Conserved densities

(Hamiltonian systems/constants of motion/linearized equations)

K. M. CASE

The Rockefeller University, New York, New York 10021

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ABSTRACT Conserved densities for Hamiltonian systems are shown to satisfy simple linear equations. These are essentially the original equations linearized around a particular solution. A number of examples are displayed, which pertain to many systems of physical interest.

I (1, 2) have shown that if an evolution equation can be written in Hamiltonian form then from any constant of motion one can find a solution of the evolution equation linearized around any particular solution of the full equation. Here the argument is turned around. We construct the linear equation that conserved densities must satisfy. [The point is that this linear equation can be very useful in finding general properties of the full evolution equation (3)].

Consider nonlinear evolution equations of the form

$$\frac{\partial u}{\partial t} = K(u). \quad [1]$$

Suppose that between any two functionals (F_1, F_2) of u we can associate a symplectic form which we will call the Poisson bracket and denote as $[F_1, F_2]$. This form is to be antisymmetric and satisfy the Jacobi identity. Further suppose that there is a functional $H[u]$ such that Eq. 1 can be written in the form

$$u_t = [u, H]. \quad [2]$$

Then we say the equation is in Hamiltonian form. Our theorem states that if F is a conserved functional,

$$\text{i.e., } \frac{dF}{dt} = 0, \quad [3]$$

then

$$\delta u = [u, F] \quad [4]$$

is a solution of Eq. 2 linearized around a fixed solution u . The result can be expressed as a linear equation for conserved densities. By this we mean the following:

Let F satisfy Eq. 3; then the "conserved density" ρ is given by

$$\rho = \frac{\delta F}{\delta u}. \quad [5]$$

The simplicity of application is best seen from some examples.

Generalized Korteweg-de Vries equation

Consider equations of the form

$$u_t = -\partial_x \{u^{m+1} + G(\partial_x u)\}, \quad [6]$$

where

$$G(\phi) = \int_{-\infty}^{\infty} G(x' - x)\phi(x')dx' \quad [7]$$

and we require $G(x) = -G(-x)$. (Eq. 6 includes, among other equations the Korteweg-de Vries, modified Korteweg-de Vries, and Benjamin-Ono equations.) We choose the Poisson bracket to be

$$[F_1, F_2] = \int_{-\infty}^{\infty} \frac{\delta F_1}{\delta u} \partial_x \frac{\delta F_2}{\delta u} dx. \quad [8]$$

Then Eq. 6 is of the form of Eq. 2 when we take

$$H = - \int_{-\infty}^{\infty} \left\{ \frac{u^{m+2}}{m+2} + \frac{1}{2} u_x \hat{G}(u_x) \right\} dx, \quad [9]$$

where $\partial/\partial x \hat{G}(x) = G(x)$.

Let F be a constant of motion for Eq. 6 and ρ be given by Eq. 5. By our theorem $\partial_x \rho$ satisfies the linearized form of Eq. 6—i.e., the equation

$$(\delta u)_t = -\partial_x \{(m+1)u^m \delta u + G(\partial_x \delta u)\}. \quad [10]$$

Inserting $\partial_x \rho$ for δu gives

$$\partial_x \partial_t \rho = -\partial_x \{(m+1)u^m \partial_x \rho + G(\partial_x^2 \rho)\}. \quad [11]$$

Assuming u vanishes sufficiently rapidly at infinity, we can integrate this with respect to x , obtaining

$$\partial_t \rho = -\{(m+1)u^m \partial_x \rho + G(\partial_x^2 \rho)\}. \quad [12]$$

This is a linear equation that all conserved densities corresponding to Eq. 6 satisfy.

Generalized Boussinesq equation

Consider the equation

$$\Phi_{tt} - \Phi_{xx} = \left(\frac{\Phi^2}{2} \right)_{xx} + G(\Phi_{xxx}), \quad [13]$$

where G is an integral operator of the same form as above.

It is convenient to transform this. Let $\phi^{(1)} \equiv \Phi$ and introduce a new function $\phi^{(2)}$ by

$$\phi_t^{(1)} = -\phi_x^{(2)}. \quad [14]$$

Then Eq. 13 is obtained if

$$\phi_t^{(2)} = -\partial_x \left[\phi^{(1)} + \frac{(\phi^{(2)})^2}{2} + G(\phi_x^{(1)}) \right]. \quad [15]$$

Eqs. 14 and 15 are seen to be of Hamiltonian form when we take the Poisson bracket to be given as

$$\begin{aligned} [F_1(\phi^{(1)}, \phi^{(2)}), F_2(\phi^{(1)}, \phi^{(2)})] \\ = \int_{-\infty}^{\infty} \left\{ \frac{\delta F_1}{\delta \phi^{(1)}} \partial_x \frac{\delta F_2}{\delta \phi^{(2)}} + \frac{\delta F_1}{\delta \phi^{(2)}} \partial_x \frac{\delta F_2}{\delta \phi^{(1)}} \right\} dx, \end{aligned} \quad [16]$$

and choose

$$H = -\frac{1}{2} \int_{-\infty}^{\infty} \left\{ \frac{(\phi^{(1)})^3}{3} - \phi_x^{(1)} G \phi^{(1)} + (\phi^{(1)})^2 + (\phi^{(2)})^2 \right\} dx. \quad [17]$$

From the theorem we know that if F is a constant of motion

$$\delta\phi^{(1)} = [\phi^{(1)}, F] = \partial_x \frac{\delta F}{\delta \phi^{(2)}} \equiv \partial_x \rho^{(2)}$$

and

$$\delta\phi^{(2)} = [\phi^{(2)}, F] = \partial_x \frac{\delta F}{\delta \phi^{(1)}} \equiv \partial_x \rho^{(1)} \quad [18]$$

satisfy the linearized form of Eqs. 14 and 15. Thus

$$\partial_x \partial_t \rho^{(2)} = -\partial_x^2 \rho^{(1)}, \quad [19]$$

and $\partial_x \partial_t \rho^{(1)} = -\partial_x \{ \partial_x \rho^{(2)} + \phi^{(1)} \partial_x \rho^{(2)} + G(\partial_x^2 \rho^{(2)}) \}$. Integrating with respect to x yields the linear equations for the conserved density $(\rho^{(1)}, \rho^{(2)})$:

$$\rho_t^{(1)} = -\{ \partial_x \rho^{(2)} + \phi^{(1)} \rho^{(2)} + G(\partial_x^2 \rho^{(2)}) \} \quad [20]$$

$$\text{and } \rho_t^{(2)} = -\partial_x \rho^{(1)}.$$

An "improved" generalized Boussinesq equation

To eliminate some unpleasant features of the Boussinesq equation, it is sometimes (4) replaced by an "improved" equation. A generalization of this is:

$$\Phi_{tt} - \Phi_{xx} - G(\Phi_{ttt}) = \left(\frac{\Phi^2}{2} \right)_{xx} \quad [21]$$

It is again convenient to replace this by two first-order (in time) equations. Thus, let

$$\phi^{(1)} \equiv \Phi. \quad [22]$$

Eq. 21 is equivalent to the set

$$\begin{aligned} \phi_t^{(1)} &= \frac{-\partial_x}{1 - G\partial_x} \phi^{(2)}, \\ \phi_t^{(2)} &= -\partial_x \left[\frac{(\phi^{(1)})^2}{2} + \phi^{(1)} \right] \end{aligned} \quad [23]$$

This is shown to be of Hamiltonian form by choosing the Poisson bracket as in Eq. 16 and

$$H = -\frac{1}{2} \int_{-\infty}^{\infty} \left\{ \phi^{(2)} \frac{1}{1 - G\partial_x} \phi^{(2)} + \frac{(\phi^{(1)})^3}{3} + (\phi^{(1)})^2 \right\} dx. \quad [24]$$

Applying my procedure, I find that corresponding to a constant F the conserved density $(\rho^{(1)}, \rho^{(2)}) \equiv (\delta F / \delta \phi^{(1)}, \delta F / \delta \phi^{(2)})$ satisfies the set of linear equations

$$\begin{aligned} \rho_t^{(1)} &= (1 - \phi^{(1)}) \partial_x \rho^{(1)}, \\ \rho_t^{(2)} &= -\partial_x \rho^{(1)} + G(\rho_{xt}^{(2)}). \end{aligned} \quad [25]$$

An "improved" generalized Korteweg-de Vries equation

We can also consider an equation related to the Korteweg-de Vries equation as the "improved" generalized Boussinesq

equation is related to the Boussinesq equation. This is the equation

$$u_t - G(u_{xt}) = -uu_x - G(u_{xx}). \quad [26]$$

It is of Hamiltonian form with

$$[F_1, F_2] = \int_{-\infty}^{\infty} \frac{\delta F_1}{\delta u} \frac{\partial_x}{1 - G\partial_x} \frac{\delta F_2}{\delta u} dx, \quad [27]$$

and

$$H = -\frac{1}{2} \int_{-\infty}^{\infty} \left\{ \frac{u^3}{3} - u_x G(u) \right\} dx. \quad [28]$$

The conserved density $\rho \equiv \delta F / \delta u$ corresponding to a constant F satisfies the linear equation

$$\rho_t = u \frac{\partial_x}{1 - G\partial_x} \rho - \frac{G\partial_{xx}}{1 - G\partial_x} \rho. \quad [29]$$

Discussion

Although the theorem applies to any Hamiltonian system the main interest is the case when there are more constants than the classical ones. (By classical we mean the constants associated with space and time translational invariance and, if it occurs, mass conservation.) Typically, this is the case when we have a completely integrable system with constants that are in involution. In this situation two questions arise:

(i) In such cases there are frequently more than one Hamiltonian formulation (5). Does this give rise to a different linear equation for conserved densities? Obviously not, because the linear equation is essentially the linearized form of the original equation. However, it is amusing to work this out for a simple example, such as the standard Korteweg-de Vries equation.

(ii) Because the constants are all in involution we can, with fixed Poisson bracket, consider any of the constants as Hamiltonian. This gives us many (let us say infinitely many) new Hamiltonian systems. But the constants are constants for all these systems. Therefore, we have the remarkable result that the infinite number of conserved densities each satisfy an infinite number of linear equations! (Of course, the same density that satisfies two different linear equations is not the same function of x and t . Rather it is the same functional of the u values—but the u values are different.)

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1. Case, K. M. (1978) *Phys. Rev. Lett.* **40**, 351–354.
2. Case, K. M. (1978) in *Topics in Nonlinear Dynamics*, ed. Jorna, S. (Am. Inst. Phys., New York), pp. 360–373.
3. Gardner, C. S., Greene, J. M., Kruskal, M. D. & Miura, R. M. (1974) *Commun. Pure Appl. Math.* **27**, 97–110.
4. Makhankov, V. G. (1978) *Phys. Rep.* **35**, 1–128.
5. Magri, F. (1978) *J. Math. Phys.* **19**, 1156–1162.