On a Nonlinear Master Equation and the Haken-Kelso-Bunz Model

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Abstract. A nonlinear master equation (NLME) is proposed based on general information measures. Classical and cut-off solutions of the NLME are considered. In the former case, the NLME exhibits uniquely defined stationary distributions. In the latter case, there are multiple stationary distributions. In particular, for classical solutions, it is shown that transient solutions converge to stationary distributions that maximize information measures (H-theorem). Cut-off distributions are studied numerically for the Haken-Kelso-Bunz model. The Haken-Kelso-Bunz model is known to describe multistable human motor control systems. It is shown that a stochastic Haken-Kelso-Bunz model based on a NLME can exhibit multiple stationary cut-off distributions. In doing so, we illustrate that multistability in stochastic biological systems can be established by means of cut-off distributions.

Key words: Haken-Kelso-Bunz model, information measures, nonlinear master equation, q -information

1. Introduction

A fundamental assumption of standard thermodynamics is that long-range interactions between macroscopic parts of the systems being studied can be neglected. From this assumption it follows that the entropy measure of these systems, the so-called Boltzmann-Gibbs-Shannon (BGS) entropy and information measure, is an extensive or additive quantity, cf. [39, Chap. 3] or [31, Chap. 1]. The breakdown of standard thermodynamics in the context of statistical systems with long-range interactions has repeatedly been pointed out, cf., e.g., *Terletsii* [45, Chap. 7], *Haken* [24, Sec. 4.11], and *Landsberg* [33]. With the advent of an one-parametric nonextensive entropy proposed by *Tsallis* [46], generalized thermodynamical concepts have been developed aiming at descriptions of nonextensive systems characterized by equilibrium statistics different from the extensive Boltzmann-Gibbs-Shannon statistics [1, 3, 4, 6, 34, 35, 37, 48]. In this context, transient stochastic processes that converge to generalized equilibrium distributions have been described in terms of Fokker-Planck equations which are nonlinear with respect to their probability densities [7, 8, 10, 13, 15, 16, 18, 20, 36, 38, 49]. Similar stochastic descriptions in terms of master equations, however, have not been discussed so far. In this article,

we discuss a nonlinear master equation (NLME) that exhibits multiple or uniquely defined stationary distributions different from Boltzmann distributions. We are in particular interested in the application of this nonlinear master equation to biological systems such as human motor control systems. Therefore, our departure point is information theory rather than phenomenological thermodynamics. Accordingly, the proposed NLME is based on information measures. In line with information theory [26, 29, 42], we assume that biological systems converge to stationary states that are characterized by a maximum of information and are consistent with the constraints imposed by their environments. In line with the aforementioned recent development of a nonextensive thermodynamics, and in line with earlier developments in information theory [2, 40], we will consider general information measures that may include the BGS entropy and information measure as a special case.

This article is organized as follows. First, we will discuss a NLME for classical probability distributions. By generalizing the H-theorem known from the theory of linear master equations, we will demonstrate the convergence of transient solutions to uniquely defined stationary distributions. Subsequently, we will shift the focus to cut-off solutions. To this end, the NLME for classical solutions will be generalized. The existence of stationary cut-off distributions and the role of cut-off distributions for stochastic multistable biological systems will be illustrated by means of the Haken-Kelso-Bunz model and the q -information measure proposed by *Aczel* and *Daroczy* [2] and *Tsallis* [46].

2. On a Nonlinear Master Equation for Classical Distributions

2.1. DEFINITION OF THE NONLINEAR MASTER EQUATION

Let $p_i(t) \in [0, 1]$ denote the elements of a complete set of probabilities with $i = 1, \dots, N$ observed at time t (i.e., $\sum_{i=1}^N p_i(t) = 1$). Let S denote a general concave information measure of the form

$$S = \sum_{i=1}^M s(p_i) \quad \text{with} \quad \left. \frac{d^2 s(x)}{dx^2} \right|_z < 0 \quad \text{for } z \in (0, 1], \quad (1)$$

where p_i denote a complete set of probabilities $p_i(t)$ (i.e., $\forall t : \sum_{i=1}^N p_i(t) = 1$). We assume that the kernel $s(x)$ is an infinitely differentiable function defined on the interval $(0, 1]$. The inequality in Eq. (1) implies the concavity of the information measure S [53]. In addition to the concavity property, we require that the derivative $g(x) = ds(x)/dx$ is unbounded in the limit of vanishing probabilities, that is, we have

$$x \rightarrow 0 \Rightarrow g(x) \rightarrow \infty. \quad (2)$$

As we will show below, this property guarantees that the probabilities $p_i(t)$ cannot assume negative values. We study now the evolution of the probability distribution $\{p_i(t)\}_{i=1,\dots,N}$ described by the NLME

$$\dot{p}_i = \varphi_i(p_1, \dots, p_N; S) , \quad t \geq t_0 \quad (3)$$

under the initial condition $\{p_i(t_0)\}_{i=1,\dots,N}$. In what follows, we assume that $p_i(t_0) > 0$ holds for all states i . The dot on the left hand side of Eq. (3) denotes the derivative with respect to t . The function φ_i depends on the information measure S via the kernel s and is defined by

$$\varphi_i(p_1, \dots, p_N; S) = \sum_{k=1}^N A_{i,k} f(p_k) - f(p_i) \sum_{k=1}^N A_{k,i} \quad (4)$$

with

$$f(x) = \frac{1}{e} \exp \left\{ - \left. \frac{ds(z)}{dz} \right|_x \right\} , \quad (5)$$

where e stands for the Euler constant $\exp\{1\}$. Finally, we require the positivity of the matrix elements $A_{i,k}$ (i.e., $\forall i, k : A_{i,k} > 0$). In this article the elements of $A_{i,k}$ are considered as constants, that is, they neither depend on the probabilities p_i nor on the time variable t . (In this regard, a more general NLME is discussed in, e.g., [15].) For the BGS information measure $^{\text{BGS}}S = - \sum_{i=1}^N p_i \ln p_i$, we obtain $g(x) = ds(x)/dx = -\ln(x) - 1$. In this case, the NLME defined by Eqs. (4) and (5) recovers the conventional linear master equation. In this special case, the matrix elements $A_{i,k}$ describe transition rates between the individual states of the system in question. The requirement $\forall i, k : A_{i,k} > 0$, which has been formulated above, implies that all states are connected. In the context of linear master equations, this property guarantees the existence of unique stationary solutions, cf., e.g., *Haken* [24, Sec. 4.9] or *van Kampen* [52, Sec. 5.3]. We refer to the evolution equation (3,...,5) as a *nonlinear* master equation because the right hand side of Eq. (3) depends, in general, in a nonlinear fashion on the probabilities $p_i(t)$.

We refer to positive definite probability distributions (i.e., distributions with $p_i > 0$ for all i) as classical distributions. We demonstrate now that for any classical initial distribution $p_i(t_0) \in (0, 1]$ with $\sum_{i=1}^N p_i(t_0) = 1$ the relations $p_i(t) \in (0, 1]$ and $\sum_{i=1}^N p_i(t) = 1$ hold for $t \geq t_0$. That is, for classical initial distributions, the solutions of the NLME (3) correspond to (time-dependent) classical probability distributions, again. The normalization of the probability distribution $\{p_i(t)\}_{i=1,\dots,N}$ can be verified using Eqs. (3) and (4). From Eq. (4), we obtain $\sum_{i=1}^N \varphi_i = 0$. Then, Eq. (3) implies that $d \sum_{i=1}^N p_i / dt = 0$. Next, we show that the probabilities $\{p_i(t)\}_{i=1,\dots,N}$ are positive definite for $t \geq t_0$ provided that the initial distribution $\{p_i(t_0)\}_{i=1,\dots,N}$ has only positive probabilities (i.e., $\forall i : p_i(t_0) > 0$). To this end, we exploit a fundamental property of first order differential

equations. Let $\dot{q}(t) = h(q)$ with $h \in C^\infty(0, \infty)$ describe the evolution of an one-dimensional state variable q and let us further assume that the conditions $q(t_0) > 0$ and $\lim_{z \rightarrow 0} h(z) > 0$ hold. Then, we can conclude that $q(t)$ is positive for all times $t \geq t_0$. This proposition can be proven, for example, by considering the evolution of $q(t)$ as an overdamped motion in a potential $V(z)$ with $h(z) =: -dV(z)/dz$ and $\lim_{z \rightarrow 0} dV(z)/dz < 0$. The latter inequality implies that the slope of the potential close to the origin is negative and that $q(t)$ has to ‘climb up’ the potential in order to approach the origin when being initially located in the domain of positive numbers. This is inconsistent with the notion of an overdamped motion, that is, with a gradient dynamics. Consequently, the state variable $q(t)$ is confined to positive numbers. Now, let $m(t)$ denote a particular index for which at time t the relation $\forall l \in \{1, 2, \dots, N\} : p_{m(t)} \leq p_l$ holds. There might be more than one such index (degeneracy of smallest probabilities), which does, however, not affect the following proof. In line with the above considerations, the positivity of the probabilities $\{p_i(t)\}_{i=1, \dots, N}$ is guaranteed if the right hand side of Eq. (3) satisfies for $i = m(t)$ the condition $p_i \rightarrow 0 \Rightarrow \varphi_i > 0$. Since φ_i does not explicitly depend on the time variable t , it is sufficient to prove this inequality for one particular instance t^* and we can simply write m instead of $m(t^*)$. The limit $\lim_{p_m \rightarrow 0} \varphi_m$ can be computed from Eqs. (2), (4), and (5) and we obtain

$$\begin{aligned}
\lim_{p_m \rightarrow 0} \varphi_m &= \lim_{p_m \rightarrow 0} \frac{1}{e} \left\{ \sum_{k=1}^N A_{m,k} \exp \{-g(p_k)\} - \exp \{-g(p_m)\} \sum_{k=1}^N A_{k,m} \right\} \\
&= \lim_{p_m \rightarrow 0} \frac{1}{e} \left\{ \sum_{k=1, k \neq m}^N A_{m,k} \exp \{-g(p_k)\} - \exp \{-g(p_m)\} \sum_{k=1, k \neq m}^N A_{k,m} \right\} \\
&= \frac{1}{e} \sum_{k=1, k \neq m}^N A_{m,k} \exp \{-g(p_k)|_{p_m \rightarrow 0}\} - \frac{1}{e} \exp \left\{ -\lim_{p_m \rightarrow 0} g(p_m) \right\} \sum_{k=1, k \neq i}^N A_{k,m} \\
&= \frac{1}{e} \sum_{k=1, k \neq m}^N A_{m,k} \exp \{-g(p_k)|_{p_m \rightarrow 0}\} . \tag{6}
\end{aligned}$$

Since the conditions $\sum_{l=1}^N p_l(t) = 1$ and $\forall l : p_l \geq p_m$ hold in the limit $p_m \rightarrow 0$, there exists at least one nonvanishing probability $p_{l^*} > 0$ with $l^* \neq m$. As mentioned earlier, we assume that $\forall i, k : A_{i,k} > 0$ holds. We can then conclude that there is at least one nonvanishing term $A_{i,l^*} \exp \{-g(p_{l^*})|_{p_m \rightarrow 0}\}$ of the sum $\sum_{k=1, k \neq i}^N A_{i,k} \exp \{-g(p_k)|_{p_m \rightarrow 0}\}$, which implies $\lim_{p_m \rightarrow 0} \varphi_m > 0$. In sum, we have shown that for $t \geq t_0$ the relations $\sum_{i=1}^N p_i(t) = 1$ and $\forall i : p_i(t) > 0$ are satisfied. From this it also follows that $\forall i : p_i(t) \leq 1$ for $t \geq t_0$.

2.2. H-THEOREM AND UNIQUENESS OF STATIONARY SOLUTIONS

In this section we assume the existence of at least one normalized stationary solution $\{p_{i,\text{st}}\}_{i=1,\dots,N}$ with $p_{i,\text{st}} \in (0, 1]$ for all states i . This probability distribution satisfies $\forall i : \varphi_i(p_{1,\text{st}}, \dots, p_{N,\text{st}}; S) = 0$ or, alternatively,

$$\forall i : \sum_{k=1}^N A_{i,k} \exp \{-g(p_{k,\text{st}})\} = \exp \{-g(p_{i,\text{st}})\} \sum_{k=1}^N A_{k,i} . \quad (7)$$

We claim that the assumed stationary solution is a unique stationary solution of the NLME (3,...,5) and that arbitrary classical initial distributions $\{p_i(t_0)\}_{i=1,\dots,N}$ with $\forall i : p_i(t_0) \in (0, 1]$ converge to this distribution in the limit $t - t_0 \rightarrow \infty$. Consequently, if we succeed in solving Eq. (7) with respect to the probabilities $p_{i,\text{st}}$, then the solution obtained is the only stationary solution of the NLME (3,...,5). We prove our proposition by means of an H-theorem. In line with a similar study on an H-theorem for nonlinear Fokker-Planck equations [19], we generalize the conventional Kullback distance measure [32] according to

$$K(\{p_i\}, \{q_i\}; S) = -S(\{p_i\}) + S(\{q_i\}) + \sum_{i=1}^N \frac{ds(x)}{dx} \Big|_{q_i} (p_i - q_i) , \quad (8)$$

where $\{p_i\}_{i=1,\dots,N}$ and $\{q_i\}_{i=1,\dots,N}$ denote two complete sets of probabilities. Note that alternative definitions for generalized Kullback measures have also been discussed [9, 44, 47]. The generalized Kullback measure $K(\{p_i\}, \{q_i\}; S)$ is semi-positive definite for every concave information measure S of the form (1) because $K(\{p_i\}, \{q_i\}; S)$ simply represents the concavity inequality of S . That is, from the inequality $\forall x > 0, y > 0 : k(x, y) = -s(x) + s(y) + \sum_{i=1}^N ds(y)/dy \cdot (x - y) \geq 0$ it follows that $K(\{p_i\}, \{q_i\}; S) = \sum_{i=1}^N k(p_i, q_i) \geq 0$. For the special case of the BGS information ^{BGS}S , the generalized Kullback measure $K(\{p_i\}, \{q_i\}; S)$ recovers the conventional Kullback measure, that is, $K(\{p_i\}, \{q_i\}; ^{BGS}S) = \sum_{i=1}^N p_i \ln(p_i/q_i)$. We show now that the relations

$$\frac{d}{dt} K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) \leq 0 , \quad (9)$$

$$\frac{d}{dt} K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) = 0 \Leftrightarrow \forall i : p_i(t) = p_{i,\text{st}} \quad (10)$$

are satisfied when the evolution of $p_i(t)$ is governed by the NLME (3,...,5). Eqs. (9) and (10) in combination with $K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) \geq 0$ constitute an H-theorem for the NLME (3,...,5) which states that any initial distribution $\{p_i(t_0)\}_{i=1,\dots,N}$ converges to the stationary distribution $\{p_{i,\text{st}}\}_{i=1,\dots,N}$. On account of the implication $dK(\{p_i(t)\}, \{p_{i,\text{st}}\}; S)/dt = 0 \Rightarrow \forall i : p_i(t) = p_{i,\text{st}}$ we can further conclude that $\{p_{i,\text{st}}\}_{i=1,\dots,N}$ is the only stationary distribution of the NLME. (Consider a second stationary distribution $\{p'_{i,\text{st}}\} \neq \{p_{i,\text{st}}\}$. Next, identify $\{p'_{i,\text{st}}\}$ with $\{p_i(t)\}$. From the implication $dK(\{p_i(t)\}, \{p_{i,\text{st}}\}; S)/dt = 0 \Rightarrow \forall i : p_i(t) = p_{i,\text{st}}$ it then follows

that $\{p'_{i,\text{st}}\} \equiv \{p_{i,\text{st}}\}$, which is in contradiction with the previously made assumption $\{p'_{i,\text{st}}\} \not\equiv \{p_{i,\text{st}}\}$.)

Let us derive the inequality (9). To this end, we differentiate the generalized Kullback measure (8) with respect to t and obtain

$$\frac{d}{dt} K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) = - \sum_{i=1}^N [g(p_i) - g(p_{i,\text{st}})] \dot{p}_i . \quad (11)$$

For the sake of convenience, we introduce the variables $\tilde{p}_i(t) = \exp\{-g(p_i) - 1\} = f(p_i)$ and $\tilde{p}_{i,\text{st}}(t) = \exp\{-g(p_{i,\text{st}}) - 1\} = f(p_{i,\text{st}})$. Then, we can rewrite Eqs. (4) and (11) as

$$\begin{aligned} \frac{d}{dt} K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) &= - \sum_{i=1}^N \ln\left(\frac{\tilde{p}_{i,\text{st}}}{\tilde{p}_i}\right) \varphi_i, \\ \varphi_i &= \sum_{k=1}^N A_{i,k} \tilde{p}_k - \tilde{p}_i \sum_{k=1}^N A_{k,i} . \end{aligned} \quad (12)$$

Eliminating φ_i , we obtain

$$\begin{aligned} \frac{d}{dt} K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) &= - \sum_{i,k=1}^N A_{i,k} \tilde{p}_k(t) \ln\left(\frac{\tilde{p}_{i,\text{st}}}{\tilde{p}_i}\right) + \sum_{i,k=1}^N A_{k,i} \tilde{p}_i(t) \ln\left(\frac{\tilde{p}_{i,\text{st}}}{\tilde{p}_i}\right) \\ &= - \sum_{i,k=1}^N A_{i,k} \tilde{p}_k(t) \left[\ln\left(\frac{\tilde{p}_{i,\text{st}}}{\tilde{p}_i}\right) - \ln\left(\frac{\tilde{p}_{k,\text{st}}}{\tilde{p}_k}\right) \right] . \end{aligned} \quad (13)$$

In line with the H-theorem for linear master equations, cf. *van Kampen* [52, Sec. 5.5], we introduce the new variables $x_i(t) = \tilde{p}_i(t) / \tilde{p}_{i,\text{st}}$ and rewrite Eq. (13) as

$$\frac{d}{dt} K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) = - \sum_{i,k=1}^N \underbrace{A_{i,k} \tilde{p}_{k,\text{st}}}_{> 0} x_k \ln\left(\frac{x_k}{x_i}\right) . \quad (14)$$

Note that the transformed probabilities $\tilde{p}_i(t)$ and $\tilde{p}_{i,\text{st}}$ are positive which implies that $x_i(t)$ can assume only positive values. Using the inequality $x_k \ln(x_k / x_i) \geq x_k - x_i$ (see, e.g., [43, Sec 3.A]), which holds for any pair $x_i > 0$ and $x_k > 0$, we obtain from Eq. (14) the inequality

$$-\frac{d}{dt} K(\{p_i(t)\}, \{p_{i,\text{st}}\}; S) \geq \sum_{i,k=1}^N A_{i,k} \tilde{p}_{k,\text{st}} (x_k - x_i) . \quad (15)$$

On account of the identity $\sum_{k=1}^N A_{i,k} \tilde{p}_{k,\text{st}} = \tilde{p}_{i,\text{st}} \sum_{k=1}^N A_{k,i}$ (cf. Eq. (7)), the left hand side of Eq. (15) vanishes, that is, $\sum_{i,k=1}^N A_{i,k} \tilde{p}_{k,\text{st}} (x_k - x_i) = 0$ (cf. also *van*

Kampen [52, Sec. 5.5.]). Consequently, we arrive at the inequality $dK(\{p_i(t)\}, \{p_{i,st}\}; S)/dt \leq 0$, which is the proposition (9).

We turn now our attention to Eq. (10). The implication $\forall i : p_i(t) = p_{i,st} \Rightarrow dK(\{p_i(t)\}, \{p_{i,st}\}; S)/dt = 0$ follows from the fact that the generalized Kullback measure (8) does not depend explicitly on the time variable t . We still need to show the opposite implication which states that if the relation $dK(\{p_i(t)\}, \{p_{i,st}\}; S)/dt = 0$ holds, then we find $\forall i : p_i(t) = p_{i,st}$, where the stationary distribution $\{p_{i,st}\}_{i=1,\dots,N}$ satisfies Eq. (7). From Eq. (14) we obtain $dK(\{p_i(t)\}, \{p_{i,st}\}; S)/dt = 0 \Rightarrow \sum_{i,k=1}^N A_{i,k} \tilde{p}_{k,st} x_k \ln(x_k/x_i) = 0$. We can exploit now a technique used in the theory of linear master equations, cf., e.g., [52, Sec. 5.5], and add a zero in the form $\sum_{i,k=1}^N A_{i,k} \tilde{p}_{k,st} (x_i - x_k) = 0$ to the result $\sum_{i,k=1}^N A_{i,k} \tilde{p}_{k,st} x_k \ln(x_k/x_i) = 0$. Thus, we obtain

$$0 = \sum_{i,k=1}^N \underbrace{A_{i,k} \tilde{p}_{k,st}}_{> 0} \underbrace{\left[x_k \ln\left(\frac{x_k}{x_i}\right) + (x_i - x_k) \right]}_{\omega_{i,k} \geq 0}. \quad (16)$$

The expressions $\omega_{i,k}$ represent again the concavity inequality of the logarithm (i.e., $\forall z > 0 : \ln(1+z) \leq z$, e.g., [43, Sec 3.A]). Since this concavity inequality states that $\omega_{i,k} > 0$ for $x_k \neq x_i$ ($x_i > 0, x_k > 0$) and $\omega_{i,k} = 0$ for $x_k = x_i$, Eq. (16) is satisfied if and only if $\forall i, k : \omega_{i,k} = 0$, which implies that $\forall i, k : x_i = x_k$. Using the transformed probabilities $\tilde{p}_i(t)$ and $\tilde{p}_{i,st}$, this identity can equivalently be expressed as $\forall i, k : \tilde{p}_k(t)\tilde{p}_{i,st} = \tilde{p}_i(t)\tilde{p}_{k,st}$. Inserting this result into φ_i as given by Eq. (12) and taking Eq. (7) into account, we find that $\forall i : \dot{\varphi}_i = 0$. Consequently, we get $\forall i : \dot{p}_i = 0$. In sum, if the derivative of the generalize Kullback measure $K(\{p_i(t)\}, \{p_{i,st}\}; S)$ with respect to the time variable t vanishes, then the probabilities $p_i(t)$ correspond to a stationary solution of the NLME. Let us denote this stationary solution with $\{p'_{i,st}\}_{i=1,\dots,N}$. We need to show now that $\{p'_{i,st}\}_{i=1,\dots,N}$ agrees with the assumed stationary distribution $\{p_{i,st}\}_{i=1,\dots,N}$. To this end, we can exploit the relation $\forall i, k : \tilde{p}_k(t)\tilde{p}_{i,st} = \tilde{p}_i(t)\tilde{p}_{k,st}$, which reads now $\forall i, k : \tilde{p}'_{k,st}\tilde{p}_{i,st} = \tilde{p}'_{i,st}\tilde{p}_{k,st}$. Alternatively, we can write $\forall i : \tilde{p}_{i,st} = C \tilde{p}'_{i,st}$, where C is a positive constant. Using $\tilde{p}'_{i,st} = f(p'_{i,st})$ and $\tilde{p}_{i,st} = f(p_{i,st})$, we arrive at $\forall i : f(p_{i,st}) = C f(p'_{i,st})$. Since $f(x)$ is a strictly monotonically increasing function (cf. Eq. (5) and concavity of information measures), we can distinguish between three cases: a) $C = 1$ and $\forall i : p'_{i,st} = p_{i,st}$, b) $C > 1$ and $\forall i : p'_{i,st} > p_{i,st} \Rightarrow \sum_{i=1}^N p'_{i,st} > 1$, and c) $C < 1$ and $\forall i : p'_{i,st} < p_{i,st} \Rightarrow \sum_{i=1}^N p'_{i,st} < 1$. Only the first case is consistent with the normalization condition proven in Sec. 2.1. Therefore, we can finally conclude that, indeed, $dK(\{p_i(t)\}, \{p_{i,st}\}; S)/dt = 0$ implies that $\forall i : p_i(t) = p_{i,st}$ holds.

2.3. NLME WITH GENERALIZED STEADY-STATE STATISTICS

In the previous section we assumed the existence of a stationary distribution of the NLME (3,...,5). We study now a special case of this NLME in which the existence of stationary distributions can be guaranteed. In line with the theory of linear master equations, e.g., [39, Sec. 15.2] or [11, Sec. 1.7], we decompose the matrix $A_{i,k}$ according to

$$A_{i,k} = \lambda_{i,k} \exp\{\alpha \epsilon_k\} \quad (17)$$

with $\alpha > 0$ and $\forall i, k : \lambda_{i,k} = \lambda_{k,i} > 0$. We may consider ϵ_i as the energy of the state i . For linear master equations the parameter α can be identified with a reciprocal temperature and the symmetry of the matrix $\lambda_{i,k}$ results from the assumption that the detailed balance condition holds at thermodynamic equilibrium [39, Sec. 15.2]. A discussion about the relevance of the detailed balance condition for the NLME (3,...,5) is beyond the scope of this paper. For the linear case the reader is referred to, for example, *Haken* [24].

We can immediately verify that in the special case of the generalized transition rates (17) the steady-state condition (7) is solved by

$$g(p_{i,\text{st}}) = \alpha \epsilon_i + \beta, \quad (18)$$

where β is a normalization constant fixed by the condition $\sum_{i=1}^N p_{i,\text{st}} = 1$. As stated earlier $g(x) = ds(x)/dx$ is a strictly monotonically decreasing function with a positive singularity at the origin, cf. Eqs. (1) and (2). Consequently, $g(1)$ is the smallest possible value that can be assumed by the left hand side of Eq. (18). Therefore, if there is a particular state i^* with ϵ_{i^*} such that $\alpha \epsilon_{i^*} + \beta < g(1)$, then Eq. (18) cannot be solved for $p_{i^*,\text{st}} \in (0, 1]$. In other words, a necessary condition for Eq. (18) to be satisfied is that β obeys the constraint $\forall i : g(1) \leq \alpha \epsilon_i + \beta$. Now, let $c_{\min}$ denote the smallest value of the set $\{\alpha \epsilon_i\}_{i=1,\dots,N}$. Then, the constraint $\forall i : g(1) \leq \alpha \epsilon_i + \beta$ can equivalently be expressed as $\beta \geq g(1) - c_{\min}$. In the following, we demonstrate that the normalization constant $\beta \geq g(1) - c_{\min}$ exists for any finite energy spectrum $\{\epsilon_i\}_{i=1,\dots,N}$ (i.e., for $N < \infty$). As a consequence of the monotony of $g(x)$, there exists an inverse function $g^{-1} \in C^\infty[g(1), \infty)$ which maps the interval $[g(1), \infty)$ onto the interval $(0, 1]$. In particular, we find the limit behavior $g^{-1}(x \rightarrow \infty) = 0$, cf. Eq. (2). The function g^{-1} is a strictly monotonically decreasing function and can be exploited to compute the stationary distribution according to

$$p_{i,\text{st}} = g^{-1}(\alpha \epsilon_i + \beta). \quad (19)$$

We consider now the expression $\sum_{i=1}^N p_{i,\text{st}}$ as a function of β , that is, we introduce the function $I(\beta) = \sum_{i=1}^N g^{-1}(\alpha \epsilon_i + \beta)$. Then, we need to prove the existence of a parameter value $\beta = \beta^*$ such that $I(\beta^*) = 1$ and $\beta^* \geq g(1) - c_{\min}$. Let i' denote the index i which corresponds to $\alpha \epsilon_i = c_{\min}$. Then, we can rewrite $I(\beta = g(1) - c_{\min})$ as $I(\beta = g(1) - c_{\min}) = g^{-1}\{g(1)\} + \sum_{i \neq i'} g^{-1}(\cdot) = 1 + \sum_{i \neq i'} g^{-1}(\cdot)$, which

gives us the inequality $I(\beta = g(1) - c_{\min}) \geq 1$. Second, we differentiate $I(\beta)$ with respect to β and obtain

$$\left. \frac{dI(\beta)}{d\beta} = \sum_{i=1}^N \underbrace{\frac{dg^{-1}(z)}{dz}}_{< 0} \right|_{\alpha \epsilon_i + \beta} < 0 \quad (20)$$

for $\beta \geq g(1) - c_{\min}$. Finally, we compute the limiting case $\beta \rightarrow \infty$ and find $\lim_{\beta \rightarrow \infty} I(\beta) = \sum_{i=1}^N \lim_{\beta \rightarrow \infty} g^{-1}(\alpha \epsilon_i + \beta) = 0$ (because of the asymptotic behavior $g^{-1}(x \rightarrow \infty) = 0$, see above). In sum, $I(\beta)$ is a strictly monotonically decreasing function with $I(\beta = g(1) - c_{\min}) \geq 1$ and $I(\beta \rightarrow \infty) = 0$. Consequently, there exists a parameter value β^* with $I(\beta^*) = 1$ and $\beta^* \geq g(1) - c_{\min}$ and the stationary distribution $\{p_{i,\text{st}}\}_{i=1,\dots,N}$ exists and can be computed from Eq. (19).

In closing this section, we show that stationary distributions determined by Eq. (19) correspond to maximum information distributions of the information measure (1). Let E denote the mean value of a potential function given by the levels ϵ_i . The level ϵ_i may be interpreted as some kind of energy or effort associated with the state i . Then, E may be regarded as the mean energy or mean effort of a system distributed according to the probabilities p_i . We make the information measure (1) extremal under the constraint $E = \sum_{i=1}^N p_{i,\text{st}} \epsilon_i$ and the normalization condition $\sum_{i=1}^N p_{i,\text{st}} = 1$. The corresponding variational problem reads

$$\mathcal{J} = S + \alpha' \cdot \left(E - \sum_{i=1}^N p_{i,\text{st}} \epsilon_i \right) + \beta' \cdot \left(1 - \sum_{i=1}^N p_{i,\text{st}} \right) = \text{extremal} , \quad (21)$$

where α' and β' denote Lagrange multipliers. We can solve the variational problem by equating the variational derivative $\delta \mathcal{J}$ to zero for all possible variations $\delta p_{i,\text{st}}$ and obtain

$$g(p_{i,\text{st}}) = \alpha' \epsilon_i + \beta' . \quad (22)$$

If we identify the Lagrange parameters α' and β' , respectively, with α and β , the steady-state statistics of the canonical ensemble described by Eq. (22) coincides with the stationary distribution (18) of the NLME given by Eqs. (3,...,5) and (17). Using the maximum principle (21) it can, again, be shown that α' is related to a reciprocal temperature [17]. In the context of information theory this implies that α' and α can be seen as a measures for strength of fluctuations in a system. We would like to point out that the constraint $E = \sum_{i=1}^N p_{i,\text{st}} \epsilon_i$ used in Eq. (21) is linear with respect to the probabilities p_i . Alternative energy constraints that are nonlinear with respect to the probabilities p_i have recently been considered [54]. Therefore, we realize now that the proposed NLME is rooted in the traditional concept of linear energy constraints while it generalizes the linear master equation with respect to its entropy measure: the BGS entropy.

2.4. HAKEN-KELSO-BUNZ MODEL IN THE FRAMEWORK OF THE q -INFORMATION: $q \leq 1$

In a seminal paper [46], *Tsallis* introduced in physics the q -entropy

$$\begin{aligned} {}^T S &= \frac{1 - \sum_{i=1}^N (p_{i,st})^q}{q - 1}, \quad q \neq 1, \\ {}^T S &= {}^{\text{BGS}} S = - \sum_{i=1}^N p_{i,st} \ln p_{i,st}, \quad q = 1, \end{aligned} \quad (23)$$

which depends on the parameter q – an indicator of the degree of nonextensivity or nonadditivity. The q -entropy is also known in information theory as q -information measure [2]. Since our main focus is on open biological system that operate far from the thermodynamic stationary, in what follows, we will refer to ${}^T S_q$ as a information measure. Obviously, the q -information measure (23) includes the Shannon information ${}^{\text{BGS}} S$ as a special case. Indeed, it can be shown that the limiting case $\lim_{q \rightarrow 1} {}^T S = {}^{\text{BGS}} S$ holds [46]. Let us now write ${}^T S$ as ${}^T S = \sum_{i=1}^N {}^T s(p_i)$. The derivative ${}^T g(x) = d {}^T s(x) / dx$ of the kernel ${}^T s$ reads

$$\begin{aligned} {}^T g(x) &= \frac{q x^{q-1} - 1}{1 - q}, \quad q \neq 1, \\ {}^T g(x) &= -\ln x - 1, \quad q = 1. \end{aligned} \quad (24)$$

For $q < 0$ the information measure ${}^T S$ is not concave, that is, ${}^T S$ does not assume the form (1) – a key property used in the present study. For $q > 0$ the entropy ${}^T S$ is concave, that is, ${}^T S$ can be cast into the form (1). In addition, for $q \in (0, 1]$ the function ${}^T g$ satisfies Eq. (2). Therefore, we focus in this section on the parameter region $q \in (0, 1]$. Inserting Eq. (17) and ${}^T S$ into the NLME (3, ..., 5), we obtain

$$\begin{aligned} \dot{p}_i &= \sum_{k=1}^N \frac{\lambda_{i,k}}{e} \exp \left\{ \alpha \epsilon_k - \frac{q (p_k)^{q-1}}{1-q} \right\} - \exp \left\{ \alpha \epsilon_i - \frac{q (p_i)^{q-1}}{1-q} \right\} \sum_{k=1}^N \frac{\lambda_{i,k}}{e}, \\ &\quad q \in (0, 1), \\ \dot{p}_i &= \sum_{k=1}^N \lambda_{i,k} \exp \{ \alpha \epsilon_k \} p_k - p_i \exp \{ \alpha \epsilon_i \} \sum_{k=1}^N \lambda_{i,k}, \quad q = 1. \end{aligned} \quad (25)$$

Note that in the derivation of Eq. (25) we used the assumed symmetry property $\lambda_{i,k} = \lambda_{k,i}$. The linear master equation for $q = 1$ is well-known in literature [11, 39]. It can be shown that the NLME (25) for $q < 1$ converges to the linear master equation in the limit $q \uparrow 1$. (To this end, exploit the relation $\lim_{q \rightarrow 1} (x^{q-1} - 1) / (1 - q) = -\ln x$ for $x > 0$). On account of the preceding analysis, we can conclude that the NLME (25) displays a unique stationary distribution which corresponds to the generalized statistics induced by the information measure

^TS (23). Furthermore, any initial distribution with $\forall i : p_i(t_0) > 0$ converges to this stationary probability distribution.

In a far-reaching paper, Haken, Kelso, and Bunz proposed a model to describe multistable rhythmic finger movements [5, 25, 27, 28, 30]. In literature, this model is referred to as the HKB model. The model describes the phase difference ϕ between two index fingers of a human who oscillates his or her index fingers with the same frequency. It is nowadays well-documented that for low oscillation frequencies humans can perform bimanual oscillatory index finger movements in only two stable stationary patterns: an in-phase pattern (synchronized activity of homologous muscle groups with zero degree phase-difference $\Rightarrow$ index fingers move anti-parallel) and an anti-phase pattern (synchronized activity of homologous muscle groups with 180 degrees phase-difference $\Rightarrow$ index fingers move parallel). For high oscillation frequencies there is only one coordination pattern that can be performed in a stable fashion: the in-phase pattern [5, 25, 27, 28, 30]. We assign to the in-phase pattern the stationary phase difference $\phi_{st} = 0$ and to the anti-phase pattern the stationary phase difference $\phi_{st} = \pi$. According to the HKB model, the dynamics of the phase difference ϕ is given by

$$\dot{\phi}(t) = -\frac{dV(\phi)}{d\phi}, \quad V(\phi) = -a \cos(\phi) - b \cos(2\phi) \quad (26)$$

for $a \geq 0$ and $b \geq 0$. The parameters a and b are assumed to depend on the oscillation frequency. In the low frequency regime, the parameters satisfy $a < 4b$. Then, the potential dynamics (26) admits for two stationary solutions $\phi_{st} = 0$ and $\phi_{st} = \pi$ corresponding to the two coordination patterns that can be performed in a stable fashion. In the high frequency regime, the parameters satisfy $a > 4b$. Then, there is only one stationary solution given by $\phi_{st} = 0$ indicating that under these circumstances humans can perform only one stable coordination pattern. The transition from bistability to monostability occurs at $a = 4b$.

Several stochastic versions of the HKB model have been proposed, see, for example, [12, 21, 22, 23, 41]. So far, however, the HKB model has not been discussed in the context of master equations and general information measures. Using the NLME (25), we consider now the stochastic HKB model defined by Eq. (25) with N energy levels ϵ_i given by

$$\epsilon_i = -a \cos(i \Delta\phi) - b \cos(2i \Delta\phi) \quad (27)$$

for $i = 1, \dots, N$ and $\Delta\phi = 2\pi/N$. In doing so, the discrete energy levels i are related to the continuous potential energy V according to $\epsilon_i = V(\phi(i))$ with $\phi(i) = i \Delta\phi$. Furthermore, we consider a global coupling between all states: $\forall i, k : \lambda_{i,k} = 1$. According to this model, the stationary solution is given by

$$p_{i,st} = \left[\frac{q}{1 + (1-q)[\alpha\epsilon_i + \beta]} \right]^{1/(1-q)}, \quad q \in (0, 1),$$

$$p_{i,st} = \frac{\exp\{-\alpha\epsilon_i\}}{\sum_{i=1}^N \exp\{-\alpha\epsilon_i\}}, \quad q = 1, \quad (28)$$

Figure 1. Stationary solution of the stochastic HKB model (25b). Solid thin line: HKB potential (27) for $a = b = \alpha = 1$. Solid thick line: Boltzmann potential (28b). Crosses: Initial in-phase distribution. Circles: stationary classical distribution obtained by a numerical simulation of Eq. (25b). Note that the scale of the y-axis describes the probabilities $p_{i,\text{st}} \in [0, 1]$ and $p_i(t_0) \in [0, 1]$. The energy levels (27) are rescaled such that they can be plotted in the same figure. The scale of the x-axis runs from $-\pi + 1$ (≈ 2) to $\pi + 1$ (≈ 4). With this choice, the two critical regions at $\phi = 0$ and $\phi = \pi$ do not coincide with a boundary region of the x-axis.

where β is determined by the normalization condition $\sum_{i=1}^N p_{i,\text{st}} = 1$. For $q = 1$ (Shannon information case) Figures 1 and 2 show the Boltzmann distribution (28b) and the stationary solutions of the stochastic HKB model given by Eqs. (25) and (27) obtained by a numerical simulation of the model for two different initial distributions. In our example, the parameters a and b satisfy the low frequency condition, that is, we deal with the bistable regime of the HKB model. In line with our theoretical considerations there is a single stationary distribution. The distribution is bimodal. The two peaks indicate the two possible stationary points of the deterministic model (26). For $q \neq 1$ but $q \in (0, 1)$ the result of a simulation is shown in Figure 5. The stationary distribution (thin flat line) differs from the Boltzmann distribution (thick flat line). The distribution is flatter than

Figure 2. Simulation of the model as used in Figure 1. Here, the system is initially prepared according to an anti-phase cut-off distribution (crosses).

the Boltzmann distribution and assume the typical shape of the Lévy-like distributions related to the q -statistics [50]. Note that the distributions (28) derived by the maximum entropy principle (21) using a linear energy constraint correspond to the maximum entropy distributions that can be derived from TS using the energy constraint $E = \sum_{i=1}^N [p_{i,st}]^q \epsilon_i / \sum_{k=1}^N [p_{k,st}]^q$ when we replace $1 - q$ by $1/(1 - q)$ [54]. Consequently, the results discussed here for TS , the linear energy constraint, and $q < 1$ correspond to results derived in the literature for TS , the aforementioned nonlinear energy constraint, and $q > 1$.

Stochastic HKB models that exhibit unique stationary probability distributions in the bistable regime such as the model (25) and (27) for $q \in (0, 1]$ and the models discussed in [14, 21, 41] can be regarded as useful descriptions of stochastic multistable motor control system if the mean escape time between the minima of the HKB potential (26) is small in comparison with the duration of the performance of the coordination task [51]. In this condition, the two peaks represent two quasi-stationary distributions that are separately when subjects initially perform in-phase

finger movements (peak at $\phi = 0$) or anti-phase finger movements (peak at $\phi = \pi$) [14, 41].

3. On a Nonlinear Master Equation for Cut-Off Distributions

3.1. DEFINITION OF THE NONLINEAR MASTER EQUATION

The objective now is to generalize the NLME (3) in order to describe systems with stationary cut-off distributions. Cut-off distributions are semi-positive probability distributions for which some states i are definitely not occupied. In other words, we deal with distributions $\{p_i\}_{i=1}^N$ that can be decomposed into two sets I_0 and I_+ defined by $I_0 = \{i \mid p_i = 0\}$ and $I_+ = \{i \mid p_i > 0\}$ which satisfy $I_0 \cup I_+ = \{1, \dots, N\}$. We will discuss cut-off distributions in the context of stochastic systems with expansible and concave information measures [2]. That is, we require that the measures $S(p_1, \dots, p_N)$ under consideration satisfy Eq. (1) and

$$S(p_1, \dots, p_N, p_{N+1} = 0) = S(p_1, \dots, p_N) . \quad (29)$$

We do not require that Eq. (2) is satisfied. For expansible information measures (such as the Shannon measure), only those states contribute to the amount of information of a system, that are occupied with a finite probability. Consequently, for cut-off distributions, expansible information measures depend only on the states described by the set I_+ . In line with this observation, we propose a generalization of the NLME (3) given by

$$\dot{p}_i = \begin{cases} \varphi_i(p_i; I_+(t), S) & \text{for } i \in I_+(t) \\ 0 & \text{for } i \in I_0(t) \end{cases} \quad (30)$$

with $I_0(t) = \{i \mid p_i(t) = 0\}$ and $I_+(t) = \{i \mid p_i(t) > 0\}$ and $\varphi_i(p_i; I_+(t), s)$ given by

$$\varphi_i(p_i; I_+, S) = \sum_{k \in I_+} A_{i,k} f(p_k) - f(p_i) \sum_{k \in I_+} A_{k,i} , \quad (31)$$

$$f(x) = \frac{1}{e} \exp \left\{ - \left. \frac{ds(z)}{dz} \right|_x \right\} . \quad (32)$$

If S satisfies Eq. (2) and we deal with classical distributions (i.e., we have $I_0 = \emptyset$ and $I_+ = \{1, \dots, N\}$) the NLME (30,...,32) reduces to the NLME for classical distributions defined by Eqs. (3,...,5). The generalized NLME (30) takes only ‘active’ states i into account (i.e., states with a finite occupation probability). During the evolution of a distribution $\{p_i(t)\}_{i=1}^N$ some of the probabilities p_i may vanish. In this case, the respective states are regarded as ‘passive’ states that do not contribute to the stochastic dynamics of the system any more. The system evolves only on the subspace of the phase space given by its active states. For any semi-positive

initial distribution $\{p_i(t_0)\}_{i=1}^N$ the time-dependent solution of the NLME is semi-positive as well. The reason for this is that by definition of the NLME the variables $p_i(t)$ simply cannot become negative. We can also verify that the normalization of solutions of the NLME (30) is time-invariant. To this end, we compute the derivative of $\sum_{i=1}^N p_i(t)$ with respect to t . Then, we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N p_i(t) &= \sum_{i=1}^N \dot{p}_i(t) = \sum_{i \in I_0(t)} \dot{p}_i(t) + \sum_{i \in I_+(t)} \dot{p}_i(t) = \sum_{i \in I_+(t)} \dot{p}_i(t) \\ &= \sum_{i \in I_+(t)} \sum_{k \in I_+(t)} [A_{i,k} f(p_k) - f(p_i) A_{k,i}] = 0. \end{aligned} \quad (33)$$

In sum, the NLME (30) describes a well-defined evolution equation for normalized positive-definite probability distributions. Furthermore, the NLME (30) admits for stationary cut-off distributions. We illustrate this issue for the transition rate matrix (17). For the matrix (17), Eq. (31) becomes

$$\begin{aligned} \varphi_i(p_i; I_+, S) &= \sum_{k \in I_+} \lambda_{i,k} \exp\{\alpha \epsilon_k - g(p_k) - 1\} \\ &\quad - \exp\{\alpha \epsilon_i - g(p_i) - 1\} \sum_{k \in I_+} \lambda_{i,k} \end{aligned} \quad (34)$$

with $g(z) = ds(z)/dz$. It is clear from Eq. (34) that φ_i vanishes for $p_i = p_{i,st}$ satisfying Eq. (18). Consequently, if Eq. (19) describes a cut-off distribution, then the NLME (30) exhibits this cut-off distribution as a possible stationary probability distribution. Once again, we would like to emphasize that for classical initial distributions the NLME described by Eqs. (30,...,32) recovers the NLME given by Eqs. (3,...,5). Therefore, if Eq. (19) describes a classical distribution, then this solution represents a stationary distribution of the NLME (30). In short, the NLME (30) can have both cut-off and classical stationary solutions.

3.2. HAKEN-KELSO-BUNZ MODEL IN THE FRAMEWORK OF THE q -INFORMATION: $q > 1$

For $q > 1$ the q -information measure (23) allows for cut-off distributions [46, 48]. From Eq. (19) and (24) it follows that

$$p_{i,st} = \left[\frac{1 - (q-1)[\alpha \epsilon_i + \beta]}{q} \right]_+^{1/(q-1)}, \quad q > 1 \quad (35)$$

with $[a]_+ = a$ for $a \geq 0$ and $[a]_+ = 0$ for $a < 0$ [18, 46, 48]. β is determined by the condition $\sum_{i=1}^N p_{i,st} = 1$. From Eq. (35) we can read off that the distribution (35) has a maximum at the minimum energy level $\epsilon_i = \epsilon_{\min}$ (note that $\alpha > 0$). Furthermore, for energy levels ϵ_i larger than $\epsilon_{\min}$ the distribution assumes values smaller than $p_i(\epsilon_{\min})$. In view of these findings, we can now propose a stochastic

Figure 3. Stationary solutions of the stochastic HKB model (30) for $q = 2$ and $a = b = \alpha = 1$. Crosses: initial in-phase distribution. Solid line: stationary cut-off distribution obtained by a numerical simulation of the model.

HKB model for cut-off distributions. We consider the NLME given by Eqs. (30) and (34) for the HKB potential (27) and the q -information (23) involving the g -function (24). The model is solved numerically for $q = 2 > 1$, $a = b = \alpha = 1$, and two different initial cut-off distributions. Figure 3 shows the stationary solution obtained from a simulation with a peak-shaped initial distribution centered at $\phi = 0$. Figure 4 shows the stationary solution of the model for a peak-shaped initial distribution centered at $\phi = \pi$. In line with our theoretical considerations, we obtain stationary cut-off distributions. The stochastic model is bistable in the sense that for the same parameter values we obtain two stationary distributions. Therefore, the stochastic HKB model based on the NLME (30) can describe a motor control system that is subjected to noise and exhibits multistable coordination patterns that can be performed over long periods. This model does not require the assumption that the mean escape time from a potential well to another potential well is small in comparison with the observation time of a system.

Figure 4. Simulation of the model as used in Figure 3. Here, the initial distribution corresponds to a anti-phase cut-off distribution (crosses).

4. Summary and Outlook

The theory of linear master equations was extended and a nonlinear master equation related to general information measures was proposed. It was shown that the nonlinear master equation exhibits unique classical stationary solutions that act as stable attractors to which transient solutions converge for arbitrary classical initial distributions. For a particular kind of transition rate matrices the existence of stationary distributions is guaranteed. These stationary distributions coincide with the maximum information distributions obtained from information theory. Based on the master equation for classical probability distributions a nonlinear master equation has been proposed that can exhibit both classical and cut-off distributions. In the case of cut-off distributions, the master equation can exhibit multiple stationary solutions. Our findings have been illustrated for a multistable biological system: the human motor control system involved in the performance of coordinated oscillatory finger movements. To this end, we examined the solutions of a stochastic HKB model described by the nonlinear master equation for classical and cut-off distri-

Figure 5. Summary of possible stationary solutions of the stochastic HKB models (25) and (30). Solid flat line: stationary Lévy-like distribution for $q = 0.8$. Solid thick line: stationary Boltzmann distribution for $q = 1$. Peak-shaped distribution centered at the origin. Stationary cut-off distribution for $q = 2$ given an in-phase initial distribution. Peak-shaped distribution centered at π . Stationary cut-off distribution for $q = 2$ given an anti-phase initial distribution. Parameter values: $a = b = \alpha = 1$.

butions and the q -information measure proposed by Tsallis and Aczel and Daroczy. The results are summarized in Figure 5. For $q \in (0, 1)$ we obtain uniquely defined Lévy-like distributions that are flatter than Boltzmann distributions. $q = 1$ recovers the case of a linear master equation with a uniquely defined stationary solution: the Boltzmann distribution. For $q > 1$ we obtain multiple stationary cut-off distribution. For each potential well we obtain one stationary probability distribution. Consequently, in the bistable regime of the HKB model, there are two stationary probability distributions. The stochastic HKB model for cut-off distribution illustrates that sustained multistable performance in the presence of noise may involve cut-off distributions and non-additive (non-extensive) information measures.

For $q > 1$ the model that has been proposed so far fails to describe nonequilibrium phase transitions related to changes of environmental parameters. It cannot describe a nonequilibrium phase transition from the anti-phase cut-off distribution

centered at $\phi = \pi$ to the in-phase cut-off distribution centered at $\phi = 0$. The reason for this is that in the case of cut-off distributions the model can deal with the shrinking of the effective phase space (set of states i with $p_i > 0$) but it cannot deal with an expansion of the effective phase space. The set of active state I_+ is a nonincreasing set with respect to its number of elements. Future studies may be devoted to develop more sophisticated models based on nonlinear master equation that include an ‘activation’ of passive states (i.e., states i with $p_i = 0$). If we succeed in the derivation of such a model, we could also describe nonequilibrium phase transitions between stationary cut-off distributions.

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