar{\mathbf{S}} \bar{\mathbf{C}}) \\ I_7 &= \text{tr}(\bar{\mathbf{S}} \bar{\mathbf{E}}) & I_8 &= \text{tr}(\bar{\mathbf{C}}^{-1} \bar{\mathbf{E}} \bar{\mathbf{S}} \bar{\mathbf{E}}) \\ I_9 &= \text{tr}(\bar{\mathbf{S}} \bar{\mathbf{E}} \bar{\mathbf{S}} \bar{\mathbf{E}}) & I_{10} &= \text{tr}(\bar{\mathbf{S}} \bar{\mathbf{E}} \bar{\mathbf{S}} \bar{\mathbf{E}} \bar{\mathbf{S}} \bar{\mathbf{E}}) \end{aligned} \quad (26)$$

which are their invariants, therefore, $W_s = \mathcal{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}}) = \mathcal{W}_s(I_1, I_2, \dots, I_{10})$. By definition, $I_4 = \text{tr}(\bar{\mathbf{S}} \bar{\mathbf{C}}) = 0$; however, $I_1 = \text{tr}(\bar{\mathbf{C}}^{-1} \bar{\mathbf{E}}) \neq 0$ in general. In other words, $\bar{\mathbf{S}}$ is deviatoric, but is not (however, $\bar{\mathbf{d}}$ is deviatoric).

Examining the governing constitutive equation (17) allows one to argue that the pure strain invariants cannot be realized. This removes I_1, I_2, I_3 from further consideration. Furthermore, for terms where stress and strain are coupled and, in particular, for the Fung solid per Defn. 4, contributions to the strain energy must be quadratic in stress and linear in strain, which removes I_7, I_8, I_{10} from further consideration. Indeed, of the ten invariants, only I_9 satisfies these requirements. Invariants I_5 and I_6 remain viable for modeling the low-strain behavior; they are hyperelastic in origin. Hence, the elastic strain-energy function that governs the shear response of a Fungian solid can specify, at most, a functional form where $W_s = \mathcal{W}_s(I_5, I_6, I_9)$.

We suppose that the shear behavior of our Fungian solid extends the elastic strain-energy function of a Hookean solid in that

$$\mathfrak{W}_s(I_5, I_6, I_9) = \frac{1}{4\mu} I_5 - \frac{\beta}{4\mu} I_9 \quad (27)$$

where μ is a shear modulus, in the sense of a Hookean solid, while the Fung parameter β is dimensionless. No dependence upon invariant I_6 is considered in our modeling. Placing this strain-energy function into its constitutive equation (17) produces

$$\frac{d \bar{\mathbf{S}}}{d \bar{\mathbf{E}}} = 2\mu \left(\mathbf{C} \odot \mathbf{C} - \beta \frac{1}{2} (\mathbf{C} \odot \bar{\mathbf{E}} + \bar{\mathbf{E}} \odot \mathbf{C}) \right)^{-1} : \left(\mathbf{I} \odot \mathbf{I} + \frac{\beta}{2\mu} \frac{1}{2} (\mathbf{I} \odot \mathbf{C} \bar{\mathbf{S}} + \mathbf{C} \bar{\mathbf{S}} \odot \mathbf{I}) \right) : d \bar{\mathbf{E}}, \quad (28)$$

or, as a gradient, one obtains

$$\frac{d \bar{\mathbf{S}}}{d \bar{\mathbf{E}}} = 2\mu \left(\mathbf{C} \odot \mathbf{C} - \beta \frac{1}{2} (\mathbf{C} \odot \bar{\mathbf{E}} + \bar{\mathbf{E}} \odot \mathbf{C}) \right)^{-1} : \left(\mathbf{I} \odot \mathbf{I} + \frac{\beta}{2\mu} \frac{1}{2} (\mathbf{I} \odot \mathbf{C} \bar{\mathbf{S}} + \mathbf{C} \bar{\mathbf{S}} \odot \mathbf{I}) \right) \quad (29)$$

where the first expression can be used for a direct integration of deviatoric stress, while the second expression is applicable for insertion into the deviatoric tangent modulus $\mathbb{D}$ of

Corollary 2 for use in numerical applications like finite elements. Here $\beta \frac{1}{2} (\mathbf{C} \odot \bar{\mathbf{E}} + \bar{\mathbf{E}} \odot \mathbf{C})$ is the source of a Rajagopal effect. In an absence of this effect, the response will become hypo-elastic [15].

5. Application

In this section, we apply our theory for a Fungian solid to the parenchyma of lung. Our characterization of parenchyma is based on published data for dog lung ([43] and [45]). The goal of these early experiments was to experimentally accomplish the separation of bulk and shear responses and, thus, to separately characterize the bulk and elastic moduli. The former was addressed by means of perturbations in pressure during lung deflation, graphic A in Fig. 1, and will be described in this text. These inflation data were independent of any constitutive assumption. Data acquired for the elastic modulus, in contrast, were dependent not only on a specific hyperelastic model but also on an assumption of linear elasticity. This test consisted of a punch test and will not be described in this manuscript (details can be found in Lai-Fook et al. [43]).

Since we do not wish to be bound by the earlier constitutive assumptions, and we eschew linear elasticity for this problem, we analyse a separate inflation protocol from Lai-Fook et al. [43] by solving its boundary value problem, graphic B in Fig. 1. The results of this analysis show that separate characterization of the elastic modulus may be achieved directly from the inflation test without any constitutive assumptions and without invoking linear elasticity, provided that force and area data are recorded. This analysis is new. Unfortunately, these data were not reported in the manuscripts at the time.

However, the analysis of the boundary value problem does afford us some insight into behavior of our model and, more importantly, enables the characterization of our elastic modulus against the published data (with all its limitations) by means of a somewhat back-door but, otherwise, theoretically sound route. Another consequence of this analysis is that we lay out a path for a more complete future experimental characterization of parenchyma.

5.1. Bulk Response

To measure the bulk tangent modulus from excised lobes of canine lungs Lai-Fook et al. [43] and Lai-Fook [45] carried out a probing protocol that was exercised at various locations

along the pressure-volume ($P V$) curve. After equilibrating each lobe of lung at TLC (a pressure of 25-30 cm-H₂O) a static $P V$ deflation maneuver dropped the transpulmonary pressure P (the pleural pressure was atmospheric in their experiments) down to target pressures of 16, 12, 8, and 4 cm H₂O. Small perturbations in pressure were then imposed such that the effective (or tangent) bulk modulus in the neighbourhood of these target pressures could be determined. These bulk moduli and related data from Lai-Fook et al. [43] are reproduced in Figure 1A and tabulated in Table 2. Model predictions are plotted in Fig. 2 and compared to an independent set of data from Lai-Fook [45].⁷

An examination of the data in Table 2 suggests a simple law of geometric progression: the bulk tangent modulus will double with any fixed increase in pressure, here observed to be 4 cm H₂O. This phenomenology suggests an elastic strain-energy function of the form

$$\mathfrak{B}_b(\ln \Delta, \wp) = \frac{1}{\kappa} \left(\frac{\alpha}{\ln 2} \right)^2 2^{-\wp/\alpha} \quad (30)$$

where κ is a bulk modulus, in the sense of a Hookean solid, and α is the doubling interval in pressure. Both parameters have units of stress, so $\mathfrak{B}_b$ has units of stress, too. Substituting this strain-energy function into its constitutive equation (15) leads to its governing differential equation

$$\frac{d\wp}{d \ln \Delta} = \kappa 2^{\wp/\alpha} \quad (31)$$

that, after a separation of variables, integrates to

$$\ln \Delta = \ln \Delta_{\max} \left(1 - \frac{1}{2^{\wp/\alpha}} \right) \quad \text{where} \quad \ln \Delta_{\max} = \frac{\alpha}{\ln(2) \kappa} \quad (32)$$

with $\ln \Delta_{\max}$ being the dilatation at TLC. Pressure causes dilatation. This particular bulk, elastic, strain-energy function does not require the full implicit capability of our theory because $\mathfrak{B}_b$ only depends upon $\wp$ in Eq. (30).

Formula (31) has been fit to the data in Table 2 via Eq. (25) yielding parameters $\alpha = 4$ cm H₂O and $\kappa = 7$ cm H₂O, whose response is displayed as the curve in Fig. 2. Given these parameters, one determines for dog lung a dilatation at TLC of $\ln \Delta_{\max} = 0.824$ implying a dilatation at TLC of $\Delta_{\max} = 2.28$, therefore, the volume of lung can expand to at most 228% of its unstressed reference configuration—a reference state that is derived, not specified.

The dilations and dilatations predicted by Eq. (32) allow percentages of volume fraction to be estimated via $(\ln \Delta / \ln \Delta_{\max}) 100$, which is contrasted against its experimental equivalent of $(V/V_{TLC}) 100 \times$ listed in Table 3. The predictions of $(\ln \Delta / \ln \Delta_{\max}) \times \times 100$ all lie within two standard deviations from their experimental means over the range of pressures investigated. These predictions provide us with a sense of confidence regarding the validity of our proposed bulk, elastic, material model (30).

5.2. Elastic Modulus: The Lai-Fook Experiments Revisited

Following the static pressure perturbations described above, Lai-Fook et al. [43] subjected the excised lung lobes to two additional tests. One a so-called *uniaxial* test and another a so-

⁷The hypo-elastic theory developed by the authors' in their earlier paper [15] leads to a material model that predicts a linear relationship between P and $\bar{\kappa}$, which was an improvement over classical elasticity where $\bar{\kappa} = k$, a constant. Clearly, the data in Table 2 suggest that an even more flexible nonlinear relationship between P and $\bar{\kappa}$ is required.

called *indentation* or *punch* test. It was the latter that Lai-Fook et al. [43] and Lai-Fook [45] used to characterize the shear response of the lung. A mechanical analysis of the *punch* problem can be found in [46]. Unfortunately, this analysis requires the assumption of a specific constitutive model and the invocation of linear elasticity, both of which we eschew. Instead, we turn to the former *uniaxial* test (Figure 3), which Lai-Fook et al. [43] used to determine extension ratios. We are not interested in extension ratios, but rather we analyse the boundary value problem (cf. Appendix D) that describes this test and show that it can yield an effective (tangent) elastic modulus without any *a priori* constitutive assumptions and, more importantly, without the assumption of linear elasticity.

After equilibrating a whole lobe of lung at TLC, the Lai-Fook [43] *uniaxial* protocol begins

by stabilizing a lobe at a static pressure of inflation ($\varphi = \frac{1}{3} \text{tr}(\mathbf{SC})$) and then, while holding this pressure fixed, one applies an incremental compressive load to the lung along the axis of gravity. This requires a release of some air from the lung in order to maintain a constant pressure. (They accounted for gravity in their analysis.) Of their method for data analysis Lai-Fook et al. [43] said:

“In reducing the loads to stresses and the displacements to strains in the uniaxial loading test, the (canine) lobes have been assumed to be right cylinders with plane-parallel upper and lower surfaces. The left upper lobe and the left and right lower lobes were used because their shapes best approximated the assumed shape. The similar stress-strain curves for different lobes provide some evidence that the errors introduced by making this assumption are small. Nevertheless, it is likely that the most significant source of error in evaluating these data is in assuming that the loads were uniformly supported.”

Similar to the bulk tangent modulus of Eq. (25), the elastic tangent modulus of Young is defined by

$$\bar{E} = L \frac{d(T/A)}{dL} \equiv \lambda \frac{d(T/A)}{d\lambda} \quad (33)$$

where T is the axial force of traction imposed on the lobe whose current cross-sectional area is A , with L being the current distance between the plates of the loading fixture (Figure 3). We define a stretch $\lambda = L/\lambda_v$ in Appendix D, with λ being the total axial stretch and λ_v being the volumetric stretch caused by pressure P , as established in Appendix C. This relative axial stretch λ equates with $\lambda = L/L_0$, where L_0 is a gage length—in other words, at the height of the lobe when inflated at an isotropic pressure of P at its reference state $\{P, \lambda_v\}$. Because $A = A_0 \lambda_w^2$ where A_0 is the cross-sectional area at $P = 0$ and λ_w is the transverse

stretch, which for this boundary value problem is described by $\lambda_w = \frac{1}{2} \lambda_v (3 - \lambda)$ (cf. Appendix D), it follows that

$$L \frac{d(T/A)}{dL} = \frac{4\lambda}{A_0 \lambda_v^2} \frac{d(T/(3 - \lambda)^2)}{d\lambda} \quad (34)$$

which, when the derivative is expanded upon, becomes

$$\bar{E} = \lambda \frac{d(T/A)}{d\lambda} = \frac{4}{(3 - \lambda)^2} \left(\frac{2\lambda}{3 - \lambda} \frac{T}{A_p} + \frac{dT/A_p}{d \ln \lambda} \right) \quad (35)$$

where $A_p = A_0 \lambda_v^2$ is the cross-sectional area associated with the reference state $\{P, \lambda_v\}$, which is constant during any given loading perturbation of a traction T about some fixed pressure P .

The desired tangent modulus resides at the limit $\lambda \rightarrow 1$ where $T \rightarrow 0$ so that

$$\bar{E}|_{\lambda=1} = \left. \frac{dT/A_p}{d \ln \lambda} \right|_{T=0, \lambda=1} \quad (36)$$

which is gotten by extrapolating an experimental response curve acquired via Eq. (35) back to its reference state of $\{P, \lambda_v\}$.

Now that we have a formula that allows us to experimentally quantify the elastic tangent modulus for parenchyma, we seek a solution for the same boundary value problem imposed onto any general elastic solid that admits a tangent modulus M . In components, one has

$$dS^{11} = \sum_{i=1}^3 M^{11ii} dE_{ii} \text{ so that}^8$$

$$\bar{E} = \lambda_\ell^4 \frac{dS^{11}}{dE_{11}} = \lambda_\ell^4 \left(M^{1111} + M^{1122} \frac{dE_{22}}{dE_{11}} + M^{1133} \frac{dE_{33}}{dE_{11}} \right) \quad (37)$$

where λ_ℓ^4 pushes dS^{11}/dE_{11} forward into the current configuration where it can be correctly interpreted as being Young's, elastic, tangent modulus. From an assumption of isotropy, and imposing the kinematics that describe the Lai-Fook experiment (cf. Appendix D), one determines that

$$M^{1122} = M^{1133} \quad \text{and} \quad \frac{dE_{22}}{dE_{11}} = \frac{dE_{33}}{dE_{11}} = -\frac{3-\lambda}{4\lambda} \quad (38)$$

with $dE_{22}/dE_{11}|_{\lambda=1} = -1/2$, thereby simplifying Eq. (37) to

$$\bar{E} = \lambda_\ell^4 \left(M^{1111} - \frac{3-\lambda}{2\lambda} M^{1122} \right) \quad \text{with} \quad \begin{matrix} M^{1111} = V^{1111} + D^{1111} \\ M^{1122} = V^{1122} + D^{1122} \end{matrix}, \quad (39)$$

so that

$$\bar{E}|_{\lambda=1} = \bar{E}_v + \bar{E}_D \quad (40)$$

wherein

$$\bar{E}_v = \lambda_v^4 (V^{1111} - V^{1122})|_{\lambda=1} \quad \text{and} \quad \bar{E}_D = \lambda_v^4 (D^{1111} - D^{1122})|_{T=0, \lambda=1}, \quad (41)$$

The above expressions apply for any elastic material model written in the form of $dS = \mathbb{M}$ as it applies to the Lai-Fook experimental protocol of Appendix D.

5.3. Elastic Modulus of Parenchyma

Taking the kinematic and kinetic results of Appendix D and incorporating them into Corollary 2 while using the strain-energy functions that are pertinent for parenchyma, viz.,

⁸For the case of the finite Hookean solid of Sec. 4.1, Eq. (37) returns the classic result of $E = 9\kappa\mu/(3\kappa + \mu)$ whenever stretch is $\lambda = 1 + \varepsilon$ with ε being of infinitesimal extent.

Eqs. (27 & 30), allows us to quantify the four relevant components found in Eqs. (40 & 41) for the tangent modulus.

Volumetric Contribution. From the volumetric tangent modulus of Eq. (14), using constitutive equation (31), one finds that

$$\lambda_\ell^4 V^{1111} = \kappa 2^{P/\alpha} - 2P \quad \text{and} \quad \lambda_\ell^4 \frac{3-\lambda}{2\lambda} V^{1122} = \kappa \frac{2\lambda}{3-\lambda} 2^{P/\alpha} \quad (42)$$

that when summed result in an $\bar{E}_V$ for Eq. (41) of simply

$$\bar{E}_V = -2P \quad (43)$$

whose source arises from the $-2P C^{-1} \odot C^{-1}$ term in Eq. (14).

Deviatoric Contribution. From the deviatoric tangent modulus of Eq. (16) one finds that

$$\begin{aligned} \lambda_\ell^4 D^{1111} &= \frac{2}{3} \lambda_\ell^4 \left(\frac{dS}{dE_{11}} - \frac{(3-\lambda)^2}{4\lambda^2} \frac{dS}{dE_{22}} \right), \\ \lambda_\ell^4 \frac{3-\lambda}{2\lambda} D^{1122} &= -\frac{1}{3} \lambda_\ell^4 \left(\frac{2\lambda}{3-\lambda} \frac{dS}{dE_{11}} - \frac{3-\lambda}{2\lambda} \frac{dS}{dE_{22}} \right) \end{aligned} \quad (44)$$

so that

$$\bar{E}_D = \lambda_\ell^4 \left(\frac{dS}{dE_{11}} - \frac{dS}{dE_{22}} \right) \Big|_{T=0, \lambda=1}. \quad (45)$$

Solving our constitutive equation (29) for insertion into the above expression, subject to the kinematics and kinetics of Appendix D that govern this experiment, results in an $\bar{E}_D$ for Eq. (41) of

$$\bar{E}_D = 2\mu + \beta P. \quad (46)$$

Combining Eqs. (43 & 46) with Eq. (40) provides the following theoretical result for our material model when determining the elastic response of parenchyma subject to Lai-Fook's experimental protocol

$$\bar{E} \big|_{\lambda=1} = 2\mu + (\beta - 2) P, \quad (47)$$

whose ability to correlate with data is shown in Fig. 4. Note that the result is not independent of pressure but, importantly, it is independent of the tangent bulk modulus.

6. Summary and Discussion

To address the need for accurate and efficient constitutive models of parenchyma to be included in fluid-structure interaction computations, we derived a novel, implicit, elastic theory from thermodynamics. This led to the development of a generic template for implicit strain energy that is an exact analogue to the classic Fung model. One consequence of this analysis was the understanding that thermodynamics requires that a Fung elastic solid exhibits a Rajagopal effect. Hitherto, this effect has been overlooked in the biomechanics literature.

To support this theory, we also proposed a novel definition of Lagrangian strain rate (Section 2.2). Unlike the classic definition of Lagrangian strain rate (discussed in Appendix A), this new definition is separable into volumetric and deviatoric terms, a separation that is both mathematically and physically justified. The proposed approach leads to a linear prediction for describing the experimental data of Fig. 4; whereas, the other more common approach leads to a nonlinear prediction for describing these data, which they are not. For the purpose of modeling lung, the dilatoric/deviatoric splitting of strain rate proposed in Section 2.2 applies.

On the foundation of our new theory, and with our definition of strain rate, we proposed strain-energy functions for modeling the parenchyma of lung. Invariant theory was called upon to assist in this process. Considerations of thermodynamics and the character of a Fung elastic solid enabled us to eliminate all but three of the ten invariants.

The resulting model has four parameters, two for the bulk response and two for the deviatoric response. For canine lung, these parameters were determined against published data to have values of $\mu = 1.5 \text{ cm H}_2\text{O}$, $\kappa = 7 \text{ cm H}_2\text{O}$, $\alpha = 4 \text{ cm H}_2\text{O}$, and $\beta = 3.85$. To illustrate the importance of selecting a definition of strain rate that was appropriate for our model, we also show the deleterious impact that adopting the classical definition of strain rate would have had, given the same data and the same constitutive model (Figure A.1).

In contrast, adopting the definition of strain rate herein, the model was shown to have excellent correspondence with all the data that it was fit against. Specifically, the bulk tangent modulus was correctly predicted to be non-linear (Figure 2), while the elastic tangent model was correctly predicted to be linear. This represents an improvement over the authors' previous hypo-elastic model [15]. For both moduli predictions of the present model were well within the experimental limits of reported error ($\pm 1 \text{ SD}$). For validation, predictions of lung volume were compared against independent data and shown to lie within error tolerances of $\pm 2 \text{ SD}$.

Appendix A. Alternative Deviatoric Strain-Rate Formulation

A set of isochoric tensors for the deformation gradient $\tilde{\mathbf{F}}$, the Green deformation $\tilde{\mathbf{C}}$, and the Green strain $\tilde{\mathbf{E}}$ can be defined by (cf. e.g., Simo and Hughes [35, pg. 358])

$$\tilde{\mathbf{F}} = \Delta^{-1/3} \mathbf{F}, \quad \tilde{\mathbf{C}} = \tilde{\mathbf{F}}^T \tilde{\mathbf{F}}, \quad \text{and} \quad \tilde{\mathbf{E}} = \frac{1}{2} (\tilde{\mathbf{C}} - \mathbf{I}).$$

Such deformations are isochoric because $\det \tilde{\mathbf{F}} = 1$. Within the constructs of these definitions, strain can be decomposed into an isotropic-isochoric split according to

$$\mathbf{E} = e\mathbf{C} + \tilde{\mathbf{E}} \quad \text{wherein} \quad e = \frac{1}{2} (1 - \Delta^{-2/3})$$

where e is a dilatoric strain, defined to satisfy the trace of the above strain decomposition (note that $\text{tr}(\mathbf{C}^{-1}\tilde{\mathbf{E}}) \neq 0$, i.e., $\tilde{\mathbf{E}}$ is not deviatoric). Taking the derivative of the above definition for $\tilde{\mathbf{E}}$ produces a dilatoric-deviatoric split in the strain rate of the form

$$d\mathbf{E} = \frac{1}{3} d \ln(\Delta) \mathbf{C} + \Delta^{2/3} d\tilde{\mathbf{E}}$$

where $\Delta^{2/3} d\tilde{\mathbf{E}}$ is this formulation's deviatoric strain rate, not $d\tilde{\mathbf{E}}$. This strain rate is deviatoric (even though $\tilde{\mathbf{E}}$ is not) because it is traceless, i.e., $\Delta^{2/3} \text{tr}(\mathbf{C}^{-1} d\tilde{\mathbf{E}}) = 0$ follows because $\text{tr}(\mathbf{C}^{-1} d\mathbf{E}) = \text{tr}(\mathbf{F}^{-1} d\mathbf{F}) = d \ln \Delta$ from Eq. (2). The advantage of this approach is that $\tilde{\mathbf{E}}$ is easy to compute.

Adopting the above construction for deviatoric strain-rate allows one to express the rate at which work is being done on a material element as

$$dW = \text{tr}(\mathbf{S} \cdot d\mathbf{E}) = \Delta^{2/3} \text{tr}(\bar{\mathbf{S}} \cdot d\tilde{\mathbf{E}}) + \wp d \ln \Delta$$

which is distinct from the expression derived in later in Appendix B. There is a subtle coupling between these distortional $\Delta^{2/3} \text{tr}(\bar{\mathbf{S}} \cdot d\tilde{\mathbf{E}})$ and dilational $\wp d \ln \Delta$ strain energies with respect to dilation Δ . These two strain energies are not independent of one another. This coupling manifests itself in this theory's deviatoric tangent modulus

$$\mathbb{D} = \Delta^{-2/3} \frac{d\bar{\mathbf{S}}}{d\tilde{\mathbf{E}}} : \left(\mathbf{I} \odot \mathbf{I} - \frac{1}{3} \mathbf{C} \otimes \mathbf{C}^{-1} \right)$$

which continues to be felt in its constitutive response

$$\Delta^{-2/3} \frac{d\bar{\mathbf{S}}}{d\tilde{\mathbf{E}}} = \left(\frac{\partial^2 \mathfrak{W}_s(\tilde{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{S}} \partial \bar{\mathbf{S}}} \right)^{-1} : \left(\mathbf{I} \odot \mathbf{I} - \Delta^{-2/3} \frac{\partial^2 \mathfrak{W}_s(\tilde{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{S}} \partial \tilde{\mathbf{E}}} \right)$$

which are the equivalents to Eqs. (16 & 17) in Corollary 2 wherein, unlike the formulæ above, the deviatoric and dilatoric strain energies are completely independent of one another.

From these two alternative formulæ one is lead to

$$\bar{E}|_{\lambda=1} = 2\mu + \beta \frac{P}{\lambda_v^2} - 2P \quad \text{where} \quad \lambda_v = \exp \left(\frac{\ln \Delta_{\max}}{3} \left(1 - \frac{1}{2^{\wp/\alpha}} \right) \right)$$

instead of Eq. (47) for the predicted shear response of parenchyma, the outcome of which is illustrated in Fig. A.1. Here the same value for μ has been adopted as in Fig. 4, with β then being selected to correlate its formula for $\bar{E}$ with the experimentally observed value at pressure $P = 16 \text{ cm H}_2\text{O}$. The nonlinear curve predicted by using the deviatoric strain-rate of this appendix does not match the linear response suggested by the data.

In this regard, we have relied on experimental data to down-select which deviatoric strain-rate construction to adopt in our theory.

Appendix B. Proof of Corollary 2

Given that $dW = \text{tr}(\mathbf{S} \cdot d\mathbf{E})$, we begin the proof by decomposing the work being done into two separate constituents

$$\begin{aligned} dW &= dW_b + dW_s, \\ dW_b &= \varphi d \ln \Delta, \\ dW_s &= \text{tr}(\bar{\mathbf{S}} d \bar{\mathbf{E}}). \end{aligned}$$

Placing $d\mathbf{E}$ from Eq. (10) into dW gives $dW = \frac{1}{3} \text{tr}(\mathbf{S}\mathbf{C}) d \ln \Delta + \text{tr}(\mathbf{S} d \tilde{\mathbf{E}})$. Then substituting $\mathbf{S}$ from Eq. (12) into this result gives $dW = \varphi d \ln \Delta + \varphi \text{tr}(\mathbf{C}^{-1} d \tilde{\mathbf{E}}) + \text{tr}(\bar{\mathbf{S}} d \bar{\mathbf{E}}) = 0$. The first term on the right-hand side is dW_b ; the second term does not contribute because $\text{tr}(\mathbf{C}^{-1} d \tilde{\mathbf{E}}) = 0$ from Eq. (10); while the third term is dW_s . These strain energies are independent of each other. This allows the Clausius-Planck inequality (3) to be rewritten as two separate contributions

$$\begin{aligned} \mathfrak{D} &= \mathfrak{D}_b + \mathfrak{D}_s, \quad \mathfrak{D} \geq 0, \\ \mathfrak{D}_b &= \varphi d \ln \Delta - dW_b, \\ \mathfrak{D}_s &= \text{tr}(\bar{\mathbf{S}} d \bar{\mathbf{E}}) - dW_s \end{aligned}$$

with an elastic response requiring that $\mathfrak{D} = 0$ from Defn. 1 for which a sufficient condition is that $\mathfrak{D}_b = \mathfrak{D}_s = 0$. Expressing the bulk response W_b as an implicit function of its state, viz., $W_b = \mathfrak{W}_b(\ln \Delta, \varphi)$, and, similarly, expressing the shear response W_s as an implicit function of its state, viz., $W_s = \mathfrak{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}})$, allows the differentials dW_b and dW_s appearing in the above dissipation equations to be expanded by the chain rule so that

$$\begin{aligned} \mathfrak{D}_b &= \left(\varphi - \frac{\partial \mathfrak{W}_b(\ln \Delta, \varphi)}{\partial \ln \Delta} \right) d \ln \Delta - \frac{\partial \mathfrak{W}_b(\ln \Delta, \varphi)}{\partial \varphi} d\varphi = 0, \\ \mathfrak{D}_s &= \left(\bar{\mathbf{S}} - \frac{\partial \mathfrak{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{E}}} \right) : d \bar{\mathbf{E}} - \frac{\partial \mathfrak{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{S}}} : d \bar{\mathbf{S}} = 0. \end{aligned}$$

Now, differentiating both expressions by their thermodynamic force from the left leads to constitutive expressions of the Rajagopal and Srinivasa [26, 27] type specified in Eq. (1); in particular,

$$\begin{aligned} \left(1 - \frac{\partial^2 \mathfrak{W}_b(\ln \Delta, \varphi)}{\partial \varphi \partial \ln \Delta} \right) d \ln \Delta - \frac{\partial^2 \mathfrak{W}_b(\ln \Delta, \varphi)}{\partial \varphi^2} d\varphi &= 0, \\ \left(\mathbf{I} \odot \mathbf{I} - \frac{\partial^2 \mathfrak{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{S}} \partial \bar{\mathbf{E}}} \right) : d \bar{\mathbf{E}} - \frac{\partial^2 \mathfrak{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{S}} \partial \bar{\mathbf{S}}} : d \bar{\mathbf{S}} &= 0 \end{aligned}$$

where the fourth-order identity tensor $\mathbf{I} \odot \mathbf{I}$ has components $\frac{1}{2}(\delta_i^K \delta_j^L + \delta_i^L \delta_j^K)$ with $\odot$ being the inner dyadic operator. Assuming invertibility of their respective compliances, the above expressions can be rewritten as

$$\begin{aligned} dp &= \left(\frac{\partial^2 \mathfrak{W}_b(\ln \Delta, \varphi)}{\partial \varphi^2} \right)^{-1} \left(1 - \frac{\partial^2 \mathfrak{W}_b(\ln \Delta, \varphi)}{\partial \varphi \partial \ln \Delta} \right) d \ln \Delta \\ d \bar{\mathbf{S}} &= \left(\frac{\partial^2 \mathfrak{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{S}} \partial \bar{\mathbf{S}}} \right)^{-1} : \left(\mathbf{I} \odot \mathbf{I} - \frac{\partial^2 \mathfrak{W}_s(\bar{\mathbf{E}}, \bar{\mathbf{S}})}{\partial \bar{\mathbf{S}} \partial \bar{\mathbf{E}}} \right) : d \bar{\mathbf{E}} . \end{aligned}$$

When rearranged as gradients these expressions become Eqs. (15 & 17).

To derive the remaining two formulæ, Eqs. (14 & 16), one differentiates Eq. (12) with respect to strain arriving at

$$\frac{d\mathbf{S}}{d\mathbf{E}} = \mathbf{C}^{-1} \otimes \frac{d\varphi}{d\mathbf{E}} + \varphi \frac{d\mathbf{C}^{-1}}{d\mathbf{E}} + \frac{d\bar{\mathbf{S}}}{d\mathbf{E}}$$

from which one defines

$$\mathbb{M} = \frac{d\mathbf{S}}{d\mathbf{E}}, \quad \mathbb{V} = \mathbf{C}^{-1} \otimes \frac{d\varphi}{d\mathbf{E}} + \varphi \frac{d\mathbf{C}^{-1}}{d\mathbf{E}}, \quad \text{and} \quad \mathbb{D} = \frac{d\bar{\mathbf{S}}}{d\mathbf{E}}$$

with, e.g., $d\mathbf{S}/d\mathbf{E}$ and $\mathbf{C}^{-1} \otimes d\varphi/d\mathbf{E}$ having components of dS^I/dE_{KL} and $C^{IJ}d\varphi/dE_{KL}$, respectively, where $\otimes$ is the outer dyadic operator. The above formulæ establish Eq. (13).

An application of the chain rule allows construction of the volumetric tangent modulus $\mathbb{V}$ by expanding

$$\frac{d\varphi}{d\mathbf{E}} = \frac{d\varphi}{d \ln \Delta} \frac{d \ln \Delta}{d\Delta} \frac{d\Delta}{d\mathbf{C}} : \frac{d\mathbf{C}}{d\mathbf{E}}$$

where $d\varphi/d \ln \Delta$ comes from the strain-energy function via Eq. (15) while, from Eq. (2) and from $\det \mathbf{F} = \sqrt{\det \mathbf{C}}$ and $d(\det \mathbf{M}/d\mathbf{M}) = \det(\mathbf{M}) \mathbf{M}^{-T}$ for any invertible tensor $\mathbf{M}$, one determines that

$$\frac{d\Delta}{d\mathbf{C}} = \frac{d \sqrt{\det \mathbf{C}}}{d\mathbf{C}} = \frac{1}{2} \sqrt{\det \mathbf{C}} \quad \mathbf{C}^{-1} = \frac{1}{2} \Delta \mathbf{C}^{-1}$$

while, from the definition for Green strain $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, one gets

$$\frac{d\mathbf{C}}{d\mathbf{E}} = 2\mathbf{I} \odot \mathbf{I}.$$

Collecting terms, recalling that $d \ln \Delta / d\Delta = 1/\Delta$, we find that

$$\frac{d\varphi}{d\mathbf{E}} = \frac{d\varphi}{d \ln \Delta} \mathbf{C}^{-1}.$$

Another expansion via the chain rule leads to

$$\frac{d\mathbf{C}^{-1}}{d\mathbf{E}} = \frac{d\mathbf{C}^{-1}}{d\mathbf{C}} : \frac{d\mathbf{C}}{d\mathbf{E}}$$

wherein

$$\frac{d\mathbf{C}^{-1}}{d\mathbf{C}} = -\mathbf{C}^{-1} \odot \mathbf{C}^{-1}$$

follows from differentiating the identity $\mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$ where $\mathbf{C}^{-1}$ $\mathbf{C}^{-1}$ which has components $\frac{1}{2}(C^{IK}C^{JL} + C^{IL}C^{JK})$ and, therefore,

$$\frac{d\mathbf{C}^{-1}}{d\mathbf{E}} = -2\mathbf{C}^{-1} \odot \mathbf{C}^{-1}.$$

Substituting the above two products into the prior formula defining the volumetric tangent modulus leads to Eq. (14).

Another application of the chain rule also allows construction of the deviatoric tangent modulus $\mathbb{D}$ by expanding

$$\frac{d\bar{\mathbf{S}}}{d\mathbf{E}} = \frac{d\bar{\mathbf{S}}}{d\bar{\mathbf{E}}} : \frac{d\bar{\mathbf{E}}}{d\mathbf{C}} : \frac{d\mathbf{C}}{d\mathbf{E}}$$

where $d\bar{\mathbf{S}}/d\bar{\mathbf{E}}$ comes from the strain-energy function via Eq. (17) while, from differentiating Eq. (11) with respect to the deformation $\mathbf{C}$, one gets

$$\frac{d\bar{\mathbf{E}}}{d\mathbf{C}} = \frac{d\mathbf{E}}{d\mathbf{C}} - \frac{1}{6}\mathbf{C} \otimes \mathbf{C}^{-1} - \frac{1}{6}\ln(\det \mathbf{C}) \frac{d\mathbf{C}}{d\mathbf{C}} + \frac{1}{6}\ln(\det \mathbf{C}) \frac{d\mathbf{C}}{d\mathbf{C}}$$

where $d \ln(\det \mathbf{C})/d\mathbf{C} = \mathbf{C}^{-1}$. Because

$$\frac{d\mathbf{E}}{d\mathbf{C}} = \frac{1}{2}\mathbf{I} \odot \mathbf{I} \quad \text{and} \quad \frac{d\mathbf{C}}{d\mathbf{E}} = 2\mathbf{I} \odot \mathbf{I},$$

one finally arrives at

$$\frac{d\bar{\mathbf{E}}}{d\mathbf{E}} = \mathbf{I} \odot \mathbf{I} - \frac{1}{3}\mathbf{C} \otimes \mathbf{C}^{-1}$$

where $\mathbf{C} \otimes \mathbf{C}^{-1}$ has components $C^{KL}C^{IJ}$. Substituting this result into the formula defining the deviatoric tangent modulus leads to Eq. (16). This completes the proof of Corollary 2.

Figure A.1.

Elastic tangent modulus of Young for canine lung. Data are from Lai-Fook et al. [43] and are tabulated in Table 2. A representation of these data using the alternative model presented in Appendix A is drawn using parameters $\mu = 1.5 \text{ cm H}_2\text{O}$ and $\beta = 6.6$.

Appendix C. Inflation

The purpose of this appendix is to associate the theoretical parameters in Eq. (25) with experimentally measurable values.

The deformation gradient $\mathbf{F}$ of an inflation has components F_j^i of

$$F_j^i = \begin{bmatrix} \lambda_v & 0 & 0 \\ 0 & \lambda_v & 0 \\ 0 & 0 & \lambda_v \end{bmatrix} \text{ so } C_{IJ} = \begin{bmatrix} \lambda_v^2 & 0 & 0 \\ 0 & \lambda_v^2 & 0 \\ 0 & 0 & \lambda_v^2 \end{bmatrix}$$

where λ_v is the volumetric stretch and $C_{IJ} = F_I^i \delta_{ij} F_J^j$ is the deformation tensor of Green, wherein δ_{ij} is the metric of Cartesian space—the Kronecker delta. From this definition one determines the dilation Δ , dilatation $\ln \Delta$, and its derivative $d \ln \Delta$ to have values of

$$\Delta = \det \mathbf{F} = \lambda_v^3, \quad \ln \Delta = 3 \ln \lambda_v, \quad \text{and} \quad d \ln \Delta = 3 d \lambda_v / \lambda_v$$

that when averaged over the whole volume V of a lung, normalized against some reference volume V_0 , become

$$\Delta = V/V_0, \quad \ln \Delta = \ln(V/V_0), \quad \text{and} \quad d \ln \Delta = dV/V.$$

These formulæ establish the kinematics of inflation.

During inflation, the first Piola-Kirchhoff stress $\mathbf{P} = \mathbf{F}\mathbf{S}$ has components

$$\mathbf{P}^{IJ} = \begin{bmatrix} P/\lambda_v & 0 & 0 \\ 0 & P/\lambda_v & 0 \\ 0 & 0 & P/\lambda_v \end{bmatrix} \text{ so } \mathbf{S}^{IJ} = \mathbf{F}_k^I \mathbf{P}^{kJ} = \begin{bmatrix} P/\lambda_v^2 & 0 & 0 \\ 0 & P/\lambda_v^2 & 0 \\ 0 & 0 & P/\lambda_v^2 \end{bmatrix}$$

where P is the transpulmonary pressure. The inverse of the deformation gradient $\mathbf{F}^{-1}$ has components denoted by $\mathbf{F}_j^I$ (recall that $\mathbf{F}$ has components F_j^I). From Eq. (12) comes

$$\varphi = \frac{1}{3} \text{tr}(\mathbf{S}\mathbf{C}) = P \quad \text{and} \quad \bar{\mathbf{S}}^{IJ} = \mathbf{S}^{IJ} - \varphi \mathbf{C}^{IJ} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where the hydrostatic pressure φ equates with the transpulmonary pressure P used by physicians.⁹ The Cauchy deformation tensor $\mathbf{C}^{-1}$ has components C^{IJ} (recall that $\mathbf{C}$ has components C_{IJ}). These are expected results. This completes the demonstration of Eq. (25).

Appendix D. Inflation Followed by Compression

The purpose of this appendix is to provide the fundamental kinematic and kinetic fields needed to describe the Lai-Fook compression experiment.

The Lai-Fook et al. [43] experimental protocol of a compressive axial perturbation being superimposed onto an isotropic dilatation for the purpose of quantifying the shear and/or elastic moduli is described by a deformation gradient with components

$$\mathbf{F}_j^i = \begin{bmatrix} \lambda_\ell & 0 & 0 \\ 0 & \lambda_w & 0 \\ 0 & 0 & \lambda_w \end{bmatrix} \text{ so } \mathbf{C}_{IJ} = \begin{bmatrix} \lambda_\ell^2 & 0 & 0 \\ 0 & \lambda_w^2 & 0 \\ 0 & 0 & \lambda_w^2 \end{bmatrix}$$

where λ_ℓ and λ_w are the respective longitudinal and transverse (width) stretches.

Since the transpulmonary pressure P is held fixed during these uniaxial perturbations, the first Piola-Kirchhoff stress $\mathbf{P}$ will have components of

$$\mathbf{P}^{IJ} = \begin{bmatrix} (P - T)/\lambda_v & 0 & 0 \\ 0 & (P - cT)/\lambda_v & 0 \\ 0 & 0 & (P + cT)/\lambda_v \end{bmatrix}$$

wherein $\{P, \lambda_v\}$ serves as a kind of reference state, cf. Appendix C, onto which an incremental compressive traction T is superimposed (T is assigned here so as to be positive valued), and where c is a parameter to be determined. Therefore, the second Piola-Kirchhoff stress $\mathbf{S}$ has components of

⁹This is why we defined pressure as the Lagrangian field $\varphi = +\frac{1}{3} \text{tr}(\mathbf{S}\mathbf{C}) = P$ instead of how it is normally defined, viz., the Eulerian field $p = -\frac{1}{3} \text{tr}(\mathbf{T}) = -\frac{1}{3} \Delta^{-1} \text{tr}(\mathbf{S}\mathbf{C}) = -P/\lambda_v^3$. This follows because tissue pressure φ is being interpreted as the gauge pressure P of the air within a lung, which must have opposing signs from Newton's third law, i.e., the tissue is in tension because the air is in compression.

$$S^{IJ} = F_k^I P^{kJ} = \begin{bmatrix} (P - T) / \lambda_\ell \lambda_v & 0 & 0 \\ 0 & (P - cT) / \lambda_w \lambda_v & 0 \\ 0 & 0 & (P + cT) / \lambda_w \lambda_v \end{bmatrix}$$

whose state of hydrostatic pressure is quantified by

$$\varphi = \frac{1}{3} \text{tr}(SC) = \frac{1}{3} \lambda_\ell (P - T) / \lambda_v + \frac{2}{3} \lambda_w (P + cT) / \lambda_v$$

which is distinct from the imposed transpulmonary pressure P .

Lai-Fook's testing protocol explores compressive perturbations from intermediate states of reference, viz., at specified transpulmonary pressures, holding these pressures constant throughout each compressive excursion. Therefore, equating the above pressure φ with the isotropic pressure of inflation P , which is fixed over each loading perturbation, provides a constraint of

$$P \equiv \varphi = \frac{1}{3} \lambda_\ell (P - T) / \lambda_v + \frac{2}{3} \lambda_w (P + cT) / \lambda_v$$

that upon equating like terms leads to

$$\lambda_w = \frac{1}{2} \lambda_v (3 - \lambda) \quad \text{and} \quad c = \frac{\lambda}{3 - \lambda} \quad \text{so that} \quad \frac{\lambda_\ell}{\lambda_w} = \frac{2\lambda}{3 - \lambda}$$

leaving λ and λ_v as the sole kinematic variables required for analysis, whereby

$$\lambda = \lambda_\ell / \lambda_v$$

is an axial stretch whose reference state associates with the intermediate isotropic configuration of $\{P, \lambda_v\}$, which is intuitive from the viewpoint of an experimentalist. Stretches λ and λ_v are referenced against different states.

The results of this appendix are independent of constitutive expression.

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Figure 1.

Reproduction (with permission) of the raw data from Lai-Fook et al. [43] used in this manuscript for material characterization: **A)** perturbations in pressure at target pressures along the static deflation curve of dog lung; **B)** load-extension data corresponding to the experimental setup illustrated in Figure 3 and analysed in the boundary value problem of Section 5.2 and Appendix D

Figure 2.

Bulk tangent modulus $\bar{\kappa}$ for canine lung. Data are from Lai-Fook et al. [43] and Lai-Fook [45] and are tabulated in Table 2. Correlations are from Eqs. (25 & 31) where $\alpha = 4$ cm H₂O and $\kappa = 7$ cm H₂O.

Figure 3.

Schematic of the *uniaxial* experimental setup of Lai-Fook et al. [43]. At target static pressures, an incremental load, T , is applied to the lung, resulting in a reduction in lung

volume required to hold the pressure constant. Lai-Fook et al. [43] used the extension $\left(\frac{L}{L_0}\right)$ and load data from this experiment to determine extension ratios. In this manuscript, we analyse the boundary value problem associated with this experiment and show that it can instead be made to yield the elastic modulus, without the limitations of an indentation test such as was used in Lai-Fook et al. [43] and Lai-Fook [45].

Figure 4.

Elastic tangent modulus of Young for canine lung. Data are from Lai-Fook et al. [43] and are tabulated in Table 2. A correlation of these data to the model of Eq. (47) are drawn for $\mu = 1.5$ cm H₂O and $\beta = 3.85$.

Table 1

These are averaged values for men and women reported by Levitzky [42]. TLC is the total lung capacity, typically at a pressure of 25-30 cm H₂O. RLV is the volume in lungs after maximum expiration. FVC is the maximum volume expired after maximum inspiration. TV is the tidal volume. FRC is the volume in lungs after tidal expiration. IC is the maximum volume inspired following tidal expiration. ERV is the maximum expiration at the end of tidal expiration. IRV is the maximum inspiration at the end of tidal inspiration.

Volumetric Property	Men (mL)	Women (mL)	Men % TLC	Women % TLC
TLC	6000	4200	100	100
RLV	1200	1000	20	24
FVC	4800	3200	80	76
TV	600	500	10	12
FRC	2400	1800	40	43
IC	3600	2400	60	57
ERV	1200	800	20	19
IRV	3000	1900	50	45

Table 2

These experimental data were digitized from figures published in Lai-Fook et al. [43, Figs. 6 & 7] and Lai-Fook [45, Fig. 2] for the transpulmonary pressure P , volume V , and bulk and elastic tangent moduli $\bar{\kappa}$ and $\bar{\kappa}'$ of canine lung, where $\bar{\kappa}' = V dP/dV$ while $\bar{\kappa}$ was established using an elastic solution for the indentation of a punch on a half-space.

Pressure P (cm-H ₂ O)	Volume V (% TLC)	Bulk Modulus $\bar{\kappa}$ (cm-H ₂ O)	Elastic Modulus (cm-H ₂ O)
4	50 ± 3	15 ± 5	10.8 ± 1.5
8	70 ± 2.5	28 ± 7	17.7 ± 2
12	84 ± 2	56 ± 15	24.5 ± 3.5
16	90 ± 2	107 ± 12	33.6 ± 4.5

Table 3

The experimental data in columns 1 & 2 are from Table 2. The predictions in the remaining two columns are from Eq. (32) using parameters $\alpha = 4$ cm H₂O and $\kappa = 7$ cmH₂O so that $\ln\Delta_{\max} = 0.824$ and $\Delta_{\max} = 2.28$.

Pressure P (cm-H ₂ O)	Volume $\frac{V}{V_{TLC}} \times 100$	Dilatation $\ln\Delta$	Prediction $\frac{\ln\Delta}{\ln\Delta_{\max}} \times 100$
4	50 ± 3	0.412	50
8	70 ± 2.5	0.613	74
12	84 ± 2	0.721	87
16	90 ± 2	0.773	94