On the Problem of Asymptotic Positivity of Solutions for Dissipative Partial Differential Equations

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Abstract. The objective of this paper aims to prove positivity of solutions for the following semilinear partial differential equation

$$u_t = -\alpha u_{xxxx} + (u^2)_{xx} + u(1 - u^2).$$

This equation represents a generalised model of the so-called porous medium equation. It arises in a variety of meaningful physical situations including gas flows, diffusion of an electron-ion plasma and the dynamics of biological populations whose mobility is density dependent. In all these situations the solutions of the equation must be positive functions.

Key words: Dissipative partial differential equations, Interpolation inequalities, Ladder methods, Positivity.

1. Introduction

An important question in the study of the behaviour of solutions of dissipative partial differential equations is whether solutions with positive initial data will remain positive at any later time. In dissipative equations such as reaction-diffusion equations where the highest order differential operator is the Laplacian, it is often possible to establish positivity results by using the maximum-principle [1]. However, the maximum principle does not apply in the study of dissipative partial differential equations which contain differential operators of higher order than that of the Laplacian, and, consequently, it is a much more difficult problem to establish results on positivity of solutions. In this paper we shall investigate a particular class of dissipative partial differential equations which possess a uniform steady state solution. For this class we shall prove under a certain condition that solutions are eventually (asymptotically) positive.

Positivity of solutions is important in many fields of applied mathematics, especially in mathematical biology and in particle dynamics. In such fields the solution of the equation may represent some physical quantity that cannot be negative, so a basic requirement of any realistic model is that its solutions should remain positive for all times. The present paper only establishes *asymptotic* positivity of solutions

of our equation. Nevertheless achieving such a partial result constitutes progress in an equation where maximum principles are not available.

The class of equations which we study naturally extend the result on (asymptotic) positivity of solutions already established for a particular class of equations [2], (see also [3, 4]). This class of equations are relevant also to biological applications (see for example [5]) and they can be represented as follows:

$$u_t = -\alpha u_{xxxx} + (\phi(u))_{xx} + f(u), \quad (1)$$

where α is a positive parameter, u is a scalar valued function of (x, t) , ϕ and f are smooth functions of u . The class of equations (1) define a particular class of reaction-diffusion systems, where the $(\phi(u))_{xx}$ term is the nonlinear diffusion and the $f(u)$ term is the reaction. The theory outlined below works for any reasonable smooth functions ϕ and f , but we restrict ourselves to the following biologically interesting case:

$$\phi(u) = u^2, \quad f(u) = u(1 - u^2).$$

Thus (1) becomes

$$u_t = -\alpha u_{xxxx} + (u^2)_{xx} + u(1 - u^2). \quad (2)$$

For the above equation we shall use periodic boundary conditions on the spatial domain $\Omega = [0, 1]$.

We define a set of quantities as follows:

$$H_0 := \|u\|_2^2 = \int u^2 dx, \quad H_1 := \|u_x\|_2^2 = \int u_x^2 dx, \quad (3)$$

where the integration is over the spatial domain $\Omega = [0, 1]$.

The reader may ask why we are using this quantities: it is because it is well known [6, 7] that in one spatial dimension it is sufficient to have control on the H_1 norm in order to be able to control the L^∞ (essentially the superior norm) of the solutions, thereby giving a-priori estimates on any solution of our model. We begin with the analysis of H_0 . We first differentiate it with respect to time and we then insert the right hand side of (2) obtaining (for details see [8, 9]),

$$\frac{1}{2} \dot{H}_0 = -\alpha H_2 + 2 \int (uu_x^2) dx + 2 \int (u^2 u_{xx}) dx + H_0 - \int u^4 dx, \quad (4)$$

where the dot denotes differentiation with respect to time. The $-\alpha H_2$ term has been obtained by integration by parts (notice that the integrated terms vanish on periodic boundary conditions).

Now we make some transformation of some of the terms in (4); more precisely we integrate by parts the second term in the right hand side and we add the result to the third term obtaining $\int u^2 u_{xx} dx$; to this term we apply first Gauchy-Schwarz inequality and then Young inequality to get $\int u^2 u_{xx} dx \leq \frac{\alpha}{2} H_2 + \frac{1}{2\alpha} \int u^4 dx$. Therefore (4) becomes

$$\frac{1}{2}\dot{H}_0 \leq -\frac{\alpha}{2}H_2 - \left(1 - \frac{1}{2\alpha}\right) \int u^4 dx + H_0. \quad (5)$$

Now we notice that $-\int u^4 dx \leq -H_0^2$ by Cauchy-Schwarz and therefore if $1 - \frac{1}{2\alpha} > 0$ (5) yields

$$\frac{1}{2}\dot{H}_0 \leq -\frac{\alpha}{2}H_2 - \left(1 - \frac{1}{2\alpha}\right) H_0^2 + H_0 \leq -\left(1 - \frac{1}{2\alpha}\right) H_0^2 + H_0. \quad (6)$$

By using an absorbing ball argument we finally obtain that the long time asymptotic behaviour (denoted here and throughout by an overbar) of H_0 is given by

$$\overline{H_0} \leq \frac{2\alpha}{2\alpha - 1}, \quad (7)$$

where we are assuming that $2\alpha - 1 > 0$.

As we are looking for regularity for the solutions of our model we need to have estimates on the H_1 seminorm. Similarly, differentiating it with respect to time and inserting the right hand side of (2) we obtain

$$\begin{aligned} \frac{1}{2}\dot{H}_1 &= -\alpha H_3 + 2 \int u_x (u_x^2)_x dx + 2 \int u_x (uu_{xx})_x dx \\ &\quad + H_1 - \int u_x (u^3)_x dx. \end{aligned} \quad (8)$$

Computing the various derivatives one can see that (8) can be cast into

$$\begin{aligned} \frac{1}{2}\dot{H}_1 &= -\alpha H_3 + 4 \int u_x^2 u_{xx} dx + 2 \int u_x^2 u_{xx} dx + 2 \int uu_x u_{xxx} dx \\ &\quad + H_1 - 3 \int u^2 u_x^2 dx. \end{aligned} \quad (9)$$

The second and third terms are zero on periodic boundary conditions and performing a Cauchy-Schwarz inequality and then Young inequality on the fourth term we can see that

$$2 \int uu_x u_{xxx} dx \leq \frac{\alpha}{2} H_3 + \frac{2}{\alpha} \int u^2 u_x^2 dx.$$

Combining these terms with the similar terms in (9) we finally obtain

$$\frac{1}{2}\dot{H}_1 \leq -\frac{\alpha}{2} H_3 + H_1 - \left(3 - \frac{2}{\alpha}\right) \int u^2 u_x^2 dx. \quad (10)$$

Therefore, provided that $3 - \frac{2}{\alpha} > 0$ we finally obtain

$$\frac{1}{2}\dot{H}_1 \leq -\frac{\alpha}{2} H_3 + H_1. \quad (11)$$

Also we know that (see [8])

$$-H_3 \leq -\frac{H_1^3}{H_0^2}. \quad (12)$$

Using (12) and an absorbing ball argument we can see that

$$\overline{H_1} \leq \sqrt{\frac{2}{\alpha}} \overline{H_0}. \quad (13)$$

By using the estimate we found for $\overline{H_0}$ we have

$$\overline{H_1} \leq \sqrt{\frac{2}{\alpha}} \left(\frac{2\alpha}{2\alpha - 1} \right), \quad (14)$$

where $\alpha > \frac{2}{3}$.

Using the Gagliardo-Nirenberg inequality [6, 7] for the $\|u\|_\infty := \sup_{x \in [0,1]} |u(x)|$ norm of the solution, namely $\|u\|_\infty^2 \leq c H_1^{\frac{1}{2}} H_0^{\frac{1}{2}}$ we can see that the asymptotic behaviour of the solutions of our model obeys the estimates

$$\overline{\|u\|_\infty^2} \leq c \left(\frac{2}{\alpha} \right)^{1/4} \frac{2\alpha}{2\alpha - 1}.$$

The constant c appearing on the right hand side in the above inequality and in similar inequalities used below depends on the spatial dimension and the volume of integration but not on the solutions of the equation.

2. Positivity of Solutions

In this section we are going to show that solutions of our model, with appropriate initial data, will eventually be positive (positive for all t exceeding some *finite* value). Our strategy first involves centering the equation on the uniform steady state solution $u \equiv 1$ and then showing that for all t sufficiently large solutions of the transformed equation are bounded (in absolute value) by unity. We therefore introduce $v(x, t)$, defined by

$$u(x, t) = 1 + v(x, t) \quad (15)$$

where u satisfies (2). If the function $v(x, t)$ satisfies

$$\|v\|_\infty \leq 1$$

then clearly u is a non-negative function. Substituting (15) into (2) we obtain for v the equation

$$v_t = -\alpha v_{xxxx} + 2v_x^2 + 2vv_{xx} + 2v_{xx} - 2v - 3v^2 - v^3. \quad (16)$$

We need to find conditions which will ensure that $\overline{\|v\|_\infty} \leq 1$. This is sufficient to ensure asymptotic positivity. Our strategy is to first obtain appropriate estimates for Equation (16), and then to use these estimate to compute $\overline{\|v\|_\infty}$ such that $\overline{\|v\|_\infty} \leq 1$ is satisfied. We start by defining quantities similar to the H_0 and H_1 , namely

$$J_0 := \|v\|_2^2 = \int v^2 dx, \quad J_1 := \|v_x\|_2^2 = \int v_x^2 dx, \quad (17)$$

where, as above, the integration is over the spatial domain $\Omega := [0, 1]$. Using the same strategy previously applied to H_0 and H_1 , we obtain

$$\frac{1}{2} \dot{J}_0 = \int v[-\alpha v_{xxx} + 2v_x^2 + 2vv_{xx} + 2v_{xx} - 2v - 3v^2 - v^3] dx. \quad (18)$$

Performing the same tricks used above we finally obtain

$$\frac{1}{2} \dot{J}_0 = -\alpha J_2 - 2J_1 - 2J_0 + 2 \int v^2 v_{xx} dx - 3 \int v^3 dx - \int v^4 dx. \quad (19)$$

Now we use again Cauchy-Schwarz inequality and then Young inequality on the $-3 \int v^3 dx$ term to get $-3 \int v^3 dx \leq \frac{9}{2} J_0 + \frac{1}{2} \int v^4 dx$. Substituting into (19) we obtain

$$\frac{1}{2} \dot{J}_0 \leq -\alpha J_2 - 2J_1 + \frac{5}{2} J_0 + 2 \int v^2 v_{xx} dx - \frac{1}{2} \int v^4 dx. \quad (20)$$

By using once more Cauchy-Schwarz inequality and then Young inequality on the $2 \int v^2 v_{xx} dx$ and doing some algebra we finally obtain for (20)

$$\frac{1}{2} \dot{J}_0 \leq -\frac{\alpha}{2} J_2 - 2J_1 + \frac{5}{2} J_0 - \left(1 - \frac{2}{\alpha}\right) \int v^4 dx. \quad (21)$$

We ignore the $-\frac{\alpha}{2} J_2$ and the $-2J_1$ terms; also we have that $-\int v^4 dx \leq -J_0^2$ by Cauchy-Schwarz; thus, by using an absorbing ball argument we arrive at

$$\overline{J_0} \leq \frac{5\alpha}{2\alpha - 4}, \quad (22)$$

where we are assuming that $\alpha > 2$.

In order to control the $\|v\|_\infty$ of our perturbed equation (16) we need to have control on the J_1 norm as well. The evolution equation for J_1 is given by (after few manipulations)

$$\frac{1}{2} \dot{J}_1 \leq -\alpha J_3 - J_2 - 2J_1 + 4 \int (v_x^2) v_{xx} dx - 2 \int v (v_{xx})^2 dx. \quad (23)$$

The fourth term on the right-hand side is zero on periodic boundary conditions; the last term is handled in the following way

$$-2 \int v (v_{xx})^2 dx \leq 2 \|v\|_\infty J_2;$$

we now interpolate $\|v\|_\infty \leq c J_1^{\frac{1}{4}} J_0^{\frac{1}{4}}$ by using the Gagliardo-Nirenberg inequality; also we have $J_2 \leq J_3^{\frac{1}{2}} J_1^{\frac{1}{2}}$. Splitting the $J_3^{\frac{1}{2}} J_1^{\frac{1}{2}}$ term and absorbing the J_3 part into $-\alpha J_3$ we obtain

$$\frac{1}{2} J_1 \leq -\frac{\alpha}{2} J_3 + \frac{c}{2\alpha} J_1^{\frac{3}{2}} J_0^{\frac{1}{2}} - J_2 - 2J_1. \quad (24)$$

Ignoring the $-J_2$ and the $-2J_1$ terms, noting that $-J_3 \leq -\frac{J_1^3}{J_0^2}$ and by using an absorbing ball argument we obtain

$$\overline{J_1} \leq c \frac{J_0^{5/3}}{\alpha^{2/3}}. \quad (25)$$

Using the estimate (22) we finally obtain

$$\overline{J_1} \leq \frac{c}{\alpha^{2/3}} \left(\frac{5\alpha}{2\alpha - 4} \right)^{5/3}, \quad (26)$$

where $\alpha > 2$. To evaluate $\overline{\|v\|_\infty}$ we use Gagliardo-Nirenberg inequality:

$$\overline{\|v\|_\infty} \leq c J_1^{\frac{1}{4}} J_0^{\frac{1}{4}}.$$

By using (22) and (26) the estimate for $\overline{\|v\|_\infty}$ reads

$$\overline{\|v\|_\infty} \leq c \frac{c}{\alpha^{1/6}} \left(\frac{5\alpha}{2\alpha - 4} \right)^{2/3}. \quad (27)$$

Therefore, if α is large enough one can see that the inequality

$$\overline{\|v\|_\infty} \leq c \frac{c}{\alpha^{1/6}} \left(\frac{5\alpha}{2\alpha - 4} \right)^{2/3} \leq 1 \quad (28)$$

is easily satisfied. The restriction (28) ensures time-asymptotic positivity for the solution of our model. In fact from $u(x, t) = 1 + v(x, t)$ and from (28) for large α it follows $u(x, t) \geq 0$ asymptotically in time.

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