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A Powerful Test for Comparing Multiple Regression Functions

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Abstract

In this article, we address the important problem of comparison of two or more population regression functions. Recently, Pardo-Fernández, Van Keilegom and González-Manteiga (2007) developed test statistics for simple nonparametric regression models: $Y_{ij} = \theta_j(Z_{ij}) + \sigma_j(Z_{ij})\epsilon_{ij}$, based on empirical distributions of the errors in each population $j = 1, \dots, J$. In this paper, we propose a test for equality of the $\theta_j(\cdot)$ based on the concept of generalized likelihood ratio type statistics. We also generalize our test for other nonparametric regression setups, e.g., nonparametric logistic regression, where the loglikelihood for population j is any general smooth function $\mathcal{L}\{Y_j, \theta_j(Z_j)\}$. We describe a resampling procedure to obtain the critical values of the test. In addition, we present a simulation study to evaluate the performance of the proposed test and compare our results to those in Pardo-Fernández et al. (2007).

Keywords

Bootstrap; Comparison of regression functions; Generalized likelihood ratio; Kernel regression; Local likelihood; Nonparametric regression

1. Introduction

In this note we address the problem of comparing two or more populations in a general regression setup. We will compare populations via regression curves and develop a testing procedure in a nonparametric context using generalized likelihood ratio statistics.

Suppose that there are J populations from which we observe data. Let n_j be the size of the sample observed from the j^{th} population and (Y_{ij}, Z_{ij}) , $i = 1, \dots, n_j$, be the observed independent and identically distributed response-predictor pairs. The problem of testing equality of two regression ($J = 2$) curves has been studied by several authors leading to a substantial literature, see for example Delgado (1993), Kulasekera (1995), Neumeyer and Dette (2003) among many others. However their focus was primarily on comparing only two regression curves. Recently, Pardo-Fernández et al. (2007) developed a testing procedure that can be used to compare more than two regression functions. They considered the model for the j^{th} population to be

$$Y_{ij} = \theta_j(Z_{ij}) + \epsilon_{ij}, \quad (1)$$

where ϵ_{ij} 's are independent random errors with $E(\epsilon_{ij}|Z_{ij}) = 0$ and $\text{var}(\epsilon_{ij}|Z_{ij}) = \sigma_j^2(Z_{ij})$, with $\sigma_j^2(\cdot)$ being an unknown conditional variance function. Their method is based on the empirical distributions of the residuals for each of the regression models. Specifically, for each population, using the estimated errors $\hat{\epsilon}_{ij}$, they construct the empirical distributions of the residuals under the null and alternative hypotheses and then compare these distributions

using Kolmogorov-Smirnov and Cramér-von Mises type statistics. They propose four test statistics: T_{KS}^1 and T_{KS}^2 based on the Kolmogorov-Smirnov test, and T_{CM}^1 and T_{CM}^2 using Cramér-von Mises type statistic. However, for any given data set, it is not clear which test statistic to use. As we see from the simulation study in Pardo-Fernández et al. (2007), these four statistics show very different powers for different setups, so a common guideline as to which statistic should be used is hard to establish.

In this paper, we take a different approach by utilizing the theory of generalized likelihood ratio type statistics. The concept of generalized likelihood ratio testing in nonparametric regression was introduced by Fan, Zhang and Zhang (2001) as a technique to overcome the drawbacks of the nonparametric maximum likelihood ratio test statistic. To motivate the generalized likelihood ratio statistic, Fan et al. (2001) used the model given in (1) for $J = 1$ and developed a test for linearity of the regression function using the residual sum of squares. The usual nonparametric likelihood ratio test uses $\widehat{\theta}_{MLE}(\cdot)$, the MLE of the common function $\theta_0(\cdot)$ under the alternative hypothesis, to construct the test statistic. Instead, Fan et al. suggested to replace $\widehat{\theta}_{MLE}(\cdot)$ by any reasonable nonparametric estimator, thus leading to the generalized likelihood ratio statistic. In this paper, we use these ideas to develop a test to compare multiple regression functions when the model is given by (1).

We further develop our test for more general regression models. Let the j^{th} population have the loglikelihood $\mathcal{L}\{Y_j, \theta_j(Z_j)\}$ where $\mu_j(\cdot)$ is the unknown but true regression function. This setup includes many important and widely used regression problems, such as nonparametric logistic, probit and Poisson regression models, which are not addressed by Pardo-Fernández et al. It is interesting to note that $\mathcal{L}(\cdot)$ can be a quasi-likelihood as well when the true likelihood of the populations is unknown. Assume that the regression functions $\mu_j(\cdot)$, $j = 1, \dots, J$, have a common support. Here we are not allowing any nuisance parameters to be present in the model. We develop a test for comparing these regression functions via testing

$$H_0: \theta_1(\cdot) = \dots = \theta_J(\cdot) = \theta_0(\cdot)$$

versus the alternative that at least two of the mean functions are not equal.

This article is organized as follows. In Section 2, we start with the simple model in (1) and propose our testing procedure using a generalized likelihood ratio type statistic and derive asymptotic results for the proposed test statistic. We then generalize our procedure for generalized nonparametric models in Section 3. We conduct a simulation study to compare our method to Pardo-Fernández et al. (2007) in the case of simple nonparametric models and present the numerical results in Section 4. We will see that compared to their tests, using the various setting considered, our method shows significant gain in power in many cases while performing quite similarly in others. In Section 5, we present a simulation study to investigate the performance of our method for logistic nonparametric regression model. Section 6 presents concluding remarks. All technical details are given in the appendices.

2. Simple Nonparametric Regression Models

Consider the model given by (1). To estimate the regression functions, we use kernel based local polynomial estimators. Our results are based on local linear estimators (see Fan, Heckman and Wand (1995)) but can be easily generalized to other degrees. Let $K(\cdot)$ be a symmetric kernel density function and $K_h(z) = K(z/h)/h$, where h is the bandwidth. For any

fixed population j , we estimate $\mu_j(z_0)$ at any fixed point z_0 by minimizing the criterion function

$$\sum_{i=1}^{n_j} K_h(Z_{ij} - z_0) \{Y_{ij} - \alpha_{0j} - \alpha_{1j}(Z_{ij} - z_0)\}^2$$

with respect to $(\alpha_{0j}, \alpha_{1j})$ and then setting $\widehat{\theta}_j(z_0) = \widehat{\alpha}_{0j}$. These estimates denote the regression function estimated under the alternative hypothesis. Under the null hypothesis, $H_0: \mu_1(\cdot) = \dots = \mu_J(\cdot) = \mu_0(\cdot)$, we estimate the common function $\mu_0(\cdot)$ at z_0 by minimizing

$$\sum_{j=1}^J \sum_{i=1}^{n_j} K_h(Z_{ij} - z_0) \{Y_{ij} - \alpha_0 - \alpha_1(Z_{ij} - z_0)\}^2$$

with respect to (α_0, α_1) and then setting $\widehat{\theta}_j(z_0) = \widehat{\alpha}_0$.

For each population j , we then construct the residual sum of squares

$$SSE_j = \sum_{i=1}^{n_j} \{Y_{ij} - \widehat{\theta}_j(Z_{ij})\}^2.$$

Under the null hypothesis, define the total sum of squares

$$SSE_{\text{null}} = \sum_{j=1}^J \sum_{i=1}^{n_j} \{Y_{ij} - \widehat{\theta}_0(Z_{ij})\}^2.$$

Then we define our test statistic to be

$$T_n = (-1/2) \left\{ \sum_{j=1}^J SSE_j - SSE_{\text{null}} \right\}.$$

Here we assumed the distribution of the errors is normal, and hence the inclusion of the factor $(-1/2)$. However, one could omit this factor and still construct a valid test statistic.

Note that to compute the test statistic, one need not estimate the unknown variance functions $\sigma_j^2(z)$ for each population and hence the heteroscedasticity in the model does not play any role in the computation of T_n . However, as we will see in Section 2.2, the presence of heteroscedasticity is accounted for when we approximate the null distribution of the proposed statistic.

2.1. Asymptotic Properties

We assume that the following conditions hold:

- $n_j/n \rightarrow p_j$ as $n \rightarrow \infty$, where $n = n_1 + \dots + n_J$.
- The bandwidth satisfies $nh^2 \rightarrow \infty$ and $nh^5 \rightarrow 0$ as $n \rightarrow \infty$.

- $Z_j, j = 1, \dots, J$ have a common density $f(z)$ with a bounded support. Also, $f(\cdot)$ is Lipschitz continuous and bounded away from 0.

This condition, however, can be relaxed so that $Z_j, j = 1, \dots, J$ have different densities, see Remark 2. We assume the stronger condition for algebraic simplicity only.

- $\mu_f(\cdot), j = 1, \dots, J$ have continuous second derivatives.
- $E(\epsilon_j^4)$ exists and is finite.
- The kernel $K(\cdot)$ is symmetric and bounded with support $[-1, 1]$.

This is a common assumption in nonparametric smoothing literature and many commonly used kernels (e.g., Epanechnikov kernel) satisfy this condition.

Let $K_c(z) = K^* K(z)$, the convolution function, and $K_{ch}(z) = (1/h)K_c(z/h)$. Define $C_3(K) = \int [K(u) - K_c(u)/2]^2 du$ and

$$\begin{aligned}\mu_n &= h^{-1} E \left\{ \sum_{j=1}^J (1 - n_j/n) \sigma_j^2(Z) / f(Z) \right\} \left\{ K(0) - (1/2) \int K^2(u) du \right\}; \\ \mathcal{G}_j &= n_j^{-1} \sum_{i \neq k=1}^{n_j} \left\{ K_h(Z_{ij} - Z_{kj}) - (1/2) K_{ch}(Z_{ij} - Z_{kj}) \right\} \epsilon_{ij} \epsilon_{kj} / f(Z_{ij}); \\ \mathcal{H}_{j\ell} &= n^{-1} \sum_{i=1}^{n_j} \sum_{k=1}^{n_\ell} \left\{ K_h(Z_{k\ell} - Z_{ij}) - (1/2) K_{ch}(Z_{k\ell} - Z_{ij}) \right\} \epsilon_{k\ell} \epsilon_{ij} / f(Z_{ij}) \text{ for } j \neq \ell, \\ \mathcal{V} &= 2C_3(K) \left[\sum_{j=1}^J (1 - p_j)^2 E \left\{ \sigma_j^4(Z) / f(Z) \right\} + 2 \sum_{j \neq \ell=1}^J p_j p_\ell E \left\{ \sigma_j^2(Z) \sigma_\ell^2(Z) / f(Z) \right\} \right].\end{aligned}$$

Theorem 2.1 : Under the null hypothesis—

$$h^{1/2} (T_n - \mu_n) = h^{1/2} \sum_{j=1}^J (1 - n_j/n) \mathcal{G}_j - h^{1/2} \sum_{j \neq \ell=1}^J \mathcal{H}_{j\ell} + o_p(1) \Rightarrow \text{Normal}(0, \mathcal{V}),$$

as $n \rightarrow \infty$.

A proof of Theorem 2.1 is provided in Appendix A.

Technically, in the spirit of Fan and Li (1996), one could estimate the unknown quantities μ_n and $\mathcal{V}$ in the result above and then construct a feasible test based on the asymptotic normality. Specifically, one could estimate μ_m and $\mathcal{V}$ using some estimators $\widehat{\mu}_n$ and $\widehat{\mathcal{V}}$, and form a test statistic $T_{\text{asyp}} = h^{1/2} (T_n - \widehat{\mu}_n) / \widehat{\mathcal{V}}^{1/2}$ and compute $p\text{-value} = \text{pr}(W > |T_{\text{asyp}}|)$, where W follows a standard normal distribution.

To see the performance of the asymptotic test based on the theoretical result in Theorem 1, we conducted a simulation study. We considered the setting $\mu_1(z) = \mu_2(z) = z$, $\sigma_1^2(z) = 0.25$ and $\sigma_2^2(z) = 0.5$. We consider four different sample sizes $n_1 = n_2 = 100, 200, 500$ and 1,000, and for each case we run 10,000 simulations. The random errors were generated from the standard Gaussian distribution and the covariates were generated from a Uniform(0,1) distribution. The Epanechnikov kernel was used to obtain the estimates of the mean functions with bandwidth $h = n^{-1/4}$. In each simulation run, we computed the theoretical mean μ_n , variance $\mathcal{V}$ and $p\text{-value}$ using asymptotic normality and true values of the parameters. Based on 10,000 simulated data sets, the observed type I errors for $\alpha = 0.05$

were 0.067, 0.067, 0.063 and 0.059 for the four sample size settings, respectively. It is evident that even for large sample sizes $n_1 = n_2 = 1000$, the type I error of the asymptotic test shows some inflation. This seems to suggest that one would need even larger sample sizes to utilize the asymptotic test.

Now consider local alternatives of the form $H_a: \mu_f(z) = \mu_0(z) + n^{-1/2}h^{-1/4}r_f(z)$, where $r_f(\cdot)$ are unknown functions such that second derivatives of r_f exists. Define $R(z) = \sum_{j=1}^J p_j r_j(z)$ and assume that $E\{r_f(Z_j) - R(Z_j)\}^4$ is finite. Then we have the following result.

Theorem 2.2 : Under the local alternative hypothesis H_1 —

$$h^{1/2} (T_n - \mu_n) \Rightarrow \text{Normal}(C_4, \mathcal{V}),$$

where $\mathcal{V}$ is defined as in Theorem 1 and $C_4 = (1/2) \sum_{j=1}^J p_j E\{r_j(Z_{ij}) - R(Z_{ij})\}^2$.

A proof of Theorem 2.2 is provided in Appendix B. From the result, it is evident the asymptotic test can detect local alternative with rate $n^{-1/2}h^{-1/4}$. This is slightly inferior rate than $n^{-1/2}$.

Remark 1: Note that the asymptotic distribution depends on the unknown variance functions and the density of Z . Hence the Wilks phenomena does not hold in this case. This is caused by the fact that our test statistic uses only total sum of squares under the null and alternative hypotheses instead of the true likelihood itself and hence is not a likelihood ratio test. Thus, to utilize the asymptotic test, one needs to estimate the unknown variance functions $\sigma_j^2(\cdot)$, $j = 1, \dots, J$. In contrast, we will propose a resampling based test in Section 2.2 that has the advantage that one is not required to estimate the unknown variance functions as one would need to perform the likelihood ratio test or the asymptotic test.

Remark 2: To prove Theorem 2.1, we have assumed that Z_j , $j = 1, \dots, J$ have a common density $f(z)$. This is not a very crucial assumption, only serving the purpose of making the algebra simpler. In fact, if the support densities are different, then the asymptotic normality still holds but with a different asymptotic variance depending on the unknown densities of Z_j . Note that in this case also, the Wilks phenomena does not hold.

2.2. Null Distribution of T_n and Computation of p-values—In terms of implementation of the test, the result in Theorem 2.1 does not provide much help because of the fact that one needs to determine μ_n and $\mathcal{V}$, which depend in turn on the unknown variance functions $\sigma_j^2(z)$. Instead, we propose to use a resampling procedure to compute the null distribution of T_n . Fan and Jiang (2005) validated this type of resampling method in the additive model context when the errors are homoscedastic. Recall that $\widehat{\theta}_0(\cdot)$ is the estimate under the null model and $\widehat{\theta}_j(\cdot)$ for $j = 1, \dots, J$ are estimates for the unrestricted model. For the homoscedastic case, the steps are as follows:

1. Construct residuals under the unrestricted model as $\widehat{e}_{ij} = Y_{ij} - \widehat{\theta}_j(Z_{ij})$.
2. Let $\widetilde{e}_j = (\widehat{e}_{1j}, \dots, \widehat{e}_{n_j, j})^T$ be the vector of estimated residuals for population j . Generate a resample (with replacement) $(\widetilde{e}_{1j}^*, \dots, \widetilde{e}_{n_j, j}^*)$ from $\widetilde{e}_j$ after centering.
3. Then generate bootstrap-type data for each population j as

$$Y_{ij}^* = \widehat{\theta}_0(Z_{ij}) + \widehat{e}_{ij}^*,$$

and compute the corresponding value of the test statistic using (Y_{ij}^*, Z_{ij}) , $j = 1, \dots, J$, $i = 1, \dots, n_j$. Let the test statistic for the b^{th} resample be $T_{n,b}^*$, for $b = 1, \dots, B$ with B being a fixed large number.

4. Then the p-value is computed by

$$\text{p-value} = B^{-1} \sum_{b=1}^B I(T_{n,b}^* > T_n).$$

However, in the presence of heteroscedasticity, the resampling step 2 is not valid. Hence in the heteroscedastic case, we propose to replace steps 2 and 3 by the following:

2'. Let $\tilde{e}_j = (\tilde{e}_{1j}, \dots, \tilde{e}_{n_j,j})^T$ be the vector of estimated residuals for population j and let $\tilde{Z}_j = (Z_{1j}, \dots, Z_{n_j,j})^T$. Generate $(\tilde{e}_{1j}^*, \dots, \tilde{e}_{n_j,j}^*)$ and $(\tilde{Z}_{1j}^*, \dots, \tilde{Z}_{n_j,j}^*)$ by sampling with replacement from the pairs $(\tilde{e}_{ij}, Z_{ij})$, $i=1, \dots, n_j$, after centering $\tilde{e}_j$.

3'. Then generate bootstrap-type data for each population j as $Y_{ij}^* = \widehat{\theta}_0(Z_{ij}) + \tilde{e}_{ij}^*$, and compute the corresponding value of the test statistic using (Y_{ij}^*, Z_{ij}^*) , $j = 1, \dots, J$, $i = 1, \dots, n_j$.

Then the p-value is computed as in step 4.

Note that in steps 2 and 2', we generate resamples from the residuals separately for each j , thus preserving the distributions of Z_j for each population.

3. General Nonparametric Regression

Our testing procedure can be readily applied to generalized nonparametric regression models. Let the loglikelihood function for the j^{th} population is given by $\mathcal{L}\{Y_j, \theta_j(Z_j)\}$. Under the full model, for a given point z_0 , we estimate the regression function $\theta_j(z_0)$ by locally weighted likelihood procedure. Using a local linear setup, we thus minimize

$$\sum_{i=1}^{n_j} K_h(Z_{ij} - z_0) \mathcal{L}\{Y_{ij}, \alpha_0 + (Z_{ij} - z_0)\alpha_1\}$$

with respect to (α_0, α_1) and set $\widehat{\theta}_j(z_0) = \widehat{\alpha}_{0j}$. Under the null hypothesis we estimate the common function $\theta_0(z_0)$ by minimizing

$$\sum_{j=1}^J \sum_{i=1}^{n_j} K_h(Z_{ij} - z_0) \mathcal{L}\{Y_{ij}, \alpha_0 + (Z_{ij} - z_0)\alpha_1\}$$

and set $\widehat{\theta}_0(z_0) = \widehat{\alpha}_0$. Then we construct our test statistic as

$$T_n = \sum_{j=1}^J \sum_{i=1}^{n_j} \mathcal{L} \{Y_{ij}, \widehat{\theta}_j(Z_{ij})\} - \sum_{j=1}^J \sum_{i=1}^{n_j} \mathcal{L} \{Y_{ij}, \widehat{\theta}_0(Z_{ij})\}.$$

As described in Section 2, we can simulate the null distribution using a resampling method. Using the null fits of $\theta_0(\cdot)$, we generate the response data Y^* for $b = 1, \dots, B$ times and calculate the corresponding test statistic $T_{n,b}^*$. Then the p-value is computed by the proportion of times the simulated values $T_{n,b}^*$ exceed the observed test statistic T_n .

In Section 5, we demonstrate the testing procedure when the data from each population are modeled by logistic regression models. We will present a simulation study to observe the performance of our test.

4. Simulation Study: Simple Regression Model

In this section we conduct a simulation study to evaluate the performance of our method and compare the results to that of Pardo-Fernández et al. (2007). We consider the model given in (1) for the simulation study.

4.1. Comparing Two Populations

First, we consider comparing two mean functions. Pardo-Fernández et al. (2007) considered the following eight settings:

- (i) $\theta_1(z) = \theta_2(z) = 1$ (v) $\theta_1(z) = 1, \theta_2(z) = 1 + z$
- (ii) $\theta_1(z) = \theta_2(z) = z$ (vi) $\theta_1(z) = \exp(z), \theta_2(z) = \exp(z) + z$
- (iii) $\theta_1(z) = \theta_2(z) = \sin(2\pi z)$ (vii) $\theta_1(z) = \sin(2\pi z), \theta_2(z) = \sin(2\pi z) + z$
- (iv) $\theta_1(z) = \theta_2(z) = \exp(z)$ (viii) $\theta_1(z) = 1, \theta_2(z) = \sin(2\pi z) + 1$

In each case they considered homoscedastic and heteroscedastic variance models:

homoscedastic case: $\sigma_1^2(z) = 0.25$ and $\sigma_2^2(z) = 0.5$;

heteroscedastic case: $\sigma_1^2(z) = \sigma_2^2(z) = \exp(z) / \left(\int_0^1 \exp(t) dt \right)$.

Three different sample size settings $(n_1, n_2) = (50, 50), (100, 50)$ and $(100, 100)$ are used. For each case we run 1000 simulations and the number of bootstraps for each simulation is $B = 1000$. The random errors are generated from the standard Gaussian distribution and the covariates are generated from a Uniform(0,1) distribution. The Epanechnikov kernel is used to obtain the estimates of the mean functions. These are the same setting as in Pardo-Fernández et al. (2007).

We use the same bandwidth for both the groups. In general, the power of the generalized likelihood ratio test depends on the choice of the bandwidth. The determination of the optimal bandwidth for this type of test is still an open problem, see Fan and Jiang (2007) for a discussion of this matter. In the simulations we use $h = Cn^{-1/4}$. To observe the effect of bandwidth, we set different values of C and compute the rejection probabilities under the settings (i)-(viii) with homoscedastic errors. The results are displayed in Table 1. We can see that the rejection probabilities stay reasonably close over a large range of bandwidths in all the cases. Hence, in our further simulations we report the results only for $C = 1$.

For cases (i)-(iv), the rejection probabilities for significance levels $\alpha = 0.05$ and 0.10 are displayed in Table 2. It is evident from our results that in both the homoscedastic and heteroscedastic cases, the actual test levels are near nominal.

We now turn our attention to power calculations under settings (v)-(viii). The power of our proposed test for different cases are reported in Table 3. We now compare these results to that of Pardo-Fernández et al. (2007) (their Table 3). They considered four test statistics: T_{KS}^1 , T_{CM}^1 , T_{KS}^2 , T_{CM}^2 . We report in Table 3 the minimum and maximum power among T_{KS}^1 , T_{CM}^1 , T_{KS}^2 , T_{CM}^2 for each case that they presented. It is evident from Table 3 that for the homoscedastic case, our test statistic achieves equivalent or higher power than theirs in all the cases considered. For example, in setting (viii) it is clear the best power is provided by T_{CM}^2 , followed by T_{KS}^2 . Comparing to our results, our proposed test statistic has greater power than that of T_{CM}^2 and T_{KS}^2 and shows a significant gain in power over T_{CM}^1 and T_{KS}^1 .

In the heteroscedastic models with settings (v) and (vii), our test shows better power than T_{KS}^1 , T_{CM}^1 , and T_{CM}^2 while performing very similarly to T_{CM}^1 . However, in settings (vi) and (viii), our test performs much better. In Case (viii), T_{KS}^1 , T_{CM}^1 shows virtually no power, even in the case $n_1 = n_2 = 100$, while T_{KS}^2 and T_{CM}^2 show moderate power for small samples. Compared to these test statistics, our proposed test shows higher power in all the sample size settings. Similar conclusions can be drawn in case (vi) as well. For example, when $n_1 = n_2 = 50$, the rejection probabilities of our test are 0.987 and 0.997 for 5% and 10% levels, compared to T_{CM}^1 (best among the four statistics in this setting) with rejection probability of 0.643 and 0.753, respectively.

4.2. Comparing Three Populations

Next, we apply our method to compare three regression functions. For comparison purposes, we choose the same settings considered in Pardo-Fernández et al. (2007). They only considered the homoscedastic case: $\sigma_j^2(z) = 0.5$ for $j = 1, 2, 3$. The three cases considered are as follows:

$$(ix) \theta_1(z) = \theta_2(z) = \theta_3(z) = 1$$

$$(x) \theta_1(z) = z, \theta_2(z) = 0.25 + z \text{ and } \theta_3(z) = 0.5 + z$$

$$(xi) \theta_1(z) = z, \theta_2(z) = 0.5 \text{ and } \theta_3(z) = 1 - z.$$

The distributions of the covariates and error variables are same as in Pardo-Fernández et al.

The results are displayed in Table 4. In case (x), our test performs much better than T_{KS}^2 and T_{CM}^2 while performing similarly to T_{KS}^1 and T_{CM}^1 . However, for case (xi), the proposed test shows substantial gain in power even for small sample sizes, where their tests shows very little to moderate power.

5. Simulation Study: Logistic Regression Model

In this section we consider the case where the data from each population are generated from a logistic regression model: for population j , we observe covariates Z_{ij} and binary responses Y_{ij} with

$$\text{pr}(Y_{ij}=1) = H\{\theta_j(Z_{ij})\},$$

where $H(\cdot)$ denotes the logistic distribution function.

For the simulation study, we use $J=2$ and with the following four cases:

- (i) $\theta_1(z) = \theta_2(z) = 1$ (iii) $\theta_1(z) = \exp(z)$, $\theta_2(z) = \exp(z) + z$
- (ii) $\theta_1(z) = \theta_2(z) = z$ (iv) $\theta_1(z) = 1$, $\theta_2(z) = \sin(2\pi z) + 1$

We generated 1000 data sets and used $B = 1000$. The Epanechnikov kernel was used to obtain the estimates. The covariates Z_{ij} are generated from the Uniform(0, 1) distribution. We estimate $\theta_1(\cdot)$, $\theta_2(\cdot)$ and the common function $\theta_0(\cdot)$ as in Section 3 and compute the test statistic T_n . To compute the null distribution of T_n , we apply the resampling procedure as follows:

- Under the null model, compute the probabilities $\widehat{p}_{ij} = H\{\widehat{\theta}_0(Z_{ij})\}$, for $j = 1, 2$.
- Generate the responses Y_{ij}^* from the Bernoulli($\widehat{p}_{ij}$) distribution. Repeat B times.
- For each set of generated data, $b = 1, \dots, B$, compute test statistic $T_{n,b}^*$ based on (Y_{ij}^*, Z_{ij}) . Compute the p-value as, $B^{-1} \sum_{b=1}^B I(T_{n,b}^* \leq T_n)$.

We use bandwidth $h = n^{-1/4}$ and different sample size settings for this simulation. The results are displayed in Table 5. It is evident that for null models, the actual level is close to nominal when $n_1 = n_2$, however the test is somewhat conservative when the two sample sizes differ. Also, under models (iii) and (iv), the power increases as sample size increases. It is important to note that, compared to the Gaussian regression setup, as one would expect, one needs to have reasonably larger sample size in the logistic case to obtain similar power.

6. Discussion

In this note, we have proposed a generalized likelihood ratio based testing procedure for comparison of multiple regression functions in nonparametric regression models. We also consider the general nonparametric regression model and develop tests to compare multiple regression functions. We compare our results to those in Pardo-Fernández et al. (2007) via a simulation study and show that our test generally has similar or higher power than theirs. Note that the proposed testing procedure uses resampling technique to approximate the null distribution of the test statistic and for computing p -values. While it is possible to rigorously determine the performance of resampling procedures in the context of nonparametric local linear regression, such an investigation is out of the scope of this paper and as such is not presented here. This is a potential future research.

Supplementary Material

Refer to Web version on PubMed Central for supplementary material.

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Appendix

Appendix A:

Proof of Theorem 2.1

We write $T_n = T_{n,\text{alt}} - T_{n,\text{null}}$, where

$$\begin{aligned} T_{n,\text{alt}} &= (-1/2) \sum_{j=1}^J \left[\sum_{i=1}^{n_j} \left\{ Y_{ij} - \widehat{\theta}_j(Z_{ij}) \right\}^2 - \sum_{i=1}^{n_j} \left\{ Y_{ij} - \theta_0(Z_{ij}) \right\}^2 \right]; \\ T_{n,\text{null}} &= (-1/2) \sum_{j=1}^J \left[\sum_{i=1}^{n_j} \left\{ Y_{ij} - \widehat{\theta}_0(Z_{ij}) \right\}^2 - \sum_{i=1}^{n_j} \left\{ Y_{ij} - \theta_0(Z_{ij}) \right\}^2 \right]. \end{aligned}$$

Define $C_2(K) = K(0) - \int K^2(u) du/2$, $C_3(K) = \int [K(u) - K_c(u)/2]^2 du$ and

$$\begin{aligned} \mathcal{G}_j &= n_j^{-1} \sum_{i \neq k=1}^{n_j} \left\{ K_h(Z_{ij} - Z_{kj}) - (1/2) K_{ch}(Z_{ij} - Z_{kj}) \right\} \epsilon_{ij} \epsilon_{kj} / f(Z_{ij}) \\ \mathcal{H}_{j\ell} &= n^{-1} \sum_{i=1}^{n_j} \sum_{k=1}^{n_\ell} \left\{ K_h(Z_{k\ell} - Z_{ij}) - (1/2) K_{ch}(Z_{k\ell} - Z_{ij}) \right\} \epsilon_{k\ell} \epsilon_{ij} / f(Z_{ij}) \text{ for } j \neq \ell. \end{aligned}$$

Then the expansion of T_n follows directly from the following results:

Lemma A.1: *Under the null hypothesis,*

$$T_{n,\text{alt}} = h^{-1} C_2(K) E \left\{ \sum_{j=1}^J \sigma_j^2(Z_j) / f(Z_j) \right\} + \sum_{j=1}^J \mathcal{G}_j + R + o_p(h^{-1/2}),$$

where $R = O_p(n^{1/2} h^2 + n h^4)$.

Lemma A.2: *Under the null hypothesis*

$$\begin{aligned} T_{n,\text{null}} &= h^{-1} C_2(K) E \left\{ \sum_{j=1}^J (n_j/n) \sigma_j^2(Z_j) / f(Z_j) \right\} + \sum_{j=1}^J (n_j/n) \mathcal{G}_j + \sum_{j \neq \ell=1}^J \mathcal{H}_{j\ell} \\ &\quad + R + o_p(h^{-1/2}). \end{aligned}$$

Proofs of these two lemmas are given in the supplementary material for this paper. Now we see that

$$\begin{aligned} h^{1/2} \mathcal{G}_j &\Rightarrow \text{Normal}(\mathcal{V}_{1,j}); \\ h^{1/2} \mathcal{H}_{j\ell} &\Rightarrow \text{Normal}(\mathcal{V}_{2,j\ell}), \end{aligned}$$

where $\mathcal{V}_{1,j}=2C_3(K)E\{\sigma_j^4(Z)/f(Z)\}$ and $\mathcal{V}_{2,j\ell}=p_j p_\ell C_3(K)E\{\sigma_j^2(Z)\sigma_\ell^2(Z)/f(Z)\}$. These results follow by a direct application of Proposition 3.2 of de Jong (1987) (see e.g., arguments in Fan et al. (2001) and Fan and Jiang (2005)) due to the fact that for $j \neq \ell \in_{j\ell} \mathcal{K}$ are independent random variables with mean zero and conditional variance

$E(\epsilon_{ij}^2 \epsilon_{k\ell}^2 | Z_{ij}, Z_{k\ell}) = \sigma_j^2(Z_{ij}) \sigma_\ell^2(Z_{k\ell})$. Also straightforward algebra gives us

$$\begin{aligned} \text{cov}(h^{1/2} \mathcal{G}_j, h^{1/2} \mathcal{H}_{j\ell}) &= 0; \\ \text{cov}(h^{1/2} \mathcal{H}_{j\ell}, h^{1/2} \mathcal{H}_{\ell j}) &= \mathcal{V}_{2,j\ell}, \end{aligned}$$

and hence we have

$$\begin{aligned} \text{var} \left\{ h^{1/2} \sum_{j=1}^J (1 - n_j/n) \mathcal{G}_j - h^{1/2} \sum_{j \neq \ell=1}^J \mathcal{H}_{j\ell} \right\} \\ = \sum_{j=1}^J (1 - p_j)^2 \mathcal{V}_{1,j} + 4 \sum_{j \neq \ell=1}^J \mathcal{V}_{2,j\ell} \\ = 2C_3(K) \left[\sum_{j=1}^J (1 - p_j)^2 E\{\sigma_j^4(Z)/f(Z)\} + 2 \sum_{j \neq \ell=1}^J p_j p_\ell E\{\sigma_j^2(Z)\sigma_\ell^2(Z)/f(Z)\} \right]. \end{aligned}$$

Hence the results follows.

Appendix B:

Proof of Theorem 2.2

Recall that we consider local alternatives of the form $\theta_j(z) = \theta_0(z) + n^{-1/2} h^{-1/4} r_j(z)$, where $r_j(\cdot)$ are unknown functions. Hence, under the alternative, $\widehat{\theta}_j(z)$ estimates $\theta_j(z)$ and $\widehat{\theta}_0(z)$ estimates $\theta_{\text{alt}}(z) = \theta_0(z) + n^{-1/2} h^{-1/4} R(z)$, where $R(z) = \sum_{j=1}^J p_j r_j(z)$. Then the test statistic can be written as $T_n = T_{n,\text{alt}} - T_{n,\text{null}} - T_{n,\text{rem}}$, where

$$\begin{aligned} T_{n,\text{alt}} &= (-1/2) \sum_{j=1}^J \left[\sum_{i=1}^{n_j} \{Y_{ij} - \widehat{\theta}_j(Z_{ij})\}^2 - \sum_{i=1}^{n_j} \{Y_{ij} - \theta_j(Z_{ij})\}^2 \right]; \\ T_{n,\text{null}} &= (-1/2) \sum_{j=1}^J \left[\sum_{i=1}^{n_j} \{Y_{ij} - \widehat{\theta}_0(Z_{ij})\}^2 - \sum_{i=1}^{n_j} \{Y_{ij} - \theta_{\text{alt}}(Z_{ij})\}^2 \right]; \\ T_{n,\text{rem}} &= (-1/2) \sum_{j=1}^J \left[\sum_{i=1}^{n_j} \{Y_{ij} - \theta_{\text{alt}}(Z_{ij})\}^2 - \sum_{i=1}^{n_j} \{Y_{ij} - \theta_j(Z_{ij})\}^2 \right]. \end{aligned}$$

Also under the alternative we have the expansions,

$$\begin{aligned} \widehat{\theta}_j(z) - \theta_j(z) &= C \theta_j^{(2)}(z) h^2 + n^{-1} \sum_{k=1}^{n_j} K_h(Z_{kj} - z) \epsilon_{kj} / f(z) + o_p(h^2), \\ \widehat{\theta}_0(z) - \theta_{\text{alt}}(z) &= C \theta_{\text{alt}}^{(2)}(z) h^2 + n^{-1} \sum_{p=1}^J \sum_{k=1}^{n_p} K_h(Z_{kp} - z) \epsilon_{kp} / f(z) + o_p(h^2) \end{aligned}$$

for any given z , where $C = \int t^2 K(t) dt / 2$.

Proceeding as in Lemma A.1 and Lemma A.2 we have that

$$\begin{aligned}
T_{n,\text{alt}} &= C_2(K) h^{-1} \sum_{j=1}^J E \left\{ \sigma_j^2(Z_j) / f(Z_j) \right\} + \sum_{j=1}^J \mathcal{G}_j + R_{1,\text{alt}} + o_p(h^{-1/2}), \\
T_{n,\text{null}} &= h^{-1} C_2(K) E \left\{ \sum_{j=1}^J (n_j/n) \sigma_j^2(Z_j) / f(Z_j) \right\} + \sum_{j=1}^J (n_j/n) \mathcal{G}_j + \sum_{j \neq \ell=1}^J \mathcal{H}_{j\ell} \\
&\quad + R_{2,\text{alt}} + o_p(h^{-1/2}),
\end{aligned}$$

where

$$\begin{aligned}
R_{1,\text{alt}} &= Ch^2 \sum_{j=1}^J \sum_{i=1}^{n_j} \theta_j^{(2)}(Z_{ij}) \epsilon_{ij} - (C^2/2) h^4 \sum_{j=1}^J \sum_{i=1}^{n_j} \left\{ \theta_j^{(2)}(Z_{ij}) \right\}^2 \\
R_{2,\text{alt}} &= Ch^2 \sum_{j=1}^J \sum_{i=1}^{n_j} \theta_{\text{alt}}^{(2)}(Z_{ij}) \epsilon_{ij} - (C^2/2) h^4 \sum_{j=1}^J \sum_{i=1}^{n_j} \left\{ \theta_{\text{alt}}^{(2)}(Z_{ij}) \right\}^2
\end{aligned}$$

Now note that $\theta_j^{(2)}(z) = \theta_0^{(2)}(z) + n^{-1/2} h^{-1/4} r_j^{(2)}(z)$ and $\theta_{\text{alt}}^{(2)}(z) = \theta_0^{(2)}(z) + n^{-1/2} h^{-1/4} R^{(2)}(z)$. Hence we have

$$\begin{aligned}
R_{1,\text{alt}} &= Ch^2 \sum_{j=1}^J \sum_{i=1}^{n_j} \theta_0^{(2)}(Z_{ij}) \epsilon_{ij} - (C^2/2) h^4 \sum_{j=1}^J \sum_{i=1}^{n_j} \left\{ \theta_0^{(2)}(Z_{ij}) \right\}^2 + C n^{-1/2} h^{7/4} \sum_{j=1}^J \sum_{i=1}^{n_j} r_j^{(2)}(Z_{ij}) \epsilon_{ij} \\
&\quad - C^2 n^{-1/2} h^{15/4} \sum_{j=1}^J \sum_{i=1}^{n_j} r_j^{(2)}(Z_{ij}) \theta_0^{(2)}(Z_{ij}) - (C^2/2) n^{-1} h^{7/2} \sum_{j=1}^J \sum_{i=1}^{n_j} \left\{ r_j^{(2)}(Z_{ij}) \right\}^2 \\
&= R + o_p(h^{-1/2}),
\end{aligned}$$

since each of the remaining terms are $o_p(h^{-1/2})$ with the assumption that $h = O_p(n^{-1/5})$. Similarly we can derive $R_{2,\text{alt}} = R + o_p(h^{-1/2})$. Hence combining all the terms, we have

$$T_{n,\text{alt}} - T_{n,\text{null}} = \mu_n + \sum_{j=1}^J (1 - n_j/n) \mathcal{G}_j - \sum_{j \neq \ell=1}^J \mathcal{H}_{j\ell} + o_p(h^{-1/2}).$$

Next we derive that

$$\begin{aligned}
T_{n,\text{rem}} &= (-1/2) \sum_{j=1}^J \left[\sum_{i=1}^{n_j} \left\{ Y_{ij} - \theta_{\text{alt}}(Z_{ij}) \right\}^2 - \sum_{i=1}^{n_j} \left\{ Y_{ij} - \theta_j(Z_{ij}) \right\}^2 \right] \\
&= (-1/2) \sum_{j=1}^J \left[\sum_{i=1}^{n_j} \left\{ \theta_j(Z_{ij}) - \theta_{\text{alt}}(Z_{ij}) \right\}^2 + 2 \sum_{i=1}^{n_j} \epsilon_{ij} \left\{ \theta_j(Z_{ij}) - \theta_{\text{alt}}(Z_{ij}) \right\} \right]
\end{aligned}$$

However we see that

$$\begin{aligned}
h^{1/2} \sum_{j=1}^J \sum_{i=1}^{n_j} \epsilon_{ij} \left\{ \theta_j(Z_{ij}) - \theta_{\text{alt}}(Z_{ij}) \right\} &= h^{1/4} n^{-1/2} \sum_{j=1}^J \sum_{i=1}^{n_j} \epsilon_{ij} \left\{ r_j(Z_{ij}) - R(Z_{ij}) \right\} = o_p(1), \\
(-1/2) h^{1/2} \sum_{j=1}^J \sum_{i=1}^{n_j} \left\{ \theta_j(Z_{ij}) - \theta_{\text{alt}}(Z_{ij}) \right\}^2 &= (-1/2) n^{-1} \sum_{j=1}^J \sum_{i=1}^{n_j} \left\{ r_j(Z_{ij}) - R(Z_{ij}) \right\}^2 = -C_4 + o_p(1),
\end{aligned}$$

where $C_4 = (1/2) \sum_{j=1}^J p_j E \left\{ r_j(Z_{ij}) - R(Z_{ij}) \right\}^2$. Thus, we have $h^{1/2} T_{n,\text{rem}} = -C_4 + o_p(1)$ and hence

$$h^{1/2}(T_n - \mu_n) = h^{1/2} \sum_{j=1}^J (1 - n_j/n) \mathcal{G}_j - h^{1/2} \sum_{j \neq \ell=1}^J \mathcal{H}_{j\ell} + C_4 + o_p(1)$$

Hence under the alternative

$$h^{1/2}(T_n - \mu_n) \rightarrow \text{Normal}(C_4, \mathcal{V})$$

in distribution and the results follows.

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Table 1

Rejection probabilities (multiplied by 1000) of the GLR test for comparison of two mean functions at 5% significance level for different bandwidths $h = Cn^{-1/4}$, for $C = 0.5, 1$ and 1.5 . The models are homoscedastic and $n_1 = n_2 = 50$.

C:	$\alpha = 0.05$				$\alpha = 0.10$			
	0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.5
(i)	69	59	56	119	122	113		
(ii)	61	60	63	125	122	114		
(iii)	61	52	50	117	111	111		
(iv)	61	58	64	125	117	115		
(v)	952	981	984	981	991	993		
(vi)	998	1000	1000	1000	1000	1000		
(vii)	944	964	939	977	982	983		
(viii)	992	996	997	995	999	999		

Table 2

Rejection probabilities (multiplied by 1000) of the GLR test under the null hypothesis for comparison of two mean functions at significance levels 0.05 and 0.10. Displayed are the results for both the homoscedastic and heteroscedastic models based on 1000 simulations. The values in the parenthesis display the powers of the four test statistics (T_{KS}^1 , T_{CM}^1 , T_{KS}^2 , T_{CM}^2) from Table 2 in Pardo-Fernández et al. (2007). The case of $n_1 = 200$ and $n_2 = 200$ was not considered in their paper.

Case	Homoscedastic models		Heteroscedastic models	
	$\alpha : 0.05$	0.10	0.05	0.10
$n_1 = 50$ and $n_2 = 50$				
(i)	59 (51, 52, 56, 49)	122 (93, 105, 104, 101)	61 (52, 50, 44, 57)	117 (97, 95, 95, 94)
(ii)	60 (51, 54, 49, 55)	122 (101, 102, 95, 100)	62 (47, 53, 43, 51)	118 (95, 103, 95, 88)
(iii)	52 (64, 71, 47, 66)	111 (110, 122, 97, 121)	58 (49, 52, 44, 47)	122 (96, 103, 88, 87)
(iv)	58 (57, 55, 48, 55)	117 (102, 108, 92, 96)	65 (42, 54, 38, 46)	120 (90, 94, 86, 89)
$n_1 = 100$ and $n_2 = 50$				
(i)	59 (55, 55, 48, 50)	110 (99, 107, 88, 105))	62 (54, 52, 42, 49)	118 (99, 114, 85, 87)
(ii)	61 (48, 50, 54, 55)	110 (100, 106, 93, 101)	62 (48, 53, 32, 38)	120 (96, 102, 83, 85)
(iii)	50 (67, 76, 67, 73)	100 (117, 127, 130, 130)	59 (60, 63, 55, 41)	116 (116, 121, 101, 90)
(iv)	61 (48, 50, 59, 55)	112 (97, 111, 99, 110)	63 (50, 53, 39, 46)	122 (99, 102, 79, 82)
$n_1 = 100$ and $n_2 = 100$				
(i)	50 (52, 50, 39, 44)	116 (88, 106, 90, 87)	59 (50, 56, 39, 43)	113 (96, 99, 67, 76)
(ii)	49 (54, 53, 44, 44)	108 (88, 107, 77, 92)	60 (52, 56, 34, 39)	111 (97, 100, 79, 73)
(iii)	48 (58, 61, 45, 60)	89 (110, 110, 91, 104)	59 (54, 56, 38, 51)	102 (98, 107, 93, 101)
(iv)	51 (52, 49, 44, 49)	106 (95, 104, 86, 95)	61 (52, 56, 38, 44)	111 (93, 104, 83, 74)
$n_1 = 200$ and $n_2 = 200$				
(i)	51	109	39	83
(ii)	52	104	39	84
(iii)	45	88	39	84
(iv)	50	102	40	85

Table 3

Rejection probabilities (multiplied by 1000) of the GLR test for comparison of two mean functions at significance levels 0.05 and 0.10. Displayed are the results for both the homoscedastic and heteroscedastic models based on 1000 simulations. The values in the parenthesis display the minimum and maximum power from Table 3 in Pardo-Fernández et al. (2007). The case of $n_1 = 200$ and $n_2 = 200$ was not considered in their paper.

Case	Homoscedastic models		Heteroscedastic models	
	$\alpha : 0.05$	0.10	0.05	0.10
$n_1 = 50$ and $n_2 = 50$				
(v)	981 (698, 972)	991 (816, 986)	606 (233, 638)	704 (369, 762)
(vi)	1000 (702, 969)	1000 (802, 986)	987 (233, 643)	997 (359, 753)
(vii)	964 (639, 963)	982 (779, 977)	596 (223, 626)	692 (355, 748)
(viii)	996 (158, 844)	999 (312, 922)	780 (086, 562)	870 (167, 692)
$n_1 = 100$ and $n_2 = 50$				
(v)	993 (858, 992)	998 (917, 998)	744 (341, 816)	820 (486, 886)
(vi)	1000 (839, 991)	1000 (901, 995)	1000 (343, 815)	1000 (481, 881)
(vii)	983 (809, 984)	996 (890, 994)	720 (321, 810)	814 (463, 870)
(viii)	999 (180, 901)	1000 (342, 960)	897 (136, 709)	953 (240, 808)
$n_1 = 100$ and $n_2 = 100$				
(v)	998 (969, 1000)	999 (989, 1000)	887 (445, 928)	931 (583, 962)
(vi)	1000 (964, 1000)	1000 (983, 1000)	1000 (440, 929)	1000 (578, 962)
(vii)	999 (952, 1000)	1000 (980, 1000)	876 (428, 920)	925 (566, 958)
(viii)	1000 (324, 998)	1000 (562, 998)	986 (136, 832)	994 (261, 932)
$n_1 = 200$ and $n_2 = 200$				
(v)	1000	1000	996	999
(vi)	1000	1000	1000	1000
(vii)	1000	1000	994	999
(viii)	1000	1000	1000	1000

Table 4

Rejection probabilities (multiplied by 1000) of the GLR test for comparison of three mean functions at significance levels 0.05 and 0.10 based on 1000 simulations. The values in the parenthesis display the powers of the four test statistics ($T_{KS}^1, T_{CM}^1, T_{KS}^2, T_{CM}^2$) from Table 5 in Pardo-Fernández et al. (2007).

Case	$\alpha : 0.05$	0.10
$n_1 = 50, n_2 = 50, n_3 = 50$		
(ix)	64 (49, 51, 41, 36)	121 (88, 96, 68, 72)
(x)	806 (818, 890, 274, 430)	849 (886, 929, 391, 555)
(xi)	843 (89, 77, 290, 475)	904 (190, 146, 434, 601)
$n_1 = 100, n_2 = 50, n_3 = 50$		
(ix)	61 (54, 58, 43, 31)	121 (90, 97, 86, 83)
(x)	912 (937, 975, 420, 596)	927 (975, 991, 541, 710)
(xi)	933 (140, 106, 462, 662)	971 (232, 193, 596, 763)
$n_1 = 100, n_2 = 100, n_3 = 50$		
(ix)	62 (50, 57, 39, 47)	120 (98, 107, 73, 94)
(x)	916 (929, 940, 365, 841)	931 (958, 962, 516, 912)
(xi)	952 (171, 90, 420, 495)	982 (283, 176, 569, 616)
$n_1 = 100, n_2 = 100, n_3 = 100$		
(ix)	56 (56, 61, 55, 59)	115 (109, 119, 88, 100)
(x)	975 (989, 993, 523, 760)	988 (996, 998, 677, 850)
(xi)	993 (201, 133, 576, 835)	997 (311, 236, 708, 907)

Table 5

Rejection probabilities (multiplied by 1000) of the GLR test for logistic regression setup. The results are for comparison of two mean functions at 5% and 10% significance levels.

	$n_1 = 200, n_2 = 200$		$n_1 = 200, n_2 = 300$		$n_1 = 300, n_2 = 300$	
α :	0.05	0.10	0.05	0.10	0.05	0.10
(i)	47	106	47	90	50	109
(ii)	51	101	46	85	55	116
(iii)	666	761	748	828	827	905
(iv)	640	740	713	823	835	917