Some spectrally isolated convex planar regions

(billiard ball map/inverse spectral problem/geodesics wave equation/curvature)

SHAHLA MARVIZI AND RICHARD B. MELROSE

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540; and Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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ABSTRACT The basic question raised by M. Kac as to whether a domain in Euclidean space is determined by its Dirichlet spectrum remains open. In this note, dealing only with convex planar regions, we introduce a new countable family of (generic) spectral invariants of wave type, discuss some asymptotic properties of the distribution of closed geodesics, describe a partial converse to the Poisson relation, and thereby construct a two-parameter family of spectrally isolated regions, including the circles.

1. Regions and spectra

Let $\Omega \subset \mathbb{R}^2$ be a bounded strictly convex open region with C^∞ boundary. Orienting $\partial\Omega$ to be counterclockwise gives a unique arc length 1-form, $ds \neq 0$, $\int_{\partial\Omega} ds = L$, the length of the boundary. This determines, up to translation, an isomorphism $f: (\partial\Omega, ds) \leftrightarrow (S_L, d\theta)$ with the circle of length L , $S_L = \mathbb{R}/L\mathbb{Z}$. The curvature of the boundary, K , is then a well-defined positive C^∞ function which, transported to S_L as $q \in C^\infty(S_L)$, $K = f^*q$, satisfies

$$\int_0^L q \, ds = 2\pi, \quad \int_0^L \cos \left[\int_0^s q(t) dt \right] ds = \int_0^L \sin \left[\int_0^L q(t) dt \right] ds = 0. \quad [1.1]$$

A straightforward rearrangement of classical theory (1) gives

LEMMA. Any positive C^∞ function $q \in C^\infty(S_L)$ satisfying [1.1] arises from a strictly convex planar domain and uniquely determines that domain up to Euclidean motion.

The Dirichlet Eigenvalue Problem.

$$\Delta u = \lambda^2 u, \quad u|_{\partial\Omega} = 0 \quad [1.2]$$

where $\Delta = -(\partial/\partial x)^2 - (\partial/\partial y)^2$ has a countable, discrete set of solutions $0 < \lambda_1^2 < \lambda_2^2 < \dots$, each of finite multiplicity

$$m(\lambda) = \dim \{u; (1.2) \text{ holds}\},$$

where we always take $\lambda_j > 0$. The information contained in the spectrum is summarized in

$$\sigma_D(\lambda) = \sum_j m(\lambda_j) \delta(\lambda - \lambda_j) \in S'(\mathbb{R})$$

the spectral enumerating measure. Note that σ_D is actually determined by the curvature function $q \in C^\infty(S_L)$. Similarly, one can define σ_N corresponding to Neumann boundary conditions (or even Robin conditions).

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2. Billiard ball map

The ball bundle over the boundary $B\partial\Omega \subset T\partial\Omega$ consists of pairs (x, v) with $x \in \partial\Omega$ and v tangent to $\partial\Omega$ at x , $|v| \leq 1$. The billiard ball map

$$\beta: B\partial\Omega \rightarrow B\partial\Omega$$

is a homeomorphism defined by associating to (x, v) the line $s \rightarrow x + sw \in \mathbb{R}^2$, $w = v + r\nu_x$ where $r \geq 0$, ν_x is the inward normal to $\partial\Omega$ at x , and $|w| = 1$, then $\beta(x, v) = (x', v')$ if $x + \bar{s}w \in \partial\Omega$, $x' \neq x$ (unless $|v| = 1$) $v' + r'\nu_{x'} = w$ ($r' \leq 0$). The Riemannian structure on $\partial\Omega$, ds , allows one to identify $T\partial\Omega$ with $T^*\partial\Omega$ so giving $B\partial\Omega$, a symplectic structure which β preserves. Although β is C^∞ away from the boundary, $S\partial\Omega$, of $B\partial\Omega$, it is not C^∞ up to the boundary, the singularity at $S_+\partial\Omega$ (the counterclockwise component) can be described precisely as follows.

THEOREM 1. There exists $\zeta \in C^\infty(B\partial\Omega)$ (interpolating Hamiltonian for β) a defining function for $S_+\partial\Omega$ such that if H_ζ is the Hamilton vector field of ζ then near $S_+\partial\Omega$

$$\exp(-\zeta^{1/2}H_\zeta) \circ \beta$$

is a C^∞ map, fixing $S_+\partial\Omega$ to all orders. The Taylor series of ζ at $S_+\partial\Omega$ is uniquely specified by this condition.

Remark. Theorem 1 follows from the results of ref. 2. It is valid in this global form for a strictly convex region in Euclidean space of any dimension. Locally near a point on $S_+\partial\Omega$ ζ can be chosen so that $\exp(\zeta^{1/2}H_\zeta) = \beta$ exactly, but such a ζ does not, in general, exist globally.

3. Closed geodesics

Geodesics (transversal to $\partial\Omega$) in Ω are piecewise linear curves with singularities only at $\partial\Omega$ where they obey Snell's equal angle reflection law. Primitive closed geodesics of this type, without parametrization, are in 1:1 correspondence with the finite orbits of β in $B\partial\Omega/S\partial\Omega$. Let Γ be the set of all closed geodesics transversal to $\partial\Omega$, with $\Gamma(p, m) \subset \Gamma$, the subset making p reflections and with rotation number m . A classical result of Poincaré shows that $\Gamma(p, m) \neq \emptyset$ precisely when $2 \leq p$, $1 \leq m \leq p/2$. For $g \in \Gamma$ let $T(g)$ be the length of g as a curve in $\mathbb{R}^2$ (possibly described several times).

THEOREM 2. If $\Omega \subset \mathbb{R}^2$ is C^∞ and strictly convex, set $t_{p,m} = \inf\{T(g); g \in \Gamma(p, m)\} \leq T_{p,m} = \sup\{T(g); g \in \Gamma(p, m)\} < mL$. Then, for each m , $k \in \mathbb{Z} \exists C = C_{k,m}(\Omega)$ such that

$$T_{p,m} - t_{p,m} \leq C p^{-k} \quad \forall p > N(m, k, \Omega).$$

Moreover, there are constants $I_{k,m}$ such that, for each m ,

$$T_{p,m} \sim mL + \sum_{k=1}^{\infty} I_{k,m} p^{-2k} \quad \text{as } p \rightarrow \infty.$$

The $I_{k,1}$ determine the $I_{k,m}$ for $m > 1$.

The $I_{k,1}$ can be algebraically rearranged to give an equivalent set of constants $I_1, I_2, \dots$:

$$I_1 = c_1 \int_0^L q^{2/3} ds; I_2 = c_2 \int_0^L (8q^2 q^{-8/3} + 9q^{4/3}) ds$$

where $c_1, c_2 \neq 0$ are universal constants $\dot{q} = dq/ds$. Similarly, all the I_k are integrals of differential expressions in q , constructed out of the Taylor series of ζ at $S_+ \partial\Omega$. In fact the I_k occur as the coefficients in the Taylor series at $t = 0$ of the C^∞ function

$$I(t) = \int_{\zeta=t} dz$$

where $d\zeta \wedge dz = w$, the symplectic form on $T^* \partial\Omega$.

4. Poisson relation

Let $\mathcal{L}$ be the length spectrum of Ω (or of q), the closure in $\mathbf{R}$ of the set of lengths of closed transversal geodesics; then, $\mathcal{L} = T(\Gamma) \cup N.L$. The Poisson relation for manifolds with boundary (3, 4) shows that the Fourier transform of σ_D satisfies

$$\text{sing. supp. } \hat{\sigma}_D \subset \mathcal{L} \cup \{0\} \cup (-\mathcal{L}).$$

THEOREM 3. *For an open dense set of strictly convex planar regions (in the C^∞ topology on q), $\mathcal{R}$, which contains a neighbourhood of the circles (constants)*

$$t_{p,1}, T_{p,1} \in \text{sing. supp } \hat{\sigma}_D \quad \forall p > N(q).$$

The set $\mathcal{R}$ can be taken to consist of those regions for which there exists $\varepsilon > 0$ such that if g is a closed geodesic with $L - \varepsilon < T(g) < L$ then $g \in \Gamma(p,1)$ and is primitive. The proof of this result, which requires no nondegeneracy assumption on the

geodesics (cf. ref. 4), is related to the results of Soga (5).

COROLLARY. *If $q \in \mathcal{R}$ then the Dirichlet (or Neumann, or Robin) spectrum of q determines all the invariants I_k .*

5. Spectral isolation

Recall from refs. 6 and 7 that the area of Ω and the length of the boundary are spectral invariants of heat type. Fixing $L, I_1 \leq (2\pi)^{2/3} L^{1/3}$ by the Schwartz inequality, with equality only for the circle. Thus, the circle is spectrally determined in $\mathcal{R}$ —in fact by the isoperimetric inequality the circle is fully determined by its spectrum.

THEOREM 4. *There exists $\bar{\varepsilon} > 0$ such that if $0 \leq \varepsilon \leq \bar{\varepsilon} I_2(q)$ has a unique, up to translation, minimum among $q \in C^\infty(S_L)$, positive functions satisfying [1.1] and $I_1(q) = (2\pi)^{2/3} L^{1/3} - \varepsilon^2$. The corresponding regions, parametrized by L, ε , are spectrally determined in $\mathcal{R}$, in consequence spectrally isolated.*

The curvature q satisfies the Euler–Lagrange equations of I_2 constrained by I_1 and length, without constraint from the second two conditions in [1.1]. It is therefore an elliptic integral in the arc length, corresponding to a four-sided region—an oval—which is not, however, an ellipse (for $\varepsilon \neq 0$).

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