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A Second-Order Conditionally Linear Mixed Effects Model With Observed and Latent Variable Covariates

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Abstract

A conditionally linear mixed effects model is an appropriate framework for investigating nonlinear change in a continuous latent variable that is repeatedly measured over time. The efficacy of the model is that it allows parameters that enter the specified nonlinear time-response function to be stochastic, whereas those parameters that enter in a nonlinear manner are common to all subjects. In this article we describe how a variant of the Michaelis–Menten (M–M) function can be fit within this modeling framework using *Mplus* 6.0. We demonstrate how observed and latent covariates can be incorporated to help explain individual differences in growth characteristics. Features of the model including an explication of key analytic decision points are illustrated using longitudinal reading data. To aid in making this class of models accessible, annotated *Mplus* code is provided.

Keywords

conditionally linear models; latent variables; nonlinear mixed effects models; second-order latent growth models; structured latent curve models

Popular approaches to the analysis of continuous nonlinear repeated measures data include the *nonlinear mixed effects model* (Davidian & Giltinan, 1995; Pinheiro & Bates, 2000). The model is especially geared toward situations in which (a) subjects exhibit a pattern of response over the repeated measures that is similar to others, but with discernible individual differences; (b) individual-specific trajectories follow the generic curvilinear time-response pattern that is indicative of the behavior, but vary on particular, salient features (i.e., leveling-off behavior at later trials differ among participants); (c) the scatter of responses around individuals' trajectories differs from person to person, which is characteristic of substantial within-subject variability; and (d) the number of data points measured on the participants might be unequal or times of measurement completely unique. As Cudeck and Harring (2007) pointed out, the analytic method should satisfactorily accommodate all of these features: nonlinear patterns of change, imbalanced collections of data, within-subject variability, and particularly individual

differences in response. Fortunately, a subject-specific model, like the nonlinear mixed effects model, is well-equipped to accommodate these data and design features.

A slightly more restrictive version of the fully nonlinear model, the *conditionally linear mixed effects model* (cf. Blozis & Cudeck, 1999; Crowder & Hand, 1990, section 9.4), can be specified in which individual regression coefficients that enter the function in a nonlinear fashion are fixed across individuals, whereas parameters that enter the function in a linear fashion are permitted to vary across individuals. For instance, initial status or potential performance in developmental processes, often modeled via linear coefficients in many nonlinear functions (Seber & Wild, 1989), frequently display considerable heterogeneity in the sample; whereas a coefficient representing rate of change, often a nonlinear parameter, shows considerably less variability. For example, the empirical illustration that is introduced subsequently to describe improvement in reading fluency uses a variant of the Michaelis–Menten function:

$$y_{ij} = \beta_{1i} + \frac{(\beta_{2i} - \beta_{1i})t_{ij}}{\rho + t_{ij}} + d_{ij}, \quad (1)$$

where t_{ij} denotes the time of measurement for the i th individual on the j th occasion, with $1 \leq j \leq m$, where m is the total number of occasions. In a particular scenario the responses might be adequately represented by subject-specific coefficients on β_{1i} , initial status, and β_{2i} , maximum performance. However, the nonlinear coefficient ρ , the value of the predictor (in this case time) for which half the total expected gain is achieved, can be taken as a constant across individuals. This type of specification means that ρ is in some way more fundamental to the change process than β_{1i} or β_{2i} because the former does not vary across individuals (Cudeck, 1996).

Occasionally, the decision to allow random effects on some regression coefficients but fix others across subjects is rendered a priori based on theoretical considerations. In other circumstances, this analytic decision is justified empirically through exploratory data analysis.¹ A full development of this model for observed repeated measures was described by Blozis and Cudeck (1999), and Harring, Cudeck, and du Toit (2006) demonstrated how the same model could be fit to data as a structural equation model using the nonlinear constraints option available in mainstream software (e.g., LISREL 8.80).

When several indicators measuring the same latent construct are obtained at each measurement occasion, change in a latent variable can be investigated within this same mixed effects methodological framework. As the name implies, a second-order model (Sayer & Cumsille, 2001) uses a second-order factor structure to investigate both linear and nonlinear longitudinal change in latent variables (see, e.g., Blozis, 2006; Ferrer, Balluerka, & Widaman, 2008; Hancock, Kuo, & Lawrence, 2001; McArdle, 1988). For illustrative purposes, imagine three manifest variables, y_1 , y_2 , and y_3 , measure latent variable η for person i at time j and that change in the latent variable can be adequately summarized by the Michaelis–Menten function in Equation 1. A second-order model can then be written as

¹Frequently, this is accomplished by plotting random samples of individuals via a spaghetti plot and fitting candidate functions to individuals using nonlinear least squares estimation (see e.g., Cudeck & Harring, 2010).

$$\begin{aligned}
 y_{1ij} &= \tau_{y_1} + \lambda_{y_1} \eta_{ij} + e_{1ij}, \\
 y_{2ij} &= \tau_{y_2} + \lambda_{y_2} \eta_{ij} + e_{2ij}, \text{ and} \\
 y_{3ij} &= \tau_{y_3} + \lambda_{y_3} \eta_{ij} + e_{3ij}, \text{ with} \\
 \eta_{ij} &= \beta_{1i} + \frac{(\beta_{2i} - \beta_{1i}) t_{ij}}{\rho + t_{ij}} + d_{ij},
 \end{aligned}$$

where τ represents the intercept, λ represents the factor loading (i.e., the linear slope relating factor η to the manifest indicator), e signifies the unique factor score, η represents the factor and d represents a disturbance term. A path diagram of this model is shown in Figure 1. In Figure 1, the same manifest variables y_1 , y_2 , and y_3 are measured over m occasions. For each occasion, the three variables consist of a latent factor η_j . Change in such factors is subsequently modeled through a second-order Michaelis–Menten function with three coefficients—two linear parameters and one nonlinear parameter.

The two parameters that enter the function linearly, β_1 and β_2 , are latent growth characteristics that are represented by circles in Figure 1. The factor loadings relating β_1 and β_2 to the repeated factor η are nonlinear functions of the parameter ρ . This type of coding is fundamental to the specification of the second-order conditionally linear mixed model and is described in more detail later.

Harring (2009) demonstrated how a marginal maximum likelihood estimation algorithm could be used to fit an entirely nonlinear version of a second-order mixed effects model, such as the model highlighted in Figure 1. However, due to the intrinsic nonlinearity in the parameters, estimation of the fully nonlinear model required special programming. Thus, even though the popularity of second-order growth models is gaining momentum, its sparseness (particularly for nonlinear functions) in the applied literature could reflect the apparent difficulty in finding familiar software for which specification and implementation of the model is straightforward.

To address this concern we demonstrate how *Mplus* (version 6; Muthén & Muthén, 1998–2010), a popular structural equation modeling (SEM) program, can be utilized to fit a conditionally linear mixed effects model to examine change in a latent variable. There are several reasons why SEM programs, like *Mplus*, should be used to fit conditionally linear growth models like that explicated here. First, the SEM framework integrates both observed and latent covariates into the overall modeling system in a natural way. Second, specification of the conditionally linear model facilitates estimation of a variety of interesting nonlinear functions via normal-theory maximum likelihood with an algorithm that is not appreciably more complicated than those used for strictly linear functions. Perhaps most compelling is that SEM programs, for many researchers, are the analytic tools of choice for most purposes. Staying within a familiar software environment is not only convenient; it avoids the inefficiencies involved in setting up a specific model correctly using a foreign syntactical language.

In this article, we outline and develop a version of the nonlinear mixed effects model for repeated measures on a latent variable that is conditionally linear in the random effects. Specification of this second-order conditionally linear mixed effects model is a bit more complicated than that of a mixed effects model for observed repeated measures. The added complexity is due to the augmentation of the first-order model with a measurement model connecting the observed variable indicators to the latent variables. In addition to the nonlinear function for the repeated measures, latent and manifest covariates are defined and alternative latent covariate models are posited to account for individual differences in growth characteristics.

To underscore the efficacy of the proposed model, longitudinal reading data are introduced that have complete data but an unbalanced design (i.e., measurement occasions are not equally spaced). The model is applied to repeated measures of reading fluency, a latent variable indicated by four observed variables, along with a battery of decoding measures. *Mplus* code for the final model can be found in the Appendix, but we discuss important aspects of the code at various junctures in the text as the situation warrants. An ongoing explication of the various analytic decision points as well as a summary of the results is highlighted.

LONGITUDINAL STUDY OF READING FLUENCY

For illustrative purposes, repeated measures data were obtained from a longitudinal study of 230 fourth-grade students in which growth in reading-related skills was examined across four time points (Speece, Ritchey, & Silverman, 2006–2011). Children were assessed in the fall of their fourth-grade year and in the spring of their fourth-, fifth-, and sixth-grade years. The same measures of decoding and fluency were administered at each time point. Latent variables were created for decoding and fluency (Silverman, Speece, Harring, Ritchey, & Cutting, 2010). At each measurement occasion, both fluency and decoding were each measured by three observed variable indicators. The latent variable for decoding included observed variables: phonological elision, word attack, and word identification (alpha reliability = .82). The latent variable for fluency included measures of oral passage reading fluency, silent reading fluency, and pseudo-word decoding efficiency (alpha reliabilities = .85, .86, .88, and .91 over the four occasions, respectively). The observed variable acronyms and corresponding variable names in *Mplus* can be found in the Appendix.

Samuels (2002) described three stages of decoding: (a) the nonaccurate stage, (b) the accurate but not automatic stage, and (c) the accurate and automatic (fluent) stage. By fourth grade, most children are typically accurate but not necessarily automatic. Therefore, their decoding skills are more established than their fluency skills. As the reading task changes from “learning to read” to “reading to learn,” children build their fluency through practice reading (Chall & Jacobs, 2003). According to the verbal efficiency theory (Perfetti, 1985), children must decode fluently to free up cognitive resources to focus on understanding and learning from what they read. Once children have reached the fluent stage, fluency likely levels out as well. Beyond a certain rate, reading too quickly could actually hinder rather than support reading comprehension (Rasinski, 2004). In this study, decoding is a static latent variable measured at the first occasion in the fall of fourth grade and is thought to be related, if not systematically linked, to growth in fluency. Table 1 displays the sample statistics between the fluency measures, and the same descriptive measures for decoding are displayed in Table 2 to give some indication of the scale and variability of the variables used in the subsequent analysis.

MODEL SPECIFICATION

Measurement Model

In the basic second-order conditionally linear mixed effects model formulation, the repeated measure to be analyzed is an unobservable construct that itself is measured by multiple indicators at each measurement occasion. This relation is typically operationalized in terms of a measurement model connecting the observed indicators with the corresponding latent

variable across time. Consider the $k \times 1$ response vector, $\mathbf{y}_{ij}^* = (y_{ij1}, y_{ij2}, \dots, y_{ijk})'$, for individual i at time j with $1 \leq j \leq m$. It is assumed that these k observed variables at time j measure a single latent variable, η_i , for the i th individual. A common factor model is specified that characterizes the relation of the observed variables to the latent variable:

$$\mathbf{y}_{ij}^* = \boldsymbol{\tau}_j^* + \boldsymbol{\lambda}_j^* \boldsymbol{\eta}_{ij}^* + \mathbf{e}_{ij}^* \quad (2)$$

where $\boldsymbol{\tau}_j^*$ is a $k \times 1$ vector of variable intercepts, $\boldsymbol{\lambda}_j^*$ is a $k \times 1$ vector of fixed or unknown factor loadings that describe the linear relation between the observed variables and latent variable, $\boldsymbol{\eta}_{ij}^*$ is the common factor, and $\mathbf{e}_{ij}^*$ is a $k \times 1$ vector of uniquenesses, or measurement errors. A standard assumption is that the common factor is independent of the unique factors and the uniquenesses are mutually independent given that the linear dependence among the observed variables is accounted for by the latent factor.

It is not unusual for the same battery of instruments to be given repeatedly across the longitudinal design. In this situation, if a complete set of the same k variables were obtained at multiple occasions—with a maximum of m potential time points ($j = 1, 2, \dots, m$) then individual i would have a response vector with a total number of $q = mk$ observations. Bollen and Curran (2006) pointed out that requiring the same k variables to be measured at each occasion can be overly restrictive. They advocated that as long as one common indicator is present to anchor the latent variable, realistic across-time comparisons are possible. Hancock and Buehl (2008) amended this notion by describing the analytic consequences of estimating a second-order growth model when a construct's indicators actually shift over time.

Working from the scenario that the observed variable indicators are identical at each time point, let $\mathbf{y}_i = (\mathbf{y}_{i1}^*, \dots, \mathbf{y}_{im}^*)'$ denote a $q \times 1$ vector of responses for individual i , stacked according to j across all m occasions. The measurement model across all m measurement occasions can be specified as:

$$\mathbf{y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda} \boldsymbol{\eta}_i + \mathbf{e}_i. \quad (3)$$

In Equation 3, $\boldsymbol{\tau}$ is a $q \times 1$ vector of intercepts, $\boldsymbol{\Lambda}$ is a $q \times m$ block diagonal matrix of factor loadings, $\boldsymbol{\eta}_i$ is a $m \times 1$ vector of latent factors corresponding to individual i , and $\mathbf{e}_i$ is a $q \times 1$ vector of unique factors. For clarity,

$$\begin{pmatrix} \mathbf{y}_{i1} \\ \vdots \\ \mathbf{y}_{im} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau}_1 \\ \vdots \\ \boldsymbol{\tau}_m \end{pmatrix} + \begin{pmatrix} \boldsymbol{\lambda}_1 & & \\ & \ddots & \\ & & \boldsymbol{\lambda}_m \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_{i1} \\ \vdots \\ \boldsymbol{\eta}_{im} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_{i1} \\ \vdots \\ \mathbf{e}_{im} \end{pmatrix}$$

$$\mathbf{y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda} \boldsymbol{\eta}_i + \mathbf{e}_i$$

The distribution of the unique factors is given as:

$$\mathbf{e}_i \sim N(\mathbf{0}, \boldsymbol{\Omega}(\boldsymbol{\omega})). \quad (4)$$

The matrix $\boldsymbol{\Omega}(\boldsymbol{\omega})$ is a $q \times q$ symmetric covariance matrix in which the diagonal elements contain the unique variances corresponding to the common factor model of the repeated measures and the off-diagonal elements are the unique covariances. Unlike conventional factor analysis where the covariance matrix of the unique factors is assumed to be strictly diagonal, specification of off-diagonal elements of $\boldsymbol{\Omega}(\boldsymbol{\omega})$ under the longitudinal design implied in Equation 3 is commonplace. For example, allowing unique covariances of temporally adjacent pairs of uniquenesses to be freely estimated would seem plausible given that the same indicators are measured repeatedly over time. $\boldsymbol{\omega}$ is a column vector containing all of the distinct

parameters in Ω . Other structures can be tailored to correspond with other design, theoretical, or empirical considerations.

Model for the Repeated Latent Variable

To represent systematic change in individuals, the latent variables are assumed to follow a conditionally linear mixed effects model of the kind described by Blozis and Cudeck (1999):

$$\eta_i = \mathbf{X}(\rho)\boldsymbol{\beta}_i + \mathbf{d}_i. \quad (5)$$

In this adaptation, the $p \times 1$ vector of individual coefficients, $\boldsymbol{\beta}'_i = (\beta_{1i}, \dots, \beta_{pi})$, enters the function linearly and varies from person to person, and the parameter vector, $\boldsymbol{\rho}$, is made up of nonlinear regression coefficients that are fixed across subjects. The $m \times p$ design matrix $\mathbf{X}(\boldsymbol{\rho})$ contains possibly nonlinear functions of $\boldsymbol{\rho}$, incorporates covariates that explain the growth characteristics of η_i , and includes measurements of time. The latter values are often taken as elapsed time from the beginning of the study, or as age of a participant at the j th occasion.

It is assumed that the distribution of the individual regression coefficients is multivariate normal:

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Phi}_{\beta\beta}). \quad (6)$$

The time-specific errors $\mathbf{d}_i$ contain the disturbances from regressing a nonlinear function on the latent variable at time j , and are uncorrelated with $\boldsymbol{\beta}_i$, t_j , and $\mathbf{e}_i$. Inclusion of random effects can often account for a preponderance of variation in the repeated measures (Davidian & Giltinan, 2003); therefore the residuals, in many applications, are assumed to have a simple structure. It is commonplace to assume that the disturbances are mutually independent with homogenous variance, σ_d^2 , having distribution:

$$\mathbf{d}_i \sim N(\mathbf{0}, \Delta) \quad (7)$$

where $\Delta = \sigma_d^2 \mathbf{I}_m$ and $\mathbf{I}$ is an $m \times m$ identity matrix.

Not all nonlinear functions can be written in the form of Equation 5, for instance, functions with only intrinsically nonlinear parameters (see, e.g., Bates & Watts, 1988); yet many of the most popular nonlinear models used to describe intraindividual change in the behavioral sciences can. For example, following Browne (1993), the elements of η for individual i could be modeled as a negatively accelerating exponential function at time t_j as:

$$\eta_j = \beta_{2i} - (\beta_{2i} - \beta_{1i}) \exp\{-\rho(t_j - 1)\} + d_j$$

The function fits into Equation 5 by setting $\boldsymbol{\beta}_i = (\beta_{1i}, \beta_{2i})'$ and coding the j th row of the design matrix as:

$$[\mathbf{X}(\rho)]_j = [\exp\{-\rho(t_j - 1)\} \ 1 - \exp\{-\rho(t_j - 1)\}].$$

As another example, the empirical illustration that is introduced subsequently to describe improvement in reading fluency uses a version of the Michaelis–Menten model. For the i th individual at time t_j ,

$$\eta_j = \beta_{1i} + \frac{(\beta_{2i} - \beta_{1i})t_j}{\rho + t_j} + d_j. \quad (8)$$

The two coefficients that enter the model linearly, β_{1i} and β_{2i} , represent initial status at $t_j = 0$ and maximum performance (i.e., as $t_j \rightarrow \infty$). The substantively interesting nonstochastic parameter, ρ , is the value of the predictor (in this case time) for which half the total expected gain is achieved; that is, $\eta = \beta_{1i} + 1/2(\beta_{2i} - \beta_{1i})$. Figure 2 displays this version of the Michaelis–Menten model.

The design matrix for this function has two columns to pick up the regression coefficients that enter linearly with typical entry: $[\mathbf{X}(\rho)]_j = [1 - t_j (\rho + t_j)^{-1} \ t_j (\rho + t_j)^{-1}]$. One primary reason for examining the function for the repeated measures in this way is that the entries in the design matrix $\mathbf{X}$ correspond directly to the elements of the *Mplus* code specification (this connection will be subsequently highlighted).

An advantage of using a conditionally linear mixed effects modeling framework for nonlinear problems is that both latent variable and manifest variable covariates can be incorporated into the modeling system in a sensible and straightforward fashion. Furthermore, because of the conditional linearity in regression parameters, this complex system can be treated with essentially the same estimation strategies used in strictly linear growth models.

Factor Analysis Covariate Model

Initially, simply summarizing the repeated measurements with a conditionally linear mixed effects model tailored to a particular research situation is valuable (Choi, Harring, & Hancock, 2009; Cudeck, 1996). At a secondary stage of the analysis, information garnered about individual differences in the repeated measures can be obtained when subject-specific covariates are introduced to account for observed differences in growth characteristics. Often these covariates are individual traits or demographic variables. If multiple indicators of a construct for each individual are available, it is useful to consider a factor analysis model for the covariates (see, e.g., Blozis & Cudeck, 1999; Browne, 1993; Browne & du Toit, 1991).

Let $\mathbf{z}_i = (z_{1i}, \dots, z_{ri})'$ denote an additional set of r observed variables that depend on v latent factors $\boldsymbol{\xi}_i = (\xi_{1i}, \dots, \xi_{vi})'$ in the regression

$$\mathbf{z}_i = \boldsymbol{\Pi} \boldsymbol{\xi}_i + \boldsymbol{\zeta}_i \quad (9)$$

where for the i th individual $\boldsymbol{\Pi}$ is the $r \times v$ factor loading matrix and $\boldsymbol{\zeta}_i$ is a vector of unique variables.

In the current context, the i th individual's regression coefficients, $\boldsymbol{\beta}_i$, are thought to be linearly dependent on both observed and latent variables in the following way:

$$\boldsymbol{\beta}_i = \boldsymbol{\beta} + \boldsymbol{\Gamma}_1 \boldsymbol{\xi}_i + \boldsymbol{\Gamma}_2 \mathbf{x}_i + \mathbf{b}_i. \quad (10)$$

where $\boldsymbol{\beta}$ is a vector of intercepts conditional on both $\boldsymbol{\xi}$ and $\mathbf{x}$, $\mathbf{x}_i$ is a vector of observed covariates associated with individual i , $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are matrices of coefficients for the regression of $\boldsymbol{\beta}_i$ on $\boldsymbol{\xi}_i$ and $\mathbf{x}_i$, respectively. The regression residuals are assumed to have distribution $\mathbf{b}_i \sim N(\mathbf{0}, \boldsymbol{\Psi})$ and are uncorrelated with both $\boldsymbol{\xi}$ and $\mathbf{x}$. Blozis and Cudeck (1999) provided possible configurations of conditional models other than the type considered in Equation 10.

PRELIMINARY ANALYSIS CONSIDERATIONS

Choosing a Nonlinear Function

When using nonlinear models, an important data analytic decision is the choice of mathematical function (Cudeck & Harring, 2007). Selection of a nonlinear function can occasionally be theoretically justified based on previous research or based on expected behavior of the underlying change process. Even if other options perform satisfactorily, one function might still be preferred because it replicates the development of the response in important ways or is parameterized in such a way that it captures interesting scientific characteristics.

A more common situation is for a function to be chosen empirically by examining the data and selecting among several candidate functions because it fits the data well or provides interesting information regarding the response. Graphical displays, like spaghetti plots, or plots of individual fitted functions vis-à-vis nonlinear least squares are invaluable in judging whether or not a particular function is appropriate. In this study, a variant of the Michaelis–Menten model was eventually chosen among three candidate² functions because it demonstrated adequate model–data fit, had interpretable parameters, and seemed to appropriately characterize both individual and the average patterns of change.

Longitudinal Invariance

When using second-order models to investigate longitudinal change, an implicit assumption is that any growth in the latent variables reflects true change in the underlying phenomena or construct and is not due to changes that might occur in the measurement model. Determining the stability or invariance of measurement properties of the latent construct over time, therefore, must be a priority to draw valid inferences regarding the change process.

In this investigation, longitudinal measurement invariance was examined through fitting a series of confirmatory factor analysis (CFA) models to the data. For growth models with multiple indicators, a continuum of invariance stringency can be established by applying equivalence constraints to the factor loadings, intercepts, and error variances of the indicators—although the latter has been questioned as needed in the longitudinal context (e.g., Sayer & Cumsille, 2001). Because models with more stringent forms of invariance are often nested within models in which those equality constraints are lifted, a typical strategy is to conduct successive chi-square difference tests—noting that a nonsignificant *p* value would indicate that the more constrained model does not fit appreciably worse than the less constrained model. Other traditional SEM fit indexes—root mean squared error of approximation (RMSEA), Comparative Fit Index (CFI), or the Nonnormed Fit Index (NNFI)—can also be computed. For the fluency variables, results of model fit statistics and model comparison tests are summarized in Table 3 for CFA models which are increasingly more constrained.

For the current example, indicators at each measurement occasion were evaluated as to whether the form of the observed–latent variable relation was the same across time. This invariance of the factor pattern is what Horn, McArdle, and Mason (1983) as well as Meredith (1993) coined *configural invariance*. The fit statistics from Model 1 show that configural invariance was met (RMSEA = 0.042, CFI = .991). Weak measurement invariance (Meredith & Horn, 2001; Widaman & Reise, 1997) is established when factor loadings are constrained to be equal over time for each observed indicator. This level of invariance is what McArdle (1988) advocated as being minimally acceptable to validate the comparability of constructs across time. The chi-

²Two alternative candidate functions were an increasing decelerating exponential function and a Gompertz function (Browne, 1993). When fit to the empirical means, the residual sum of squares for the Michaelis–Menten model was smallest among the three alternatives and was parameterized to the satisfaction of the reading scientists.

square difference test ($\Delta\chi^2_{\Delta df=6}=5.5, p=.481$) supports that weak measurement invariance holds. The next level of stringency is to test whether the intercepts are equal for each indicator over time. Establishing intercept invariance uncovers whether changes in the mean of an indicator reflect substantive changes in the latent construct, in this case fluency, or simply a shift in the intercept of the indicator. Based on the chi-square difference test in Table 3, the intercepts were constrained to be invariant over time ($\Delta\chi^2_{\Delta df=6}=12.0, p=.051$). For this model the factor means were 127.1, 144.9, 152.1, and 158.1, respectively.

Identification and scaling of the latent variables of the various CFA models were established by fixing the factor loading and intercept of an observed variable to unity and zero, respectively. As a reviewer rightly pointed out, when strong factorial invariance is established, changing which indicator scales the latent variable will have no effect on the fit of the model. If, on the other hand, only weak factorial invariance can be achieved, then the latent variable will not carry the mean structure of the observed variables and the growth process is simply modeling change in the indicator variable (see, e.g., Ferrer et al., 2008, for a detailed discussion of the issues arising when establishing longitudinal invariance).

FITTING THE MICHAELIS–MENTEN MODEL IN *Mplus*

Implementation of a model in *Mplus* involves creating and executing an input file script, which is typically made up of six sections of commands: TITLE, DATA, VARIABLE, ANALYSIS, MODEL, and OUTPUT. For succinctness, the subsequent presentation focuses mainly on the MODEL statement where specification of the conditionally linear model occurs. For clarity, all *Mplus* commands will be denoted using monospace font with keywords in capital letters. Observed and latent variables will use the same font but be presented in lowercase letters. The complete *Mplus* code for the example can be found in the Appendix.

The MODEL statement for carrying out the analysis is reproduced here but is displayed in pieces corresponding to the measurement model, the Michaelis–Menten model and its corresponding nonlinear constraints, as well as the covariate model.

Measurement Model

```
MODEL:
dec BY d1-d3;
f1 BY f11*(1)
f12 (2)
f13@1
f2 BY f21*(1)
f22 (2)
f23@1
f3 BY f31*(1)
f32 (2)
f33@1
f4 BY f41*(1)
f42 (2)
f43@1
[f13@0 f23@0 f33@0 f43@0];
[f11](3); [f21](3); [f31](3); [f41](3);
[f12](4); [f22](4); [f32](4); [f42](4);
```

Code for the measurement model reflects strong invariance across time, which was chosen based on the results summarized in Table 3 (i.e., equal factor loadings and intercepts for the same indicator across time). Measurement models for both fluency (latent repeated measures) and decoding (latent covariate) are specified using the BY command. Latent decoding, dec, is linked to three indicators d1 to d3. Latent variables f1, f2, f3, and f4 representing reading fluency over time are each connected to the same three observed variable indicators. Intercepts are denoted by listing the names of the indicator variables within square brackets [] with @0 to supersede a default that intercepts are free to be estimated (i.e., [f13@0 f23@0 f33@0 f43@0]) with the factor loadings of these indicators fixed to 1. Variances and covariances for both the unique factors as well as the repeated factor are then specified (to save space, these are not presented here but are instead can be found in the *Mplus* code in the Appendix).

Michaelis–Menten Model

Specification of the Michaelis–Menten model in Equation 8 for latent fluency is not much more complicated than that of a linear growth model. In the context of second-order models, f1, f2, f3, and f4 represent the repeated factor. As a consequence, the elements of $\mathbf{X}(\rho)$ must be constrained to adhere to the functional form of the particular nonlinear model.

In this example, grade level is a discrete variable but it will be used as a continuous indicator of age. Thus, $t_j = 5.75$ is interpreted in an obvious way as halfway through the spring semester of the fifth-grade year. Values of the independent variable for which fluency was measured are $t_1 = 4.25$, $t_2 = 4.75$, $t_3 = 5.75$, and $t_4 = 6.75$. Let $t_j^* = t_j - c$ where c is a constant used to center the function so the intercept is more interpretable (Raudenbush & Bryk, 2002). The transformation, $t_j^* = t_j - 4.25$, makes the parameter corresponding to initial status, β_{1i} , interpreted as the latent fluency of the i th child at the first measurement occasion—midway through the fall semester of Grade 4. After the transformation, the resultant Michaelis–Menten function for the i th individual is:

$$\eta_j = \beta_{1i} + \frac{(\beta_{2i} - \beta_{1i})t_j^*}{\rho + t_j^*} + d_j \quad (11)$$

It should be noted that the times used in the *Mplus* code reflect this transformation (i.e., at time 4.25, the new transformed time will be 0). Under the Michaelis–Menten function in Equation 11, the explanatory regression model in Equation 8 was fit and examined. All models were fit under strong measurement invariance with unique factors for each indicator allowed to covary across time. The error structure for d was specified as $\Delta = \sigma_d^2 \mathbf{I}_m$.

```
b1 BY f1 (L1)
f2 (L2)
f3 (L3)
f4 (L4);
b2 BY f1 (L5)
f2 (L6)
f3 (L7)
f4 (L8);
f1-f4*(vard);
[f1-f4 @0];
b1* b2*; b1 WITH b2*;
[b1*150 b2*180];
```

```

MODEL CONSTRAINT:
NEW (rho);
L1 = 1-(0/(rho + 0));
L2 = 1- (.5/(rho + .5));
L3 = 1- (1.5/(rho + 1.5));
L4 = 1- (2.5/(rho + 2.5));
L5 = 1-L1;
L6 = 1-L2;
L7 = 1-L3;
L8 = 1-L4;

```

Unlike a linear growth model in which elements of the design matrix are constants, the elements of the design matrix for the Michaelis–Menten model are mathematical expressions involving time and the coefficient, ρ , with typical row $[\mathbf{X}(\rho)]_j = [1 - t_j (\rho + t_j)^{-1} t_j (\rho + t_j)^{-1}]$. Both are manipulated into their designated location using the constraint feature with the MODEL CONSTRAINT command. The NEW (rho) command³ allows the user to define additional parameters, in the current example the nonlinear parameter, ρ , and employ it in the nonlinear expressions for specified design matrix elements. Thus, given unbalanced occasions of measurement— $t_j^* = 0, 0.5, 1.50$, and 2.50 —design matrix elements associated with b_1 are modified using constraint labels L1 to L4 and elements associated with b_2 are modified using labels L5 to L8.⁴ Using these constraints, each design matrix element is specified to have the value that would be obtained by substituting the appropriate value of t_j into the Michaelis–Menten function. For example, the expression for L3 in the model constraint command $L3 = 1 - (1.5/(\rho + 1.5))$ corresponds to the third element in the first column of the design matrix. At the third occasion, $t_3 = 1.50$,

$$[\mathbf{X}(\rho)]_{31} = L3 = 1 - \frac{1.50}{(\rho + 1.50)}.$$

The residual variances in $\mathbf{d}_i$ are assumed to have a simple independence structure and thus are constrained to be equal. This can be accomplished using the code $t1-t4*(\text{vard})$. The label vard will appear in the output designating the estimate of this common variance. Because the factor means follow a growth structure they are not estimated in the model and forced to zero using $[f1-f4@0]$. However, the means of $\boldsymbol{\beta}' = (\beta_1, \beta_2)$ are estimated using $[b1*150 \ b2*180]$ along

with the elements of $\Phi_{\beta\beta} = \begin{pmatrix} \text{Var}(\beta_{1i}) & \\ \text{Cov}(\beta_{2i}, \beta_{1i}) & \text{Var}(\beta_{2i}) \end{pmatrix}$ using $b1 * b2^*$; $b1$ WITH $b2^*$;

The values 150 and 180 are starting values for the mean parameters. One possible method of determining these values is to create a weighted composite of the multiple indicators at each time point and use the subsequent composite score as the “raw” data in a conventional first-order growth model to generate potential starting values for the second-order analysis. Starting values for $\Phi_{\beta\beta}$ can be generated in the same manner. It has been our experience that convergence issues when fitting conditionally linear second-order growth models, when they arise, can often be attributable to poor starting values for the variance and covariance parameters of the growth factors or overparameterization of the conditional model in Equation 7. For the current fitted models, there were no computational issues.

³Grimm and Ram (2009) demonstrated how to specify nonlinear latent growth curve models for observed variables using *phantom* variables. This was needed prior to the nonlinear constraints functionality in *Mplus*.

⁴These labels are completely arbitrary and could be set using any notation that was deemed acceptable.

Integration of Latent and Observed Covariates

At a secondary stage of the analysis, covariates are introduced into the modeling scheme to explain individual differences in the growth parameters, namely β_{1i} and β_{2i} from Equation 8. Inclusion of both latent and manifest variable covariates in a regression to explain variability in the individual growth parameters can be specified with the ON command.

```
b1 ON dec sex;  
b2 ON dec sex;
```

Note that the means of b1 and b2 would be different in an unconditional model with no explanatory covariates than they are in the preceding conditional specification. It is recommended that exogenous categorical variables like sex not be listed in the CATEGORICAL statement, but instead be incorporated in the model as a 0 or 1 coded dummy variable (denoting group membership) predictor (Muthén & Muthén, 1998–2010).

RESULTS

As a starting point it might be beneficial to fit an unconditional growth model for the factors, $\eta_{ij} = \beta_{1i} + d_{ij}$ and $\beta_{1i} = \beta_1 + u_i$ to get a sense of how much of the variance in the repeated fluency factor is between subjects. Note, that $Var(d_{ij}) = \sigma_d^2$ and $Var(u_i) = \phi$. This variance-accounted-for measure (the intraclass correlation coefficient [ICC]) follows Raudenbush and Bryk (2002) as

$$\rho_{icc} = \frac{\phi}{\phi + \sigma_d^2} = \frac{595.2}{595.2 + 401.4} = .597.$$

Approximately 60% of the variation in the repeated measures can be accounted for by between-subject differences in growth parameters. This serves as a baseline to compare other models that include subject-specific explanatory covariates. Parameter estimates, standard errors, and fit statistics are tabulated and summarized in Table 4.

The results from Model 2, the third column in Table 4, show that mean initial fluency at the first measurement occasion was 128.7 (1.69) and children reached an average maximum fluency score of 175.2 (4.56). Across the sample, children reached half of their growth, $151.9 = 128.7 + [.5 * (175.2 - 128.7)]$, 1.4 (.29) grades after the study began, or approximately midway through their fifth-grade year. The standard errors were small compared to the estimates, indicating that these coefficients were statistically significant at all nominal levels. The nonlinear coefficient provides important information that could enlighten teaching practices of the timing and intensity of fluency development in primary school grades. For instance, knowing the time span over which half the growth might occur could facilitate targeted programmatic initiatives that are seen as most beneficial.

Individuals' initial status and their potential performance are highly positively correlated:

$$\rho(\beta_{1i}, \beta_{2i}) = \frac{Cov(\beta_{2i}, \beta_{1i})}{\sqrt{Var(\beta_{1i})Var(\beta_{2i})}} = \frac{553.9}{\sqrt{(562.4 \times 601.8)}} = .952.$$

This correlation is indicative of a phenomenon called tracking, where subjects who had initial fluency scores above average ended with fluency scores that also were above average; that is, there is very little crossing of individual trajectories.

Model 2 considered a linear covariate model that posited differences in initial status and potential performance could be explained in part by children's decoding ability and their gender. Initially, primary interest is in computing the amount of variance in status and potential performance accounted for by these two predictors (one latent, one observed). This can be ascertained by examining the standardized⁵ residual variance for β_{1i} and β_{2i} from Model 2.

$$STD[\text{var}(\beta_{1i})]=\phi_{11}^*=.158 \quad STD[\text{var}(\beta_{2i})]=\phi_{22}^*=.263.$$

The standardized variances of β_{1i} and β_{2i} from Model 1 were 1 and 1, respectively. Thus the proportions of variance accounted for by both decoding and gender in these two growth factors are $1 - 0.158 = 0.842$ and $1 - 0.263 = 0.737$. In other words, the variances in initial status and potential performance were reduced 84% and 74%, respectively, by including gender and decoding. However, when examining the coefficients for these explanatory variables, it is clear that gender did not have an effect on either status or potential performance after controlling for differences in decoding by noticing that the standard errors for these two coefficients were large compared to their estimates. Because it is a continuous latent variable, decoding has a regression-type interpretation. One could say that for initial status, a one-unit increase in decoding would result in an approximate 3.5-unit increase in initial fluency level. Likewise, every unit increase in decoding would result in maximum performance in fluency to increase by 3.4 units.

Figure 3 illustrates how the model performs for three randomly chosen subjects, describing both individualistic patterns of fluency development and average performance of the group.⁶ Note that although individual differences in initial status and potential fluency performance are evident, the exponential curves are parallel. Of course, this is to be expected as the rate parameter was fixed to be identical across individuals.

DISCUSSION

This research study considers a conditionally linear mixed effects model for describing nonlinear change in a latent construct over time, where the latent construct is measured by a set of observed variables at each time occasion. In this context, the latent variable serves as the repeated measure and is specified for each individual. To accommodate this added complexity, a measurement model is specified that acknowledges the imperfect nature of the observed variables as indicators of the latent variables. Using this more convoluted model when multiple indicators are available is preferred over merging the indicators (some optimally weighted combination) into a single index because the former allows the latent variable variance, not attributable to the change process, to be partitioned into unique indicator variance and time-specific variance (Blozis, 2006).

The model also permits for tests of longitudinal invariance. The study of measurement invariance in second-order growth models has naturally evolved from comparing multiple populations in a SEM framework (e.g., Bollen, 1989; Meredith, 1993; Meredith & Horn,

⁵The standardized values are not shown in Table 4 but were obtained from the *standardized* output in *Mplus*.

⁶Computation of individual coefficients, β_i is based on the joint normal distribution of β_i and y_i (see Fitzmaurice, Laird, & Ware, 2011). Evaluated at the maximum likelihood estimates, the estimator has an empirical Bayes interpretation (see, e.g., du Toit & Cudeck, 2009, for more details).

2001; Widaman & Reise, 1997). Failure to accept a model that is more constrained (demonstrating a greater degree of invariance) over a less constrained model indicates that the misfit occurs with some or all of the newly constrained parameters. At a minimum, strong metric invariance should be established (Ferrer et al., 2008).

Although not as general as its fully nonlinear counterpart, the conditionally linear mixed effects model for latent variables remains an attractive alternative for many general longitudinal modeling situations encountered in practice. Its methodological value stems from observing that inherent nonlinear parameters, often representing rates of change, in many nonlinear functions do not show great subject-to-subject variability, and therefore, could be fixed across individuals. For example, Cudeck and Harring (2007) showed that in a model describing reading speed as a function of the print size, the knot, an intrinsically nonlinear parameter denoting the smallest print size that could be read with maximum reading speed, was essentially the same for all subjects in the sample. Of course, when a data analytic situation dictates, modules do exist for fitting the fully nonlinear model.

A variant of the Michaelis–Menten model was utilized to model the nonlinear relationship between reading fluency and decoding and *Mplus* was used to fit all models. Through using *Mplus*, this presentation demonstrates yet again that the class of structural models includes more analytic possibilities than is widely realized. This conclusion might please regular users of SEM software, who will not have to stray far to fit another interesting and accessible class of longitudinal models. Also, as shown in the data analysis of fluency, by keeping the analysis in an SEM framework, salient features of *Mplus*—latent covariates, well-understood indexes of model fit, and allowances for missing data and unbalanced designs—can be exploited.

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FIGURE 1.

Path diagram of a second-order conditionally linear mixed model. Manifest variables are represented by squares. Latent variables are denoted by circles. The triangle represents a constant to estimate the means and intercepts. Although not shown, the intercepts of the manifest variables, τ , are estimated. Also not depicted in this figure are the covariances among the unique factors at adjacent time points. Factor loadings are denoted as invariant over time.

FIGURE 2.

Representative Michaelis–Menten function, $y = \beta_1 + \frac{(\beta_2 - \beta_1)x}{\rho + x}$.

FIGURE 3.

Fitted individual Michaelis–Menten functions plotted against individuals' estimated factor scores for three cases. Initial value and asymptotic behavior differ for each subject, but time when half of the growth takes place is fixed across individuals. Trajectory of the population fitted function is given by a dashed line.

TABLE 1
Sample Means, Standard Deviations, and Correlations for the Three Fluency Measures across Four Measurement Occasions

Variable	1	2	3	4	5	6	7	8	9	10	11	12
f11: TOWRE	1.000											
f12: CBM-WI	0.778	1.000										
f13: CBM-PR	0.713	0.785	1.000									
f21: TOWRE	0.816	0.708	0.703	1.000								
f22: CBM-WI	0.683	0.794	0.720	0.763	1.000							
f23: CBM-PR	0.654	0.707	0.838	0.733	0.782	1.000						
f31: TOWRE	0.840	0.688	0.720	0.892	0.679	0.682	1.000					
f32: CBM-WI	0.628	0.772	0.716	0.703	0.821	0.706	0.675	1.000				
f33: CBM-PR	0.684	0.735	0.886	0.768	0.756	0.875	0.746	0.809	1.000			
f41: TOWRE	0.847	0.721	0.739	0.920	0.701	0.692	0.905	0.666	0.739	1.000		
f42: CBM-WI	0.627	0.760	0.700	0.679	0.856	0.686	0.659	0.821	0.709	0.659	1.000	
f43: CBM-PR	0.672	0.730	0.880	0.704	0.702	0.847	0.704	0.743	0.941	0.711	0.732	1.000
<i>M</i>	33.3	65.7	127.6	38.1	77.4	138.9	38.6	80.5	153.4	41.8	88.4	158.7
<i>SD</i>	10.1	15.4	27.2	10.1	17.1	39.2	10.0	16.8	30.3	10.1	18.7	28.5

Note. TOWRE = Test of Word Reading Efficiency Phonemic Efficiency; CBM-WI: Curriculum-Based Measures of Word Identification Fluency; CBM-PR = Passage Reading Fluency.

TABLE 2

Sample Means, Standard Deviations, and Correlations for the Three Decoding Measures

Variables	d1	d2	d3
d1: WJ-III Word Identification	1.000		
d2: WJ-III Word Attack	0.775	1.000	
d3: CTOPP Elision	0.541	0.525	1.000
<i>M</i>	52.9	21.8	14.2
<i>SD</i>	6.12	5.47	4.56

Note. WJ-III = Woodcock-Johnson Test of Cognitive Abilities (3rd ed.); CTOPP = Comprehensive Tests of Phonological Processing.

Tests of Model Comparison and Fit Indexes for Comparing Models Under Increasingly Stringent Measurement Invariance

TABLE 3

Model	Measurement Invariance Level	χ^2	df	Model Comparison	$\Delta\chi^2$	Δdf	p	RMSEA 90% CI	CFI
1	Configural	60.1	30					(.023, .061)	.991
2	Weak $\lambda_{q1} = \dots = \lambda_{qm}$	65.6	36	2 vs. 1	5.5	6	0.481	(.031, .068)	.980
3	Strong $\lambda_{q1} = \dots = \lambda_{qm}$ $\tau_{q1} = \dots = \tau_{qm}$	76.6	42	3 vs. 2	12.0	6	0.051	(.042, .077)	.969

Note. RMSEA = root mean squared error of approximation; CFI = Comparative Fit Index. The baseline model on which RMSEA and CFI are based is a model for which the covariances between manifest variables across time were freely estimated.

TABLE 4

Maximum Likelihood Estimates for Models for the Repeated Measures Following Michaelis–Menten Function with Various Explanatory Covariate Models

	M1: Unconditional	M2: Conditional Covariate Model
Means (β , ρ)		
β_1	127.68 (1.76)	128.73 (1.69)
β_2	175.88 (4.76)	175.19 (4.56)
ρ	1.41 (.31)	1.36 (.29)
Covariate regression coefficients (Γ_1 and Γ_2)		
$\gamma_{\beta_{1on\xi}}$	—	3.57 (.26)
$\gamma_{\beta_{2on\xi}}$	—	3.42 (.30)
$\gamma_{\beta_{1ong}}$	—	−1.84 (1.89)
$\gamma_{\beta_{2ong}}$	—	0.05 (2.68)
Covariance ($\Omega_{\beta\beta}$)		
$Var(\beta_{1i})$	562.35 (73.36)	75.55 (25.76)
$Var(\beta_{2i})$	601.78 (91.08)	130.52 (44.36)
$Cov(\beta_{2i}, \beta_{1i})$	553.97 (73.30)	80.59 (29.04)
$Var(dec_i)$	—	31.29 (3.58)
Residual variance (σ_d^2)	42.15 (6.39)	31.51 (5.74)

Note. Standard errors are shown in parentheses.