Fractal dimensionality of Lévy processes

(mean first-passage times/maxima moments)

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ABSTRACT We determine the fractal dimensionality D of the trajectories of a class of translationally invariant Markov processes. We also provide two simple operational measures to estimate D .

Recently, considerable attention has been devoted to diverse physical phenomena exhibiting a clustered behavior in space or time domains (1–8). Examples of such clustered behavior appear in processes in fluid mechanics (1–4), solid state physics (ref. 1, p. 16; ref. 5), astrophysics (ref. 9, p. 170), data transmission systems (ref. 1, p. 93), and Brownian motion (ref. 1, p. 201). There is apparently a close connection between such clustered behavior in space or time and the Hausdorff–Besicovitch (fractal; ref. 1, p. 16) dimensionality of these processes.

The fractal or Hausdorff–Besicovitch dimensionality D of a set may be defined as follows. If a finite part Ω of the set is divided into N identical parts, each of which is geometrically similar to Ω with the similarity ratio r , then $D = \ell_n N / \ell_n(1/r)$. Analytic estimates of the fractal dimensionality of dynamical processes has so far been carried out only for Brownian motion (ref. 1, p. 201). In this paper, we examine the fractal dimensionality of the trajectory of a class of translationally invariant Markov processes and also provide two operational measures for estimating time.

A simple measure of the fractal dimensionality of a model dynamical process that exhibits clustering has been developed by Hughes *et al.* (10). The model is a discrete random walk on a lattice with transition probabilities drawn from a distribution which does not have a finite variance—i.e., a Lévy distribution (11). Hughes *et al.* were able to associate the short-scale behavior of the structure function to the fractal dimensionality of the walk. In the continuous space–time limit their model process reduces to a special case of the translationally invariant processes considered herein.

The distribution functions of translationally invariant Markov processes that we consider satisfy the Bachelier–Smoluchowski–Chapman–Kolmogorov (BSCK) chain condition and are called Lévy processes (12). Lévy processes in one, two, and three dimensions* are represented by their characteristic function $\phi(k) \sim \exp\{-|k|^\mu\}$. The exponent μ of the characteristic function determines the essential physical properties of the process. In particular, we show that μ is the fractal dimensionality of the trajectories of the process.

In this paper we propose the use of mean first-passage times and maxima moments as operational measures of the fractal dimensionality of the trajectories of Lévy processes. The idea of using mean first-passage times as a measure of the fractal dimensionality has its seed in the heuristic connection, established by Mandelbrot, between the fractal dimensionality of Brownian motion and the total time spent within a region of

space. The mean first-passage time, however, is not a sharply defined quantity because the first-passage time distribution is very broad (13, 14). Maxima moments, on the other hand, are a sensitive measure of the fractal dimensionality.

Lévy distributions

Translationally invariant stationary Markov processes in continuous space are described by probability densities that satisfy the Bachelier–Smoluchowski–Chapman–Kolmogorov chain condition,

$$P(x_2 - x_1, t) = \int_{-\infty}^{\infty} P(x_2 - x, t - \tau) P(x - x_1, \tau) dx \quad [2.1]$$

where $P(x_2 - x_1, t)$ is the probability that the value of the process X changes from x_1 to x_2 in time t . The characteristic function $\phi(k, t)$ is defined as the Fourier transform of the probability density—i.e.,

$$\phi(k, t) = \int_{-\infty}^{\infty} dx e^{ikx} P(x, t). \quad [2.2]$$

For processes satisfying the chain rule (Eq. 2.1), $\phi(k, t)$ obeys the product rule

$$\phi(k, t) = \phi(k, t - \tau) \phi(k, \tau). \quad [2.3]$$

Montroll and West (12) noticed that, since $\phi(k, t)$ satisfies rule 2.3, it is an infinitely divisible stable distribution (11).

The most general form of $\phi(k, t)$ for infinitely divisible stable distributions was obtained by Lévy (15) and Khinchine and Lévy (16). For symmetric processes, the most general form (11) is given by

$$\phi(k, t) = \exp \left[-bt |k|^\mu \left(1 + ic\omega(k, \mu) \frac{k}{|k|} \right) \right]. \quad [2.4]$$

In Eq. 2.4, μ , b , and c are constants obeying the restrictions $0 \leq \mu \leq 2$, $b \geq 0$ and $-1 \leq c \leq 1$. The function $\omega(k, \mu)$ is defined by

$$\begin{aligned} \omega(k, \mu) &= \tan(\pi\mu/2) & \text{if } \mu \neq 1 \\ &= \frac{2}{\pi} \ell_n |kbt| & \text{if } \mu = 1. \end{aligned} \quad [2.5]$$

The class of processes whose characteristic functions satisfy Eq. 2.4 are called Lévy processes. The most important feature of Lévy distributions is that, except for $\mu = 2$, the distributions $P(x, t)$ do not possess finite moments of all orders. This can be seen most easily from the fact (12) that, for $t > 0$,

* We shall henceforth denote the Euclidean dimension either by the symbol E or simply by the term “dimension.” The fractal or Hausdorff–Besicovitch dimension is always indicated either as such or by the symbol D .

$$\lim_{x \rightarrow \infty} P(x, t) \sim \mu b t \Gamma(\mu) \sin(\pi\mu/2) / \pi x^{\mu+1}. \quad [2.6]$$

Thus, all α moments defined by

$$\langle |x|^\alpha \rangle = \int_{-\infty}^{\infty} |x|^\alpha P(x, t) dx \quad [2.7]$$

are finite for $\alpha < \mu$ and are infinite for $\alpha \geq \mu$. In particular, the variance is infinite.

In two and three dimensions we restrict our attention to centrosymmetric Lévy distributions defined by the equation

$$\phi(k, t) = \phi(|k|, t) = \exp[-bt|k|^\mu] \quad [2.8]$$

where $|k| = (k_x^2 + k_y^2)^{1/2}$ in two dimensions and $|k| = (k_x^2 + k_y^2 + k_z^2)^{1/2}$ in three dimensions, respectively. The radially and spherically symmetric probability distribution functions in two and three dimensions are defined, through their respective Fourier transforms, via the relations

$$P(r, t) = \frac{1}{2\pi} \int_0^\infty e^{-bt k^\mu} J_0(kr) k dk; \quad 2E, \quad [2.9]$$

and

$$P(r, t) = \frac{1}{2\pi^2 r} \int_0^\infty e^{-bt k^\mu} \sin kr k dk; \quad 3E, \quad [2.10]$$

respectively. These Lévy distributions $P(r, t)$ are similar to the one-dimensional distribution $P(x, t)$ in that for $\mu < 2$ they also do not possess finite moments of all orders. It can be shown via an asymptotic analysis similar to the one carried out by Montroll and West (12) that, for $\mu < 2$, in two and three dimensions, $P(r, t)$ has the asymptotic form

$$P(r, t) \sim (2\pi)^{1-E} b t r^{-E-\mu}; \quad E = 2, 3. \quad [2.11]$$

From Eq. 2.11 it is clear that, for $\mu < 2$, all moments of order α defined by

$$\langle r^\alpha \rangle = (2\pi)^{E-1} \int_0^\infty r^\alpha P(r, t) r^{E-1} dr; \quad E = 2, 3, \quad [2.12]$$

are finite for $\alpha < \mu$ and infinite for $\alpha \geq \mu$.

The important feature of the asymptotic behavior (Eqs. 2.6 and 2.11) of Lévy distributions for $0 \leq \mu \leq 2$ is the power-law tail. From these asymptotic forms one can see that the probability $\mathcal{P}(r > \eta)$ that the process $r > \eta$, for large values of η , is

$$\begin{aligned} \mathcal{P}(r > \eta) &= (2\pi)^{E-1} \int_\eta^\infty P(r, t) r^{E-1} dr; \quad E = 1, 2, 3, \\ &\sim \text{const} \frac{1}{\eta^\mu}. \end{aligned} \quad [2.13]$$

Thus, the asymptotic behavior of the process r is that of a hyperbolic random variable. As pointed out by Mandelbrot such hyperbolic distributions preserve self-similarity and have trajectories with fractal dimensionality μ (ref. 1, p. 133). Thus, Lévy processes with exponent μ have trajectories with fractal dimensionality μ .

Even though the Lévy distributions are characterized by apparently simple Eq. 2.4 and 2.10, there are two major difficulties in understanding the physical properties of these distributions (12). First, the probability density $P(x, t)$ or $P(r, t)$ can be evaluated in a closed form only for special choices of the parameters μ and c . Apart from the diffusion and the Cauchy cases, a few other one-dimensional cases have been discussed by Zolotarev (see ref. 12). The second difficulty is that it is not

possible to construct simple evolution equations of the diffusion type for the probability density—i.e., equations involving $\frac{\partial P(x, t)}{\partial t}$. Such first-derivative equations, in general, contain integral operators in the position variables. For example, in one dimension when $\mu \neq 2$, the evolution equation can be shown (17) to be of the form

$$\begin{aligned} \frac{\partial P(x, t)}{\partial t} &= \frac{b}{\pi} \sin(\pi\mu/2) \Gamma(\mu + 1) \\ &\times \int_{-\infty}^{\infty} dy \{1 + c \operatorname{sign}(y - x)\} \frac{P(y, t)}{|x - y|^{\mu+1}}. \end{aligned} \quad [2.14]$$

In the next section, we demonstrate how formally simpler evolution equations possessing higher time derivatives can be obtained for certain values of the parameters μ and c .

Differential evolution equations

Under certain conditions, differential evolution equations can be obtained for the probability density which, in general, involve higher time derivatives. We first consider the one-dimensional case.

Let μ be a rational number equal to m/n where m and n are integers and the parameter c in Eq. 2.4 is selected to be zero. When m is even, we differentiate Eq. 2.4 successively n times with respect to time and inverse Fourier transform the resulting equation to obtain

$$\frac{\partial^n}{\partial t^n} P(x, t) = (-1)^{n+m/2} b^n \frac{\partial^m}{\partial x^m} P(x, t). \quad [3.1]$$

To ensure that the solutions of Eq. 3.1 are real and positive—i.e., they are probability densities—they must satisfy n initial conditions $P(x, 0)$, $P^{(1)}(x, t)|_{t=0}$, ..., $P^{(n-1)}(x, t)|_{t=0}$ (the superscripts denote the order of differentiation with respect to time). Though these sufficient initial conditions may be formally obtained from Eq. 2.4, they are, in general, as hard to evaluate as the inverse Fourier transform of the function $\phi(k, t)$ itself.

When m is odd, we differentiate Eq. 2.4 successively $2n$ times and, after inverse Fourier transforming, obtain

$$\frac{\partial^{2n}}{\partial t^{2n}} P(x, t) = (-1)^m b^{2n} \frac{\partial^{2m}}{\partial x^{2m}} P(x, t). \quad [3.2]$$

Analogous to the even m case, solutions of Eq. 3.2 must satisfy $2n$ initial conditions.

When $c \neq 0$, it can be shown that closed differential equations can be obtained for $\mu = m/n \neq 1$, with value of c given by

$$c = \tan[\pi p / 2n] / \tan[m\pi / 2n];$$

$$j = \pm 1, \pm 2, \dots \pm \infty, \quad p = \pm 1, \pm 2, \dots \pm jm. \quad [3.3]$$

The above construction can be extended to higher spatial dimensions.

Eqs. 3.1 and 3.2 are not particularly useful for computing $P(x, t)$ itself because they have to be solved subject to complicated initial conditions. However, we have been able to use them to compute properties such as the mean first-passage times and maxima moments. This utilization is discussed in the next two sections.

First-passage times for Lévy processes

Mandelbrot has discussed qualitatively the connection that exists between the fractal dimensionality of the Brownian process

and the average amount of time spent by the process in a given region of space. For persistent processes (ref. 9, p. 570; ref. 10) the average amount of time spent by the process in a given region of space is infinite and a quantitative connection with the fractal dimensionality is hard to establish. A close quantitative connection exists, however, between the mean first-passage time (for definitions, see refs. 9 and 14) to escape a given region of space and the Hausdorff-Besicovitch dimensionality. For example, for Brownian motion ($\mu = 2$) starting from the origin, the mean first-passage time $T_1(\xi)$ to reach

- i the points $\pm \xi$ 1E ,
- ii a circle of radius ξ 2E ,
- iii a sphere of radius ξ 3E

is proportional to ξ^2 (13). Thus, the mean first-passage time scales with ξ with exponent 2 which is also the fractal dimensionality of the Brownian trajectories (in Mandelbrot's terminology, the Brownian line to E trail). We show below that mean first-passage times of Lévy processes also exhibit a similar characteristic scaling behavior with respect to ξ . In fact, the exponent of ξ is precisely μ which suggests that the exponent of ξ is the Hausdorff-Besicovitch dimensionality. We establish this scaling relation by using exact boundary value techniques for $\mu = 2/n$, $n = 1, \dots, \infty$ and an approximate technique for arbitrary values of μ .

Exact Mean First-Passage Times. In general it is difficult to calculate the exact mean first-passage times of arbitrary Markov processes. However, when the equation of evolution for the probability density is a differential equation possessing only a Laplacian in spatial variables, the first-passage problem can be cast as a boundary value problem. Let us first consider one-dimensional processes.

The exact first-passage time to reach the point $x = \pm \xi$, starting from the origin, can be computed for processes which satisfy ($\mu = 2/n$):

$$\frac{\partial^n}{\partial t^n} P(x, t) = (-1)^{n+1} b^n \frac{\partial^2 P(x, t)}{\partial x^2}. \quad [4.1]$$

Let $f(x, t)$ be the probability that the process has not crossed the boundaries $x = \pm \xi$, given that it starts at the origin at time $t = 0$. From the theory of first-passage times (9) $f(x, t)$ satisfies Eq. 4.1 with the boundary conditions $f(x, 0) = \delta(x)$ and $f(\pm \xi, t) = 0$. In addition, the n initial conditions on the derivatives of $f(x, t)$ must be specified to ensure that $f(x, t)$ is real and positive. The solution $f(x, t)$, using the procedure detailed in ref. 13, is then given by

$f(x, t) =$

$$\frac{1}{\xi} \sum_{\ell=0}^{\infty} \cos \left[\frac{(2\ell+1)\pi x}{2\xi} \right] \exp \left[- \left\{ \frac{(2\ell+1)}{2\xi} \right\}^{2/n} b t \right]. \quad [4.2]$$

In Eq. 4.2 only the real positive eigenvalue in the boundary value problem has been chosen so that $f(x, t)$ is real and positive. The cumulative distribution function $F(\xi, t)$ is then

$$F(\xi, t) = \int_{-\xi}^{\xi} dx f(x, t) = \frac{4}{b\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)} \exp \left[- \left\{ \frac{(2\ell+1)\pi}{2\xi} \right\}^{2/n} b t \right]. \quad [4.3]$$

This result immediately yields the mean first-passage time to reach ξ ,

$$T_1(\xi) = \int_0^{\infty} F(\xi, t) dt = \xi^{2/n} \left[\frac{2^{2+2/n}}{\pi^{1+2/n}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)^{1+2/n}} \right]. \quad [4.4]$$

The most important feature of the behavior of the first-passage time $T_1(\xi)$ is embodied in the first factor $\xi^{2/n}$, $2/n \equiv \mu$. The second factor is a monotonic increasing factor of n . When $n = 1$, we recover the result for the diffusion process (13)—i.e.,

$$T_1(\xi) = \xi^2/2b. \quad [4.5]$$

The exponent 2 of ξ is the same as the fractal dimensionality of the Brownian trail. When $n = 2$, we get the result for the Cauchy process,

$$T_1(\xi) = \xi \left[\frac{8G}{b\pi^2} \right], \quad [4.6]$$

where G is the Catalan's constant ($=0.915956 \dots$). The exponent of ξ is consistent with Mandelbrot's identification of the fractal dimensionality of the Cauchy trail as unity.

The scaling behavior, proved above for one dimension, can be extended to higher dimensions. For example, in two dimensions the mean first-passage time $T_1(\xi)$ to a circle of radius ξ , and in three dimensions to a sphere of radius ξ , can be computed from the radially and spherically symmetric cumulative distribution functions $F(\xi, t)$ given in ref. 13, respectively. The scaling result is $T_1(\xi) \sim \xi^\mu$, $\mu = 2/n$, as obtained above.

Approximate $T_1(\xi)$. An exact evaluation of $T_1(\xi)$ for arbitrary μ is, in general, complicated for two reasons. First, if μ is irrational there are no differential evolution equations. Second, even if μ is rational but cannot be expressed as $2/n$, the evolution equation possesses spatial derivatives of order higher than 2. For such cases, the boundary value technique as used above no longer applies. Hence, we proceed by using an approximation technique as follows.

The approximation procedure rests on the assumption that one can replace the conditional probability density $f(x, t)$ by $P(x, t)$ in Eq. 4.3. Such a replacement yields the correct scaling behavior of $T_1(\xi)$ but not necessarily the correct coefficient. For transient processes, such a replacement yields a coefficient of ξ^μ which is larger than the correct one but is finite. For persistent processes, the approximate coefficient turns out to be infinite. To see this, consider the one-dimensional case

$$T_1(\xi) = \int_0^{\infty} F(\xi, t) dt \approx \int_0^{\infty} dt \int_{-\xi}^{\xi} P(x, t) dx. \quad [4.7]$$

For a Lévy process, we substitute for $P(x, t)$ the Fourier transform of Eq. 2.4 to obtain in terms of the scaled variables $\hat{x} = x/\xi$, $\hat{k} = k\xi$, $\hat{t} = (bt)/\xi^\mu$,

$$T_1(\xi) \approx \frac{\xi^\mu}{b} \quad [4.8]$$

$$\times \left[\frac{1}{2\pi} \int_0^{\infty} d\hat{t} \int_{-1}^1 d\hat{x} \int_{-\infty}^{\infty} d\hat{k} e^{-i\hat{k}\hat{x}} e^{-i|\hat{k}|^\mu \hat{t}} \left(1 + ic \omega(\hat{k}, \mu) \frac{\hat{k}}{|\hat{k}|} \right) \right].$$

Thus, Eq. 4.8 has the correct scaling behavior. The coefficient of ξ^μ , however, is finite only when $\mu < 1$, as can be seen by examining the singularities in the integrand of Eq. 4.8 after integrating over $\hat{t}$. The integral is finite only for transient processes ($\mu < 1$) and is infinite for persistent processes ($\mu > 1$) (ref. 9, p. 570; ref. 10).

An entirely analogous argument can be presented for the two-

and three-dimensional cases. In two and three dimensions, we replace $F(\xi, t)$ by

$$F(\xi, t) \equiv \int_0^\xi P(r, t) r^{E-1} dr; \quad E = 2, 3. \quad [4.9]$$

In both these cases the approximation procedure carried out above yields the correct scaling behavior $T_1(\xi) \sim \xi^\mu$. The approximate coefficient of ξ^μ is finite for $0 \leq \mu < 2$ in two dimensions but is finite for all μ , $0 \leq \mu \leq 2$ in three dimensions. Thus, the approximation procedure can always be applied to transient processes.

The results obtained above are for continuous Markov processes. Weiss and Rubin (18) have considered discrete random walk processes with transition probabilities with infinite variance and have obtained scaling results for mean first-passage times, similar to Eqs. 4.4 and 4.8. In the continuum limit their results agree with those given in this section. The following point should be noted, however. In discrete random walk models, with transition probabilities with infinite variance, there exists a difference equation of evolution on the lattice. In the continuum limit of these models, no closed differential equation with $\partial P / \partial t$ exists. This is precisely the reason for constructing closed differential equations with higher order time derivatives (cf. Eq. 4.1).

Maxima moments for Lévy processes

A disadvantage of using the mean first-passage time as an indicator of the fractal dimensionality of a process is that it is not a sharp measure (14). An alternative property of a Lévy process which is a sensitive measure of the fractal dimensionality is the distribution of the maxima attained in a given time t (for a definition of maxima, see ref. 14). For Brownian motion ($\mu = 2$) all maxima moments are finite and well defined. However, this is not the case for Lévy processes with $\mu < 2$. In this latter case, maxima of certain order only are finite. This is to be expected since the distribution $P(x, t)$ or $P(r, t)$ for $\mu < 2$ has a power-law tail.

We define the fractional moment of order α of the maxima distribution as

$$Z_\alpha(t) = \alpha \int_0^\infty d\xi \xi^{\alpha-1} [1 - F(\xi, t)]. \quad [5.1]$$

For Lévy processes, $Z_\alpha(t)$ is finite iff $\alpha < \mu$, $\mu < 2$, and is infinite otherwise. Equally interesting is the scaling behavior of $Z_\alpha(t)$ —i.e.,

$$Z_\alpha(t) \sim t^{\alpha/\mu}, \quad \alpha < \mu. \quad [5.2]$$

These results can be demonstrated in an exact fashion when the equation of evolution has the Laplacian form (Eq. 4.1).

We first consider the one-dimensional case. Substituting the expression from Eq. 4.3 into Eq. 5.1 and Laplace transforming both sides, we obtain ($\mu = 2/n$)

$$\tilde{Z}_\alpha(\varepsilon) = \frac{4\alpha}{\varepsilon \pi^{1-\mu}} \int_0^\infty d\xi \xi^{\alpha-1} \sum_{\ell=0}^\infty \frac{(-1)^\ell (2\ell+1)^{\mu-1}}{[\varepsilon b^{-1} (2\xi)^\mu + (\pi(2\ell+1))^\mu]}. \quad [5.3]$$

In terms of the scaled variable $\hat{\xi} = (\varepsilon b^{-1})^{1/\mu} \xi$ we can write Eq. 5.3 as

$$\tilde{Z}_\alpha(\varepsilon) = \frac{A_\alpha}{\varepsilon^{1+\alpha/\mu}} \quad [5.4]$$

where

$$A_\alpha = 4\alpha \pi^{\mu-1} b^{\alpha/\mu} \int_0^\infty d\hat{\xi} \hat{\xi}^{\alpha-1} \sum_{\ell=0}^\infty \frac{(-1)^\ell (2\ell+1)^{\mu-1}}{(2\hat{\xi})^\mu + (\pi(2\ell+1))^\mu}. \quad [5.5]$$

It is immediately clear from the inverse Laplace transform of Eq. 5.4 that

$$Z_\alpha(t) = \frac{A_\alpha}{\Gamma(\alpha/\mu)} t^{\alpha/\mu}. \quad [5.6]$$

The coefficient A_α , however, is finite only when $\alpha < \mu$, as can be demonstrated by performing the sum in Eq. 5.5 by a contour integration.

In two and three dimensions the scaling of $Z_\alpha(t)$ with t —i.e., $Z_\alpha(t) \sim t^{\alpha/\mu}$ —can be established by a similar procedure. Below we present a heuristic argument valid in each of the three dimensions and also valid for all values of μ , $0 \leq \mu < 2$ to show that A_α diverges for $\alpha \geq \mu$.

We proceed by splitting the integral Eq. 5.1 as

$$Z_\alpha(t) = \alpha \int_0^\eta d\xi \xi^{\alpha-1} [1 - F(\xi, t)] + \alpha \int_\eta^\infty d\xi \xi^{\alpha-1} [1 - F(\xi, t)] \quad [5.7]$$

where $F(\xi, t)$ is the appropriately defined cumulative distribution function in one, two, and three dimensions (cf. Eqs. 4.3 and 4.9). In Eq. 5.7 η is arbitrarily large. The first integral is finite for any finite value of η and the divergence of $Z_\alpha(t)$, if any, arises from the second integral. For any fixed time t , for sufficiently large values of η we may replace $[1 - F(\xi, t)]$ by

$$\sim \int_\xi^\infty r^{E-1} P(r, t) dr; \quad E = 1, 2, 3. \quad [5.8]$$

The asymptotic behavior of $P(r, t)$ given by Eqs. 2.6 and 2.11 immediately leads to the result

$$\lim_{\xi \rightarrow \infty} [1 - F(\xi, t)] \sim 1/\xi^\mu \quad [5.9]$$

in each of the three cases. Substitution of Eq. 5.9 into the second integral of Eq. 5.7 yields the result that maxima moments of order α are finite iff $\alpha < \mu$, and are infinite for $\alpha \geq \mu$.

Conclusions

Herein we summarize our main results and discuss some of their physical implications:

(i) Lévy processes in one, two, and three dimensions obey the asymptotic law $P(r > \eta) \sim \eta^{-\mu}$. Thus, the exponent μ is the fractal dimensionality of the Lévy trajectories.

(ii) Differential evolution equations for the probability density have been obtained for Lévy processes with certain parameter values. These evolution equations are, in general, not of the diffusion type and involve higher-order derivatives in space as well as time. For Lévy processes with second degree spatial derivatives, we have obtained exact mean first-passage times and maxima moments.

(iii) The mean first-passage time to a boundary ξ has been shown to scale as ξ^μ in one, two, and three dimensions, $0 \leq \mu \leq 2$. Thus, the scaling of the mean first-passage times can serve as an operational measure of the fractal dimensionality of the process.

(iv) The maxima moment $Z_\alpha(t)$ in a time interval t has been

shown to scale as $t^{\alpha/\mu}$ in one, two, and three dimensions and are finite as long as $\alpha < \mu$ and are infinite when $\alpha \geq \mu$, $\mu < 2$. These moments are sensitive measures of μ and hence of the fractal dimensionality of the process.

(v) The fractal dimensionality of other properties of Lévy processes can also be discussed in terms of the characteristic function. We present heuristic arguments for determining the fractal dimensionality of the zero-crossing set and the graphs of such processes.

Zero-Crossing Set. A simple scaling calculation shows that the probability of occupation of the origin of a Lévy process is $P(0, t) \sim t^{-1/\mu}$. The cumulative time spent at the origin between times 0 and t is proportional to $t^{1-1/\mu}$, up to an additive constant. We associate the fractal dimensionality of the zero-set with the scaling of the time spent at the origin for $t > 0$ —i.e., $D = 1 - 1/\mu$. This association is valid for $\mu \geq 1$. This result is consistent with that of the Brownian process for which $\mu = 2$, $D = 1/2$ and also that of the Cauchy process for which $\mu = 1$, $D = 0$. For all values of $\mu < 1$, $D = 0$.

Graphs of Lévy Processes. We argue that the fractal dimensionality of the graph $X(t)$ vs. t for a one-dimensional Lévy process is $D = 2 - 1/\mu$, $\mu > 1$. The argument is similar in spirit to the one presented by Mandelbrot, wherein he uses the Lipschitz condition. Instead of the Lipschitz condition, we use the result for the mean magnitude of displacement for Lévy graphs, $\langle |X(t) - X(0)| \rangle \sim t^{1/\mu}$, $t > 0$, $\mu > 1$. We note that the number of square boxes of side r required to cover the length of the graph between t and $t + r$ is equal to $r^{1/\mu-1} \left(\frac{\text{area of the graph}}{\text{area of the box}} = \frac{r^{1/\mu} r}{r^2} \right)$. The total number of boxes required to cover the graph between $t = 0$ and $t = 1$ is $N = \frac{r^{1/\mu-1}}{r} = r^{1/\mu-2}$. The fractal dimensionality is then $D = \log N / \log (1/r) = 2 - 1/\mu$, $\mu > 1/2$. Thus, for the Brownian motion

$\mu = 2$, $D = 3/2$ which agrees with known results. When μ decreases from 2 to $1/2$, D decreases monotonically from $3/2$ to 0. For $\mu \leq 1/2$, $D = 0$.

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