Radial Fourier multipliers of $L^p(\mathbb{R}^2)$

(Bochner–Riesz multipliers/localized potential spaces/square functions)

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ABSTRACT We give sufficient conditions (in terms of differentiability and growth properties) for a radial function to be an $L^p(\mathbb{R}^2)$ Fourier multiplier. These conditions are in the nature of best possible.

The well-known multiplier theorem of Hörmander (ref. 1; see also p. 96 in ref. 2) may be stated as follows. Let ϕ be a fixed nonnegative smooth bump function supported in $[1, 2]$, and let D_n^α , $\alpha > 0$, denote the fractional derivative defined initially on $\mathcal{S}(\mathbb{R}^n)$ (the space of rapidly decreasing C^∞ functions) by $(D_n^\alpha g)^\wedge(\xi) = |\xi|^\alpha g^\wedge(\xi)$ and then extended to $L^1(\mathbb{R}^n)$. For $n = 1$ we write $D^\alpha = D_1^\alpha$. Suppose m is a bounded function on $\mathbb{R}^n$ that satisfies

$$\sup_{t>0} t^{2\alpha-n} \int \left| D_n^\alpha \left[\phi\left(\frac{|\cdot|}{t}\right) m(\cdot) \right] \right|^2 = A^2 < \infty \quad [1]$$

for some real number $\alpha > n/2$. Then m is a Fourier multiplier of $L^p(\mathbb{R}^n)$ [we write $m \in M_p(\mathbb{R}^n)$] for $1 < p < \infty$, in the sense that the operator T defined initially on $\mathcal{S}$ by $(Tf)^\wedge(\xi) = m(\xi)f^\wedge(\xi)$ has a bounded extension to L^p and its norm is $\leq cA$, where the constant c is independent of m . The norm of m as a multiplier, $\|m\|_{M_p}$, is defined to be the operator norm of T . [Hörmander stated the theorem for integral $\alpha > n/2$, but a slight modification of Stein's proof in his book (ref. 2) gives the theorem as stated here. Stein earlier published this method in several notes in the *Comptes Rendus* in 1966–1967; see reference 9 on p. 285 of ref. 2.]

Condition 1 for $\alpha > n/2$ actually implies that m is bounded, while for $\alpha \leq n/2$, the space of functions satisfying 1 contains unbounded functions: in this sense, Hörmander's result is sharp; but when additional assumptions about m and p are made, the differentiability conditions required of m may be relaxed (see, for example, refs. 3 and 4).

On the other hand, let us consider the Bochner–Riesz multipliers $m_\delta(\xi) = (1 - |\xi|^2)_+^\delta$ for $\delta > 0$. Carleson and Sjölin (5) [see also the work of Fefferman (6) and the contributions of

Córdoba (7)] have shown that $m_\delta \in M_p(\mathbb{R}^2)$ when $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{2} \left(\delta + \frac{1}{2} \right)$, which is better than Hörmander's theorem and

the multiplier theorems in refs. 3 and 4 predict. [Herz (8) had previously shown that the condition $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{n} \left(\delta + \frac{1}{2} \right)$ is necessary for $m_\delta \in M_p(\mathbb{R}^n)$.]

The main aim of this paper is to cast the Carleson–Sjölin theorem in a more general setting by establishing a multiplier theorem for a class of functions of which the Bochner–Riesz multipliers m_δ , $\delta > 0$, are prototypical members. For this purpose we shall employ the spaces of localized Riesz potentials $R(q, \alpha)$ suggested by the Hörmander condition [1].

$R(q, \alpha)$, for $1 \leq q < \infty$ and $\alpha > 0$, is defined to be the class of functions m defined on $(0, \infty)$ and extended to be zero on $(-\infty, 0]$ for which

$$\|m\|_{R(q, \alpha)} = \sup_{t>0} t^{\alpha-1/q} \left\{ \int \left| D^\alpha \left[\phi\left(\frac{\cdot}{t}\right) m(\cdot) \right] \right|^q \right\}^{1/q} < \infty.$$

Our main results can then be stated as follows.

THEOREM 1. If $\alpha > 1/2$ and $m \in R(2, \alpha)$, then $m(|\xi|) \in M_p(\mathbb{R}^2)$, $4/3 \leq p \leq 4$, with $\|m\|_{M_p} \leq c\|m\|_{R(2, \alpha)}$.

THEOREM 2. If $\alpha > \frac{1}{q}$, $1 \leq q < \infty$, and $m \in R(q, \alpha)$, then $m(|\xi|) \in M_p(\mathbb{R}^2)$ for $\alpha > 2 \left| \frac{1}{p} - \frac{1}{2} \right| + \left(\frac{1}{q} - \frac{1}{2} \right)_+$ with $\|m\|_{M_p} \leq c\|m\|_{R(q, \alpha)}$.

Remarks: (i) The case $q = 2$ of Theorem 2 follows by interpolating between Theorem 1 and the Hörmander upper endpoint result in ref. 9 (p. 239) that $m(|\xi|) \in M_p(\mathbb{R}^2)$, $1 < p < \infty$ if $m \in R(2, \alpha)$, $\alpha > 1$. Here we used the fact that the space of functions of weak bounded variation $WBV_{q, \alpha}$ in ref.

10 equals $R(q, \alpha)$ with equivalent norms when $\alpha > \frac{1}{q}$, $1 < q$

$< \infty$. The case $q = 2$ and Sobolev-type embeddings give the case $q < 2$. Interpolation of the case $q = 2$ with $M_2(\mathbb{R}^2) = L^\infty(\mathbb{R}^2)$ gives the case $q > 2$. These matters will be discussed in a forthcoming paper. It should be noted that while the multiplier theorems are most naturally proved in the $R(q, \alpha)$ context, the $WBV_{q, \alpha}$ conditions are easier to verify.

(ii) Theorem 2 is best possible in the sense that the “critical index” $2 \left| \frac{1}{p} - \frac{1}{2} \right| + \left(\frac{1}{q} - \frac{1}{2} \right)_+$ cannot be reduced. This

can be seen in the case $q \leq 2$ by examining the Bochner–Riesz multipliers m_δ , $\delta > 0$, while for $q > 2$ it suffices to consider the functions $m_{a, b}(\xi)$ in ref. 11 (equation 6.4), with $b = a\alpha$.

(iii) There should be analogues of Theorems 1 and 2 for $\mathbb{R}^n$, $n \geq 3$. The conjectured version of Theorem 1 is that if $\alpha > 1/2$ and $m \in R(2, \alpha)$, then $m(|\xi|) \in M_p(\mathbb{R}^n)$ for $\frac{2n}{n+1} \leq p \leq \frac{2n}{n-1}$. A solution to this problem seems to

await a better understanding of the multipliers m_δ for $n \geq 3$. However, Dappa (12) has obtained the optimal result in the case $q = 1$, $\alpha > (3n+1)/(2n+2)$ by using the Stein–Tomas restriction theorem (13).

(iv) Gasper and Trebels proved the Hankel multiplier version of Theorem 2 in ref. 9 and gave necessary conditions for Hankel multipliers (and hence for radial Fourier multipliers) in terms of WBV -spaces in ref. 14.

The proof of Theorem 1 uses square functions in the spirit of the Igari–Kuratsubo multiplier theorem (3) and Stein's proof of Hörmander's theorem. Prior to the proof, let us make the following definitions in which all Fourier transforms are in $\mathbb{R}^2$, with functions on $(0, \infty)$ extended to be radial where appropriate.

For $\alpha > 0$, let $(\theta_\alpha)^\wedge(\xi) = |\xi|(1 - |\xi|)_+^{\alpha-1}$, $\theta_t^\alpha(x) = t^{-2}\theta^\alpha(x/t)$ and

$$\mathcal{G}^\alpha(f)(x) = \left(\int_0^\infty |\theta_t^\alpha * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Let $\hat{\psi}(\xi) = \sigma(\xi) = |\xi|\phi^2(\xi)$ and

$$g(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

If $m \in R(2, \alpha)$ and $(Tf)^\wedge(\xi) = m(\xi)f^\wedge(\xi)$, Theorem 1 will follow from the following inequalities:

$$\|Tf\|_p \leq C\|g(Tf)\|_p, \quad 1 < p < \infty, \quad [2]$$

$$\|g(Tf)\|_p \leq C\|m\|_{R(2,\alpha)} \|\mathcal{G}^\alpha(f)\|_p, \quad 1 < p < \infty, \quad \alpha > 1/2, \quad [3]$$

$$\|\mathcal{G}^\alpha(f)\|_4 \leq C\|f\|_4, \quad \alpha > 1/2, \quad [4]$$

where the constants C are independent of f and m . Inequality 2 follows from the theory of vector-valued singular integrals (2), and inequality 4 follows from proposition 1 in ref.

15, as may be seen by writing $(\theta^\alpha)^\wedge = \sum_{k=0}^\infty 2^{-k(\alpha-1)} \phi_k^\alpha$ with suitably smooth radial nonnegative bump functions ϕ_k^α supported in $\{1 - 2^{-k} \leq |\xi| \leq 1 - 2^{-k-2}\}$.

Proof of Inequality 3: Let $\alpha > 1/2$. For $h \in C^\infty(\mathbb{R})$ of compact support in $(0, \infty)$, we define the operator $(d/ds)^\alpha$ by $((d/ds)^\alpha h)^\wedge(\nu) = (-i\nu)^\alpha h^\wedge(\nu)$ and observe that since $(d/ds)^\alpha$ and D^α are related by the Hilbert transform, the assumption $m \in R(2, \alpha)$ is equivalent with $\sup_{t>0} t^{2\alpha-1} \int |(d/ds)^\alpha [\phi(\cdot/t)m(\cdot)]|^2 < \infty$. Moreover, since $(-s)_+^{\alpha-1}$ has Fourier transform $C_\alpha(-i\nu)^{-\alpha}$, we obtain for $\phi(tu)m(u)$, ($t, u > 0$) the Riemann-Liouville formula

$$\phi(tu)m(u) = C \int_u^\infty (s-u)^{\alpha-1} (d/ds)^\alpha [\phi(\cdot/t)m(\cdot)] ds \quad [5]$$

valid when $m \in R(2, \alpha)$. Applying Eq. 5, we obtain

$$\begin{aligned} \sigma(tu)m(u) &= C\phi(tu)t \int_{1/t}^{2/t} \frac{u}{s} \left(1 - \frac{u}{s}\right)^{\alpha-1} \\ &\quad \times s^\alpha \left(\frac{d}{ds}\right)^\alpha [\phi(\cdot/t)m(\cdot)] ds, \end{aligned}$$

since $\text{supp}(d/ds)^\alpha[\phi(\cdot/t)m(\cdot)] \subseteq (-\infty, 2/t]$.

Thus

$$\psi_t * Tf(x) = C\rho_t * \Lambda^t f(x),$$

where $\rho^\wedge = \phi$ and

$$\Lambda^t f(x) = t \int_{1/t}^{2/t} \theta_{1/s}^\alpha * f(x) s^\alpha (d/ds)^\alpha [\phi(\cdot/t)m(\cdot)] ds.$$

Therefore

$$g(Tf)(x) = C \left(\int_0^\infty |\rho_t * \Lambda^t f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

and, by vector-valued singular integrals,

$$\|g(Tf)\|_p \leq C \left\| \left(\int_0^\infty |\Lambda^t f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_p, \quad 1 < p < \infty. \quad [6]$$

Now

$$\begin{aligned} |\Lambda_t f(x)| &\leq C \left(t \int_{1/t}^{2/t} |\theta_{1/s}^\alpha * f(x)|^2 ds \right)^{1/2} \\ &\quad \times \left(t \int_{1/t}^{2/t} |s^\alpha (d/ds)^\alpha [\phi(\cdot/t)m(\cdot)]|^2 ds \right)^{1/2} \\ &\leq C\|m\|_{R(2,\alpha)} \left(t \int_{1/t}^{2/t} |\theta_{1/s}^\alpha * f(x)|^2 ds \right)^{1/2}. \end{aligned}$$

Finally,

$$\begin{aligned} &\left(\int_0^\infty |\Lambda^t f(x)|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq C\|m\|_{R(2,\alpha)} \left(\int_0^\infty \left(\int_{1/s}^{2/s} dt \right) |\theta_{1/t}^\alpha * f(x)|^2 ds \right)^{1/2} \\ &= C\|m\|_{R(2,\alpha)} \mathcal{G}^\alpha(f)(x). \end{aligned} \quad [7]$$

Combining inequality 6 with inequality 7 finishes the proof of inequality 3, and hence of Theorems 1 and 2.

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