L and ε functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$

(automorphic form/Whittaker model/Weil representation/Eisenstein series)

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ABSTRACT The construction of L functions for automorphic cuspidal representations of $\mathrm{GSp}(4, A) \times \mathrm{GL}(2, A)$ with a Whittaker model has been given by Novodvorsky [Novodvorsky, M. (1979) *Proc. Symp. Pure Math.* 33 (2), 87–95]. In this paper, we prove that this L function has nontrivial poles if and only if the representation of $\mathrm{GSp}(4, A)$ is a lifting from split O_4 . We also introduce a different construction of L functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ that is applicable to representations that do not have a Whittaker model—for instance, those that correspond to holomorphic modular forms. This construction is based on the lifting of these automorphic forms to $\widetilde{\mathrm{Sp}}(4)$. Lifted forms will have a Whittaker model. This allows us to write integral expressions yielding these L functions.

In this paper, we study L and ε functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$. We consider two constructions. The first is a direct construction, which is applicable to pairs (π, τ) of irreducible automorphic representations of $\mathrm{GSp}(4, A)$ and $\mathrm{GL}(2, A)$, respectively (A = the adeles of a given global field k), where π is cuspidal and generic (i.e., π has a standard Whittaker model) and τ is also generic. This construction can be inferred from ref. 1. The local factors are studied in ref. 2. The L function that is obtained corresponds to the natural eight-dimensional representation of the L group of $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, which is $\mathrm{GSp}(4, C) \times \mathrm{GL}(2, C)$. The main result here is the description of the poles of $L(\pi \times \tau, s)$ (Theorem 1.3): There is a τ such that $L(\pi \times \tau, s)$ has a pole (simple) iff π can be obtained as a Weil lifting from the connected component of the split four-dimensional general orthogonal group.

The second construction of L and ε functions is for pairs (π, τ) of irreducible, automorphic, cuspidal representations of $\mathrm{Sp}(4, A)$ (the metaplectic group) and of $\mathrm{GL}(2, A)$, respectively. π is assumed to have a standard Whittaker model. This is a new construction. It is obtained by considering a Shimura type integral. In this Shimura integral, we use an Eisenstein series with “correct” parameters, which enables us to write local L and ε functions. This Eisenstein series corresponds to $\mathrm{Ind}_{\mathrm{P}_A}^{\mathrm{Sp}(4, A)} \tau$. P is the parabolic subgroup of $\mathrm{Sp}(4)$ that has an abelian radical. Assuming that τ has a trivial central character this Eisenstein series always has a (simple) pole at $s = 1/2$. The other poles are simple and may occur only at $s = 0, 1$. This happens iff $L(\tau, 1/2) \neq 0$ (Theorem 2.1).

As an application, we get a new type of L functions for $\mathrm{PGSp}(4, A)$ twisted with irreducible automorphic representations of $\mathrm{GL}(2, A)$, which is applicable for (not necessarily generic) representations π of $\mathrm{PGSp}(4, A)$, which have a certain generalized Whittaker model. We first lift representations π as above to $\widetilde{\mathrm{Sp}}(4, A)$, using Weil representations. [Recall that $\mathrm{PGSp}(4) \cong \mathrm{Split SO}(5)$.] It turns out that the lifting of such π to $\widetilde{\mathrm{Sp}}(4, A)$ is generic. Now we apply the above construction for the lifting of π . We mention here that, locally over $\mathbf{R}$, holomorphic modular forms are not generic but

their liftings to $\widetilde{\mathrm{Sp}}(4, \mathbf{R})$ are generic. One can expect that there is a similar situation in the global case. We remark here that the same construction of L functions is available for inner forms of $\mathrm{SO}(5)$ that have nontrivial parabolic subgroups.

We mention that for $\mathrm{GSp}(4, A)$ the theory of L functions twisted with characters is available for any irreducible, automorphic, cuspidal representation of $\mathrm{GSp}(4, A)$ (3, 4). It is known that, to apply the converse theorem for $\mathrm{GL}(4, A)$, one needs L functions twisted with characters and L functions twisted with irreducible infinite-dimensional automorphic representations of $\mathrm{GL}(2, A)$. In another paper, we shall use the above to prove the existence of a correspondence that attaches to irreducible, automorphic, cuspidal representations of $\mathrm{GSp}(4, A)$, automorphic representations of $\mathrm{GL}(4, A)$ —a special case of Langlands’ functoriality conjecture.

1.a. L and ε Functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$

Let k be a global field; A , its ring of adeles; and I , its group of ideles. We introduce L and ε functions $L(\pi \times \tau, s)$, $\varepsilon(\pi \times \tau, s)$ for irreducible, automorphic, generic (i.e., having a standard Whittaker model) representations π, τ of $\mathrm{GSp}(4, A)$ and $\mathrm{GL}(2, A)$. We assume that π is cuspidal. These factors have been suggested by Novodvorsky (1).

Consider the group $G = \{(g_1, g_2) \in \mathrm{GL}(2)^2 | \det g_1 = \det g_2\}$. It is easy to see that $G \subset \mathrm{GSp}(4)$. We choose the embedding (which we will represent below by an asterisk)

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \rightarrow \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ c_1 & c_2 & d_1 & d_2 \end{bmatrix}.$$

Let $C = \{(I_2, tI_2) \in G\}$. Put $\omega = \omega_\pi \omega_\tau$, the product of the central characters of π and τ . For $\phi \in S(A^2)$ (the Schwartz–Bruhat functions on A^2) and $g \in \mathrm{GL}(2, A)$, let $E^\phi(g; \omega, s)$ be the Eisenstein series used by Jacquet in ref. 5

$$\left[\text{For } \mathrm{Re}(s) > 1, E^\phi(g; \omega, s) = \sum_{\gamma \in \mathrm{B}_A \backslash \mathrm{GL}(2, k)} f_{\omega, s}^\phi(\gamma g); B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right]$$

$$\text{and, for } \mathrm{Re}(s) > \frac{1}{2}, f_{\omega, s}^\phi(g) = |\det g|^s \int_1 \phi((0, t)g) |t|^{2s} \omega(t) d^*t \Big].$$

Define for $\varphi \in \pi, \beta \in \tau, \phi \in S(A^2)$. [φ belongs to the appropriate subspace of $L^2(X, \omega_\pi)$, $X = \mathrm{GSp}(4, k) \backslash \mathrm{GSp}(4, A)$, corresponding to π . φ is made a function on G_A via $*$.]

$$L(\varphi, \beta, \phi, s) = \int_{C_A \backslash G_A} \varphi(g_1, g_2) \beta(g_2) E^\phi(g_1; \omega, s) dg;$$

$$\begin{aligned} \tilde{L}(\varphi, \beta, \phi, s) &= \int_{C_A \backslash G_A} \varphi(g_1, g_2) \beta(g_2) \omega^{-1}(\det g_2) \\ &\quad \times E^\phi(g_1; \omega^{-1}, s) dg. \end{aligned} \quad [1.1]$$

Let ψ be a nontrivial character of $k \backslash A$. Let $w_\varphi^\psi, w_\beta^{\psi^{-1}}$ be the standard Whittaker functions of φ, β with respect to ψ, ψ^{-1} .

THEOREM 1.1. (i) *The integrals 1.1 converge absolutely and are holomorphic for all s that is not a pole of the Eisenstein series and*

$$L(\varphi, \beta, \phi, s) = \tilde{L}(\varphi, \beta, \hat{\phi}, 1-s). \quad [1.2]$$

(ii) *For $\text{Re}(s) \gg 0$,*

$$\begin{aligned} L(\varphi, \beta, \phi, s) &= \int_{C_A N_A G_A} w_\varphi^\psi(g_1, g_2) w_\beta^{\psi^{-1}}(g_2) f_{\omega, s}^\phi(g_1) dg; \\ \tilde{L}(\varphi, \beta, \phi, s) &= \int_{C_A N_A G_A} w_\varphi^\psi(g_1, g_2) w_\beta^{\psi^{-1}}(g_2) \\ &\quad \times \omega^{-1}(\det g_1) f_{\omega^{-1}, s}^\phi(g_1) dg. \end{aligned} \quad [1.3]$$

The integrals on the right-hand side of Eq. 1.3 converge absolutely for $\text{Re}(s) \gg 0$. Here $N = \left(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right)$.

Proof: i is immediate because φ is cuspidal, and Eq. 1.2 is implied by the functional equation satisfied by the Eisenstein series. (ii) For $\text{Re}(s) \gg 0$, substitute in Eq. 1.1 the expression of the Eisenstein series as series, then

$$\begin{aligned} L(\varphi, \beta, \phi, s) &= \int_{C_A F_A G_A} \varphi(g_1, g_2) \beta(g_2) f_{\omega, s}^\phi(g_1) dg \\ &= \int_{C_A N_A^{(1)} D_A G_A} \varphi_0(g_1, g_2) \beta(g_2) f_{\omega, s}^\phi(g_1) dg. \end{aligned}$$

Here

$$\begin{aligned} F &= \{(b, g) \in G | b \in B\}, \quad N^{(1)} = \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, I_2 \right) \right\}, \\ D &= \left\{ \left(y^{-1} \det g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y, g \right) \mid y \in \text{GL}(1), g \in \text{GL}(2) \right\} \end{aligned}$$

and

$$\varphi_0(g) = \int_{N_k^{(1)} \backslash N_A^{(1)}} \varphi(ng) dn, \quad (g \in \text{GSp}(4, A)).$$

$$\text{Write } \varphi_0(g) = \sum_{\gamma \in Z_k \backslash \text{GL}(2, k)} w_\varphi^\psi \left(\left(\begin{pmatrix} \det \gamma & 0 \\ 0 & 1 \end{pmatrix}, \gamma \right) g \right),$$

(lemma 6.2 of ref. 6). Here $Z = \left(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right)$. Now Eq. 1.3 fol-

lows easily. To justify the formal computations one has to use the local study of ref. 2. $\square$

Let F be a local field, and π, τ be irreducible, admissible, generic representations of $\text{GSp}(4, F), \text{GL}(2, F)$, respectively. Let ψ be a nontrivial character of F . We denote by $W(\pi, \psi), W(\tau, \psi^{-1})$ the respective Whittaker models. For $w \in W(\pi, \psi), w' \in W(\tau, \psi^{-1})$, and $\phi \in S(F^2)$, we form the local inte-

grals analogous to Eq. 1.3; $L(w, w', \phi, s) = \int_{C_F N_F G_F} w(g_1,$

$g_2) w'(g_2) f_{\omega, s}^\phi(g_1) dg$ and similarly $\tilde{L}(w, w', \phi, s)$. These integrals converge absolutely for $\text{Re}(s) \gg 0$ and continue to meromorphic functions on the whole plane. There exists an Euler factor of minimal degree $L(\pi \times \tau, s)$ such that $L(w, w', \phi, s) (L(\pi \times \tau, s))^{-1}$ is holomorphic for all w, w', ϕ . A functional equation is satisfied; there exists a meromorphic function $\gamma(\pi \times \tau, s, \psi)$ such that $\gamma(\pi \times \tau, s, \psi) L(w, w', \phi, s) =$

$\tilde{L}(w, w', \hat{\phi}, 1-s)$. These L and γ functions are of degree 8. Put

$$\varepsilon(\pi \times \tau, s, \psi) = \frac{\gamma(\pi \times \tau, s, \psi)}{L(\pi \times \tau, s)} L(\hat{\pi} \times \hat{\tau}, 1-s).$$

Let k be a global field and $\pi = \otimes \pi_p, \tau = \otimes \tau_p$ as in the beginning of this section. Define

$$L(\pi \times \tau, s) = \prod_p L(\pi_p \times \tau_p, s),$$

$$\varepsilon(\pi \times \tau, s) = \prod_p \varepsilon(\pi_p \times \tau_p, s, \psi_p).$$

THEOREM 1.2. *The product for L converges absolutely for $\text{Re}(s) \gg 0$ and continues to a meromorphic function on the whole plane, having a finite number of poles, and*

$$L(\pi \times \tau, s) = \varepsilon(\pi \times \tau, s) L(\hat{\pi} \times \hat{\tau}, 1-s).$$

1.b. Poles of $L(\pi \times \tau, s)$ and the Weil Lifting

The notation of this section is as in the beginning of section 1.a. Denote by H the connected component of the general split four-dimensional orthogonal group over k .

THEOREM 1.3. *There is a τ such that $L(\pi \times \tau, s)$ has a pole iff π can be obtained as a Weil lifting from H_A . (See ref. 6 for the notion of Weil lifting.)*

Proof: We have to search for the poles of the functions $L(\varphi, \beta, \phi, s)$. These poles (if any) are simple and are poles of the Eisenstein series, which occur for $\alpha^{2s-1} \omega = \alpha, \alpha^{-1}, (\alpha t) = |t|$. The residue of $L(\varphi, \beta, \phi, s)$ is proportional to $B(\varphi, \beta,$

$$\phi) = C(\phi) \int_{C_A G_A G_A} \varphi(g_1, g_2) \beta(g_2) |\det g_2|^{s_0} dg \quad [\text{for an appro-}]$$

priate exponent s_0 and number $C(\phi)$]. From this formula for $B(\varphi, \beta, \phi)$, it is easy to prove that $L(\pi \times \tau, s)$ is entire for all

$$\tau \text{ if and only if } \int_{\text{SL}(2, k) \backslash \text{SL}(2, A)} \varphi(r, I_2) dr = 0, \quad \forall \varphi \in \pi.$$

Theorem 8.1 of ref. 7 implies that this condition is equivalent to the fact that π is orthogonal to all Weil liftings from H_A . $\square$

2.a. An Eisenstein Series

Let τ be an irreducible, automorphic, cuspidal representa-

tion of $\text{GL}(2, A)$. Let P be the parabolic subgroup $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right)$ of

$\text{Sp}(4)$. We introduce an Eisenstein series that corresponds to $\text{Ind}_{P_A}^{\text{Sp}(4, A)} \tau$ and has "convenient" parameters. Consider the reductive dual pair $(\text{Sp}(4), \text{SO}(4))$ in $\text{Sp}(16)$. [We consider the split $\text{SO}(4)$]. Let ψ be a nontrivial character of $k \backslash A$. Let ω_ψ be the Weil representation of $\text{Sp}(16, A)$ that corresponds to ψ . We may restrict it to $\text{Sp}(4, A) \times \text{SO}(4, A)$. Note that $C \backslash G \cong \text{SO}(4)$. We use this and consider ω_ψ as a representation of $\text{Sp}(4, A) \times C_A \backslash G_A$. The space of ω_ψ is $S(A^8)$. Define for $v \in S(A^8), \phi \in S(A^2), \beta \in \tau, \omega = \omega_\tau, h \in \text{Sp}(4, A)$

$$\begin{aligned} F_v^\phi(h; \beta, \omega, s; \psi) &= \int_{C_A N_A^{(1)} G_A} \omega_\psi(h, g) v(z_0) \\ &\quad \times \beta(g_2) f_{\omega, s}^\phi(g_1) dg. \end{aligned} \quad [2.1]$$

The groups $G, C, N^{(1)}$ are as in Section 1. The point z_0 is a special point in the space k^8 . [If we realize k^8 as 2×4 matrices and $\text{Sp}(4)$ acts by right multiplications, then $z_0 = (O | I_2)$.]

The integral 2.1 converges absolutely and is holomorphic for $\operatorname{Re}(s) \gg 0$. It continues meromorphically to the whole plane. It belongs to $\operatorname{Ind}_{\mathbb{P}_A^{\mathbb{A}}}^{\mathbb{S}p(4, A)}(\tau \otimes \alpha^{s-1/2})$, $(\alpha(t) = |t|)$. Define for $\operatorname{Re}(s) \gg 0$

$$E_v^\phi(h; \beta, \omega, s; \psi) = \sum_{\gamma \in \mathbb{P}_A^{\mathbb{A}} \backslash \mathbb{S}p(4, k)} F_v^\phi(\gamma h; \beta, v, s; \psi^{-1}). \quad [2.2]$$

THEOREM 2.1. *The series 2.2 converges absolutely and is holomorphic for $\operatorname{Re}(s) \gg 0$. It has a meromorphic continuation to the whole plane (by the general theory of Langlands). It has a finite number of poles. Assume for simplicity that τ has a trivial central character, then one pole always exists at $s = 1/2$. Two other poles occur at $s = 0, 1$ iff $L(\tau, 1/2) \neq 0$. It satisfies the functional equation*

$$E_v^\phi(h; \beta, \omega, s, \psi) = E_v^{\hat{\phi}}(h; \omega^{-1} \otimes \beta, \omega^{-1}, 1-s; \psi) \quad [2.3]$$

2.b. L and ε Functions for $\widetilde{\mathbb{S}p}(4, A) \times \operatorname{GL}(2, A)$

Let σ, τ be irreducible, automorphic, cuspidal, generic representations of $\widetilde{\mathbb{S}p}(4, A)$ and $\operatorname{GL}(2, A)$, respectively. We assume that σ is genuine. Let σ have a Whittaker model with

$$\text{respect to the character } \psi_\lambda \begin{pmatrix} 1 & x & a & b \\ 0 & 1 & c & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} = \psi(x + \lambda y),$$

$\lambda \in k^*$. [λ is determined modulo $(k^*)^2$.] For $\varphi \in \sigma, u \in S(A^2)$, and β, ϕ, v, ω are as in Section 2.a, define

$$\begin{aligned} L(\varphi, \beta; u, v, \phi, s) &= \int_{\mathbb{S}p(4, k) \backslash \mathbb{S}p(4, A)} \varphi(h) \\ &\quad \times \theta_{\psi; -\lambda}^u(h) E_v^\phi(h; \beta, \omega, s; \psi) dh \\ \tilde{L}(\varphi, \beta; u, v, \phi, s) &= \int_{\mathbb{S}p(4, k) \backslash \mathbb{S}p(4, A)} \varphi(h) \theta_{\psi; -\lambda}^u(h) \\ &\quad \times E_v^\phi(h; \omega^{-1} \otimes \beta, \omega^{-1}, s; \psi) dh. \quad [2.4] \end{aligned}$$

(For brevity, we do not mark the dependence on λ, ψ .) $\theta_{\psi; -\lambda}^u$ is the θ series that corresponds to the Weil representation of $\widetilde{\mathbb{S}p}(4, A)$ with respect to $\psi^{-\lambda}$.

THEOREM 2.2. (i) *The integrals 2.4 converge absolutely and are holomorphic for all s that is not a pole of the Eisenstein series and*

$$L(\varphi, \beta; u, v, \phi, s) = \tilde{L}(\varphi, \beta; u, v, \hat{\phi}, 1-s).$$

(ii) *For $\operatorname{Re}(s) \gg 0$,*

$$\begin{aligned} L(\varphi, \beta; u, v, \phi, s) &= \int_{U_A \backslash \mathbb{S}p(4, A)} w_\varphi^{\psi_\lambda}(h) \omega_{\psi; -\lambda}(h) \\ &\quad \times u(0, 1) V_v^\phi(h; \beta, \omega, s; \psi) dh \\ \tilde{L}(\varphi, \beta; u, v, \phi, s) &= \int_{U_A \backslash \mathbb{S}p(4, A)} w_\varphi^{\psi_\lambda}(h) \omega_{\psi; -\lambda}(h) u(0, 1) \\ &\quad \times V_v^\phi(h; \omega^{-1} \otimes \beta, \omega^{-1}, s; \psi) dh, \quad [2.5] \end{aligned}$$

where U is the standard maximal unipotent subgroup of $\mathbb{S}p(4)$, $\omega_{\psi; -\lambda}$ is the Weil representation of $\widetilde{\mathbb{S}p}(4, A)$ with re-

spect to $\psi^{-\lambda}$, $w_\varphi^{\psi_\lambda}$ is the Whittaker function of φ with respect to ψ_λ , and $V_v^\phi(h; \beta, \omega, s; \psi)$ is the meromorphic continuation of the following integral, which converges absolutely and is holomorphic for $\operatorname{Re}(s) \gg 0$

$$\int_{C_A N_A^{(1)} \backslash G_A} \omega_{\psi; -1}(h, g) v(z_0) w_\beta^{\psi^{-1}}(g_2) f_{\omega, s}^\phi(g_1) dg. \quad [2.6]$$

Here $\omega_{\psi; -1}$ is the Weil representation of $\widetilde{\mathbb{S}p}(16, A)$, restricted to $\mathbb{S}p(4, A) \times \operatorname{SO}(4, A)$, and $w_\beta^{\psi^{-1}}$ is the Whittaker function of β with respect to ψ^{-1} .

The proof of this theorem is in the same spirit as that of Theorem 1.1. Again, for the proof of part ii one needs first the analogous local theory.

Let F be a local field, σ and τ be irreducible, admissible, generic representations of $\mathbb{S}p(4, F)$ and $\operatorname{GL}(2, F)$, respectively. Assume that σ has a Whittaker model with respect to ψ_λ . Define for $\phi \in S(F^2)$, $v \in S(F^8)$, $\omega = \omega_\tau$, and $w' \in W(\tau, \psi^{-1})$ (the Whittaker model of τ with respect to ψ^{-1}), the local analog to integral 2.6.

$$\begin{aligned} V_v^\phi(h; w', \omega, s, \psi) &= \int_{C_F N_F^{(1)} \backslash G_F} \omega_{\psi; -1}(h, g) v(z_0) \\ &\quad \times w'(g_2) f_{\omega, s}^\phi(g_1); h \in \mathbb{S}p(4, F). \quad [2.7] \end{aligned}$$

The integral 2.7 converges absolutely and is holomorphic for $\operatorname{Re}(s) \gg 0$. It continues to a meromorphic function in the whole plane. [$\omega_{\psi; -1}$ in integral 2.7 is the Weil representation of $\widetilde{\mathbb{S}p}(16, F)$ restricted to $\mathbb{S}p(4, F) \times \operatorname{SO}(4, F)$.] We also define for $w \in W(\sigma, \psi_\lambda)$, $u \in S(F^2)$

$$\begin{aligned} L(w, w'; u, v, \phi, s) &= \int_{U_F \backslash \mathbb{S}p(4, F)} w(h) \omega_{\psi; -\lambda}(h) u(0, 1) \\ &\quad \times V_v^\phi(h; w', \omega, s; \psi) dh \\ \tilde{L}(w, w'; u, v, d, s) &= \int_{U_F \backslash \mathbb{S}p(4, F)} w(h) \omega_{\psi; -\lambda}(h) u(0, 1) \\ &\quad \times V_v^\phi(h; \omega^{-1} \otimes w', \omega^{-1}, s; \psi) dh. \quad [2.8] \end{aligned}$$

The integrals 2.8 converge absolutely and are holomorphic for $\operatorname{Re}(s) \gg 0$. They continue meromorphically to the whole plane.

THEOREM 2.3. (i) *There exist Euler factors of minimal degree $L(\sigma \times \tau, s)$, $\tilde{L}(\sigma \times \tau, s)$ such that $L(w, w'; u, v, \phi, s) \cdot L(\sigma \times \tau, s)^{-1}$ and $\tilde{L}(w, w'; u, v, \phi, s) \cdot \tilde{L}(\sigma \times \tau, s)^{-1}$ are holomorphic for all w, w', u, v, ϕ .*

(ii) *There exists a meromorphic function $\gamma(\sigma \times \tau, s, \psi)$ such that*

$$\gamma(\sigma \times \tau, s, \psi) L(w, w'; u, v, \phi, s) = \tilde{L}(w, w'; u, v, \hat{\phi}, 1-s).$$

Let k be a global field and σ, τ, ψ be as in the beginning of Section 2.b. Write $\sigma = \otimes_p \sigma_p$, $\tau = \otimes_p \tau_p$ and define

$$L(\sigma \times \tau, s) = \prod L(\sigma_p \times \tau_p, s) \quad [2.9]$$

$$\varepsilon(\sigma \times \tau, s) = \prod \varepsilon(\sigma_p \times \tau_p, s, \psi_p). \quad [2.10]$$

(We define the ε function as in Section 1.) The study of the local factors can be reduced to that of the factors of Section 1.

THEOREM 2.4. *The product 2.9 converges absolutely and is holomorphic for $\operatorname{Re}(s) \gg 0$. It has a meromorphic continuation with a finite number of poles. For almost all p , $\varepsilon(\sigma_p \times \tau_p, s) = 1$ and*

$$L(\sigma \times \tau, s) = \varepsilon(\sigma \times \tau, s) L(\hat{\sigma} \times \hat{\tau}, 1-s).$$

2.c. L and ε Functions for $\mathrm{PGSp}(4) \times \mathrm{GL}(2)$

We use Section 2.b to present a new type of L and ε functions for $\mathrm{PGSp}(4) \times \mathrm{GL}(2)$. We first recall the notion of a generalized Whittaker model. Let F be a local field. Consider the parabolic subgroup $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ of $\mathrm{GSp}(4)$. Write $P =$

MS , the Levi decomposition. $S = \left\{ \begin{pmatrix} I_2 & x \\ 0 & I_2 \end{pmatrix} : x \text{ symmetric} \right\}$.

For a nondegenerate F linear form l on S , there is a unique two-dimensional semisimple algebra K over F such that the connected component of the stabilizer of l in M is isomorphic to K^* . If $K = F(\sqrt{\lambda})$, $\lambda \in (F^*)^2 \setminus F^*$, $\lambda \neq (F^*)^2$, denote this connected component by D^λ and, if $K = F \oplus F$, denote it by D^1 . Set $R^\lambda = D^\lambda S$. Let ν be a character of D^λ and ψ be a nontrivial character of F ; then $\psi_{\nu, \lambda} = \nu \otimes (\psi \circ l)$ is a character of R^λ . A generalized Whittaker model of type $(R^\lambda, \psi_{\nu, \lambda})$ for an irreducible admissible representation π of $\mathrm{GSp}(4, F)$ is a realization of the space of π in a right translation invariant space of functions $w(g)$ on $\mathrm{GSp}(4, F)$, which satisfy $w(rg) = \psi_{\nu, \lambda}(r)w(g)$ for $r \in R^\lambda$. Given λ and ν , such a model is unique for π (if it exists). We have the similar notion over the adèles.

Let k be a global field. We restrict ourselves to $\mathrm{PGSp}(4)$. It is known that $\mathrm{PGSp}(4) \cong \mathrm{SO}(5)$, the connected component of the split five-dimensional orthogonal group. Consider the reductive dual pair $(\mathrm{SO}(5), \mathrm{Sp}(4))$ in $\mathrm{Sp}(20)$. It is possible to show that the Weil lifting carries an irreducible, automorphic, cuspidal representation π of $\mathrm{SO}(5, A)$, which has a gen-

eralized Whittaker model of type $(R^\lambda, \psi_{1, \lambda})$ (ψ is a nontrivial character of $k \setminus A$) to an automorphic representation $\sigma(\pi)$ of $\mathrm{Sp}(4, A)$, which has a standard Whittaker model with respect to ψ_λ . $\sigma(\pi)$ is cuspidal, if π is not lifted from $\mathrm{SL}(2, A)$. Thus, for such π , we may define $L(\pi \times \tau, s) = L(\sigma(\pi) \times \tau, s)$ and $\varepsilon(\pi \times \tau, s) = \varepsilon(\sigma(\pi) \times \tau, s)$.

THEOREM 2.5. *When both the definitions of the L and ε functions (of Sections 1, 2) are applicable, they coincide.*

We do not know yet an exact description of poles of the L function as that of Theorem 1.3. It remains to deal with the irreducible, automorphic, cuspidal representations of $\mathrm{PGSp}(4, A)$ that are not generic and do not have a generalized Whittaker model of type $(R^\lambda, \psi_{1, \lambda})$. We conjecture that any such representation can be obtained by use of Weil liftings from the various two-dimensional anisotropic orthogonal groups over k . Our evidence is based on considering the similar question over a finite field. Despite the fact that technical problems over a finite field are much simpler than those over local or global fields, it happens so far that properties over a finite field, when correctly interpreted, always give good insight to the situation in the local and in the global case.

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