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Finite amplitude, horizontal motion of a load symmetrically supported between isotropic hyperelastic springs

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Abstract

The undamped, finite amplitude horizontal motion of a load supported symmetrically between identical incompressible, isotropic hyperelastic springs, each subjected to an initial finite uniaxial static stretch, is formulated in general terms. The small amplitude motion of the load about the deformed static state is discussed; and the periodicity of the arbitrary finite amplitude motion is established for all such elastic materials for which certain conditions on the engineering stress and the strain energy function hold. The exact solution for the finite vibration of the load is then derived for the classical neo-Hookean model. The vibrational period is obtained in terms of the complete Heuman lambda-function whose properties are well-known. Dependence of the period and hence the frequency on the physical parameters of the system is investigated and the results are displayed graphically.

Keywords

Finite amplitude nonlinear vibrations; Hyperelastic springs; Periodicity theorem; Neo-Hookean model; Exact lambda function solution

1. Introduction

A variety of articles have appeared over a number of years that concern the application of hyperelastic models to study the finite amplitude, oscillatory motion of a load supported by rubberlike springs in various configurations. The first of these [1] reported the exact solution of the nonlinear problem of the undamped finite amplitude free vertical vibrations of a mass supported by a neo-Hookean rubber spring in both tension and compression. This work led subsequently to investigation of other finite amplitude, nonlinear vibration problems that investigate the uniaxial [2], shearing [3, 4] and torsional [5] motion of a load on hyperelastic springs. The general motion of a load on a perfectly flexible string is studied in [6]. More recently the effects of limited extensibility in longitudinal and shearing oscillations are reported in [7, 8], and the coupled effect of limited extensibility and stress-softening of rubberlike springs in a longitudinal motion are studied in [9]. The foregoing resources identify a few examples among several others. In this paper, the primary work in [1] again serves as the principal model for the problem to be studied.

Here we return to a classical problem of the finite amplitude uniaxial, horizontal motion of a mass supported symmetrically between identical prestretched, incompressible, isotropic hyperelastic springs, the ends of which are attached to a rigid frame [10]. The influence of the mass of the springs is included for the first time. The general formulation is applied to the classical neo-Hookean model for which an exact simple relation for the period, hence the frequency, is derived in terms of Heuman's complete lambda function for arbitrary values of the primary static stretch λ_0 and the amplitude $\alpha < \lambda_0$. It then follows that the vibrational period of the load is a monotone decreasing function with respect to increasing values of the amplitude, decreasing from its value for small motions about the deformed static configuration and ultimately vanishing at the finite amplitude limit $\alpha = \lambda_0$. Moreover, for each fixed amplitude α , the oscillational period of the load increases monotonically with the underlying static stretch, increasing from its null value at $\alpha = \lambda_0$ and approaching the finite classical value of a corresponding linear oscillator asymptotically as the primary stretch $\lambda_0 \rightarrow \infty$.

To reach these precise results, we begin in Section 2 with the general formulation of the problem for arbitrary isotropic hyperelastic springs. In Section 3 the solution and infinitesimal stability for small vibrations superimposed on the underlying finitely deformed equilibrium state are discussed in general terms and subsequently applied to the incompressible neo-Hookean model. The necessary and sufficient conditions for periodicity of all finite amplitude symmetric uniaxial motions of the load are derived in Section 4 in terms of the engineering stress and the strain energy function for an arbitrary incompressible, isotropic hyperelastic material. Then the complete periodic solution for the motion of a load between identical neo-Hookean springs is derived in Section 5 in terms of the beta function, i.e. the complement of the Heuman lambda-function [1], and from which the vibrational period, hence the frequency, is most simply expressed by a complete lambda function [11, 12] with parameters that depend on the static stretch and the superimposed vibrational amplitude. Bounds on the period follow at once. The dependence of the period on these physical parameters is analyzed in Section 6 and the results are illustrated graphically.

2. Formulation of the problem

A load of mass M is supported by a smooth horizontal surface and connected uniaxially between identical materially uniform, isotropic hyperelastic rods or blocks S_1 and S_2 , called springs, each having an undeformed length L , uniform cross-sectional area A and mass m . The opposite ends of the springs are attached to rigid supports, and we shall suppose that each spring has sustained an initial homogeneous deformation in either uniaxial compression or extension with static stretch λ_0 and deformed length $\ell_0 = \lambda_0 L$ as shown in Fig. 1. Details of the deformation at the attachments, gravitational effects due to the weight of the springs and potential wave motion are ignored. The deformation of the springs is thus assumed uniform throughout, and we suppose that no buckling arises in the compression mode of either spring. Let $\delta(t) = \xi(t)L$ be an arbitrary longitudinal displacement from the central equilibrium position of M at time t , and let $\ell_1(t) = \ell_0 + \delta(t)$ and $\ell_2(t) = \ell_0 - \delta(t)$ denote the instantaneous lengths of S_1 and S_2 , respectively. With $\ell_k(t) = \lambda_k(t)L$, it is evident that the current state of the system may be described completely in terms of the instantaneous total stretch ratios $\lambda_1(t)$ and $\lambda_2(t)$ of the springs as functions of the displacement variable $\xi(t)$:

$$\lambda_1(t) = \lambda_0 + \xi(t), \quad \lambda_2(t) = \lambda_0 - \xi(t). \quad (2.1)$$

Plainly, these are not independent; we have

$$\lambda_1 + \lambda_2 = 2\lambda_0, \lambda_1 - \lambda_2 = 2\xi. \quad (2.2)$$

Moreover, the physical requirement that $\lambda_k > 0$ leads to the following bounds on $\xi(t)$:

$$-\lambda_0 < \xi(t) < \lambda_0. \quad (2.3)$$

2.1. The strain energy function

A general materially uniform, isotropic hyperelastic material is characterized by a strain energy function $W = W(\nu_1, \nu_2, \nu_3)$, per unit undeformed volume, where the $\nu_k > 0$ denote the three principal stretches. We shall suppose that W is a real analytic function of ν_k . For a simple tension or compression [13], the equibiaxial lateral stretches $\nu_1 = \nu_2 = \nu(\nu_3)$ may be expressed as a certain function of the axial stretch $\nu_3 \equiv \lambda$; and hence the strain energy may be written as a function $\hat{W}(\lambda)$ of λ alone:

$$W = W(\nu(\lambda), \nu(\lambda), \lambda) = \hat{W}(\lambda). \quad (2.4)$$

We require that the strain energy vanish in the undeformed state $\hat{W}(1) = 0$. The uniaxial engineering stress Σ is now determined by the constitutive equation

$$\Sigma = \frac{\partial \hat{W}(\lambda)}{\partial \lambda}. \quad (2.5)$$

Alternatively, the corresponding uniaxial Cauchy stress is given generally by $\sigma_c = J^{-1} \lambda^{-1} \hat{W}(\lambda) / \lambda$ in which $J = \nu_1 \nu_2 \nu_3 = J(\lambda)$ is the ratio of the deformed to the undeformed material volume. For an incompressible material, however, the stress is determined only to within an arbitrary pressure and $J = 1$ for all λ . The transverse stretches in this instance are uniquely determined as $\nu_1 = \nu_2 = \lambda^{-1/2}$; otherwise, the form of $\nu(\lambda)$ for a compressible material will depend on the specific nature of the constitutive equation [13].

2.2. Kinetic energy of a spring

Before addressing the formulation of the equations of energy and motion for the load, we pause to review the dynamical effects of the mass of the springs themselves. Let us consider a homogeneous, elastic rod or block S of mass m , length L and uniform cross-sectional area A in the natural state κ_0 , and subjected to a uniform uniaxial stretch $\lambda(t)$ in the deformed state κ at time t . Then a cross-sectional element of thickness dX at X with mass $dm = (m/L) dX$ in κ_0 is mapped to an element $dx = \lambda dX$ with the same mass $dm = (m/\lambda L) dx$ at $x(X, t) = \lambda(t)X$ in κ . The uniaxial velocity of dm at x is $\dot{x}(X, t) = \dot{\lambda}(t)X$, which at $X = L$ is equal to the velocity $\dot{x}(L, t) = \dot{\lambda}(t)L$ of the point of attachment of the rod to the rigid load M , and hence equal to the longitudinal velocity $\dot{\chi}_{cm}(t)$ of its center of mass at $\chi_{cm} = x(L, t) + c$ for a

certain constant c . Thus, the total kinetic energy of S : $K_s = \frac{1}{2} \int_S \mathbf{v} \cdot \mathbf{v} dm$ is given by

$$K_s = \frac{1}{2} \frac{m}{L^3} \dot{\chi}_{cm}^2 \int_0^L X^2 dX = \frac{1}{2} \frac{m}{3} \dot{\chi}_{cm}^2. \quad (2.6)$$

This is the well-known classical result due to Rayleigh [14, art. 156, p. 250] that accounts for the inertia of S in a uniaxial motion of a load. The generally accepted common simplification in such problems is to ignore as negligible the potential effect of the mass of a

springy body in the analysis of motion of a load, usually considered more massive with $m \ll M$. Still, it is useful to learn the effects of the inertia of highly elastic rods or springs in the nonlinear problem studied here, though the affect of the weight of the springs on their transverse deflection is ignored. We now return to the formulation of the equations of energy and motion for the load.

2.3. Equations of energy and motion of the load

There are no dissipative forces acting on the load, so the system is conservative with constant total energy given by

$$E = K_M(\dot{\xi}) + 2K_S(\dot{\xi}) + ALU(\xi), \quad (2.7)$$

where K_M is the kinetic energy of the load, K_S is the kinetic energy (2.6) of a spring, and $U(\xi)$ is the total elastic potential energy, per unit undeformed volume, of the springs. With the aid of (2.1) in (2.4), we have for the two springs

$$U(\xi) = \hat{W}(\lambda_0 + \xi) + \hat{W}(\lambda_0 - \xi), \quad (2.8)$$

in which we recall (2.3). Moreover, the potential energy $U(0) = 2\hat{W}(\lambda_0) > 0$ at the equilibrium state $\xi = 0$ and $U(-\xi) = U(\xi)$ is an even function of $\xi(t)$. The center of mass of the load at time t is at $\chi_{cm} = \lambda_1 L + c$ from the left support; and hence with (2.1), we have

$$K_M = \frac{1}{2} M \dot{\chi}_{cm}^2 = \frac{1}{2} M L^2 \dot{\xi}^2. \quad (2.9)$$

In accordance with (2.6), the total kinetic energy of the springs is

$$2K_S = \frac{1}{3} m L^2 \dot{\xi}^2. \quad (2.10)$$

Thus, the constant total energy (2.7) of the system is given by

$$E = \frac{1}{2} \mu L^2 \dot{\xi}^2 + ALU(\xi), \quad (2.11)$$

wherein, by definition,

$$\mu = M + \frac{2}{3} m \quad (2.12)$$

is the effective mass of the system. It follows from (2.11) that the nonlinear differential equation of motion for M from its equilibrium state at $\xi = 0$ is given by

$$\ddot{\xi} + \frac{A}{\mu L} \frac{\partial U(\xi)}{\partial \xi} = 0. \quad (2.13)$$

This yields the equilibrium condition at $\xi = 0$;

$$\left. \frac{\partial U(\xi)}{\partial \xi} \right|_{\xi=0} = 0. \quad (2.14)$$

If the mass is released from $\xi(0) = \xi_0$ with velocity $v(0) = \dot{\xi}(0) = v_0$ at $t = 0$ so that (2.3) holds, then (2.11) determines the current velocity $v(\xi) = \dot{\xi}$ at ξ :

$$v^2(\xi) = v_0^2 + \frac{2A}{\mu L}(U(\xi_0) - U(\xi)). \quad (2.15)$$

Clearly, $|v(-\xi)| = |v(\xi)|$ and so we conclude that solutions that pass through $\xi = 0$ are symmetric about the ξ -axis $v = 0$.

The time required for M to move from its initial state (ξ_0, v_0) to a current state (ξ, v) consistent with (2.3) can be expressed as

$$t = \pm \int_{\xi_0}^{\xi} \frac{d\xi}{v(\xi)}, \quad (2.16)$$

in which the appropriate sign is to be chosen consistent with the initial data.

We shall establish later on precise conditions for which the finite motion is periodic. In the meantime, however, we may assert loosely that if the motion is symmetric with turning points $\xi = \pm \alpha$ at which $v^2(\pm \alpha) = 0$, then with the use of (2.15) in (2.16) the period τ is determined by

$$\tau = 4 \int_0^{\alpha} \frac{d\xi}{\sqrt{\frac{2A}{\mu L}(U(\alpha) - U(\xi))}}. \quad (2.17)$$

3. Small amplitude oscillations

We shall first study the small amplitude oscillations of the load about the deformed static stretch λ_0 . Writing $\xi = \xi_e + x$ with $x \ll 1$ in (2.13), expanding $U(\xi)/\xi$ in a power series about the static state $\xi_e = 0$ at which (2.14) holds and recalling (2.8), we obtain the differential equation for a simple harmonic oscillator characterizing the small amplitude motion of the load:

$$+\omega^2 x = 0, \quad (3.1)$$

accurate to terms of $O(x^3)$ and wherein

$$\omega^2 = \frac{2A\hat{W}_2(\lambda_0)}{\mu L}. \quad (3.2)$$

Here and below we adopt the abbreviation

$$\hat{W}_k(u) = \left. \frac{d^k \hat{W}(\lambda)}{d\lambda^k} \right|_{\lambda=u}. \quad (3.3)$$

Of course, (3.1) also may be readily derived by approximation from the energy Eq. (2.11). The rule (3.2) holds for both compressible and incompressible materials.

In general terms, it follows from (3.2) that the deformed equilibrium state at λ_0 is infinitesimally stable with circular solutions if and only if $\omega^2 > 0$, in which case ω is the

circular frequency of the small amplitude oscillations about λ_0 . If $\omega^2 = 0$ with $\omega_{\min}^2 < 0$, then the static state is unstable with exponential solutions. If $\omega^2 = \omega_{\min}^2 = 0$ at λ_0 , the static state is said to be neutrally stable. We shall return to this criterion momentarily.

We next recall (2.5), introduce the normalized engineering stress $\tilde{\Sigma}(\lambda) = \Sigma(\lambda)/G_0$, where here and throughout G_0 denotes the shear modulus of the natural state of the material, and write (3.2) as

$$\omega = \omega^* \sqrt{\frac{\partial \tilde{\Sigma}(\lambda_0)}{\partial \lambda_0}}, \quad (3.4)$$

wherein, by definition,

$$\omega^* = \frac{2\pi}{\tau^*} \equiv \sqrt{\frac{\kappa}{\mu}}, \kappa \equiv \frac{2G_0 A}{L}. \quad (3.5)$$

Physically, ω^* is the circular frequency of a load of mass μ on a linear elastic spring having an equivalent spring constant or stiffness κ ; and τ^* is its period. We note that ω depends not only on the geometry and material characteristics of the springs, but also on their mass and their primary static stretch λ_0 . Clearly, the effect of including the mass of the springs is to reduce the oscillational frequency, hence increase the period $\tau = 2\pi/\omega$. The effects of varying the dimensions of the springs is also evident. Of course, the shear modulus is related to Young's modulus in accordance with $E = 2G_0(1 + \nu)$ in which ν is Poisson's ratio; but we shall make no use of this here.

We thus find in accordance with (3.4) that a load symmetrically supported uniaxially by prestretched, isotropic elastic springs is infinitesimally stable with respect to small vibrations about all equilibrium states at which the engineering stress is a monotone increasing function of the static stretch λ_0 , i.e. if and only if $\tilde{\Sigma}'/\lambda_0 > 0$. Otherwise, there exists one or more values λ_0^* of the imposed stretch at which the stress response is stationary (including an inflection), the frequency vanishes, and hence the equilibrium state is unstable or neutrally stable. Clearly, $\tilde{\Sigma}'/\lambda_0 < 0$ holds for all equilibrium states that follow a global stress maximum, if any exist, and for all equilibrium states between points of maximum and minimum stress of any N-shaped engineering stress response curve, in which case $\omega(\lambda_0)$ is pure imaginary and all such equilibrium states are unstable. At a critical point λ_i^* of stress inflection, however, $\omega(\lambda_i^*) = \omega_{\min} = 0$ and the stability is neutral. In this case the stress response is monotone nondecreasing. The extreme stress at a stationary point is determined in the usual way.

Plainly, the stress response depends on the nature of the constitutive equation of the material, and hereafter we shall consider only incompressible rubberlike materials. The

molecular based neo-Hookean model [15] with strain energy function $W(I_1) = \frac{1}{2}G_0(I_1 - 3)$ provides the simplest example. In a simple extension, we have

$$\hat{W}(\lambda) = \frac{1}{2}G_0(\lambda^2 + 2\lambda^{-1} - 3) \quad (3.6)$$

from which (2.5) yields

$$\tilde{\Sigma}(\lambda) = \lambda - \frac{1}{\lambda^2}. \quad (3.7)$$

It is at once evident that the normalized uniaxial engineering stress for the neo-Hookean material is monotone increasing for all $\lambda \in (0, \infty)$. Indeed, $\tilde{\Sigma}(\lambda)/\lambda = 1 + 2/\lambda^3 > 0$ for all λ ; and hence the deformed equilibrium state of the symmetric neo-Hookean oscillator is infinitesimally stable with frequency

$$\omega(\lambda_0) = \omega^* \sqrt{1 + \frac{2}{\lambda_0^3}} \quad (3.8)$$

for all λ_0 . It thus follows that ω decreases monotonically from a vertical asymptote with $\lim_{\lambda_0 \rightarrow 0} \omega(\lambda_0) = +\infty$ to a horizontal asymptote with $\lim_{\lambda_0 \rightarrow \infty} \omega(\lambda_0) = \omega^*$, the curve passing through the origin (the undeformed configuration) with frequency $\omega(1) = \sqrt{3}\omega^*$. Hence, the small amplitude vibrational frequency of a load superimposed on the deformed configuration of a symmetric neo-Hookean oscillator increases with compression of the springs and decreases with their elongation. Needless to say, this assumes that the homogeneous uniaxial deformation is viable for all values of λ_0 ; but one must bear in mind that buckling modes may arise for sufficiently large compression and depending on the geometry of the springs. And, of course, no material can be extended indefinitely. We note that the Gent model, among others, imposes a limit on the extensibility of rubberlike materials. Similar kinds of small amplitude results may be obtained for more complex material models, including the Mooney–Rivlin [16], Gent [17] and experimentally based 3-term Ogden [18] models, as proved generally below for finite motions.

4. Finite amplitude motion

Let us return to the finite amplitude problem and recall the symmetric potential energy function $U(\xi) = U(-\xi)$ of the springs (2.8) for $-\lambda_0 < \xi < \lambda_0$, and that $U(0) = 2\hat{W}(\lambda_0) > 0$. To establish periodicity of the finite amplitude motion of the load about the deformed equilibrium state with initial stretch λ_0 for general isotropic hyperelastic springs, we shall prove that for any assigned initial values (ξ_0, v_0) consistent with the foregoing relations there always exists at most two real (symmetric) roots $\xi = \pm \gamma$ for which $v^2(\pm \gamma) = 0$ in (2.15), that is, for which

$$U(\gamma) = U(\xi_0) + \frac{\mu L}{2A} v_0^2 \equiv \mathcal{C} \quad \text{a positive constant;} \quad (4.1)$$

in fact, $\mathcal{C} > 2\hat{W}(\lambda_0)$. Hence, geometrically, we show that under conditions to be determined the horizontal line $U(\gamma) = \mathcal{C}$ for any pair (ξ_0, v_0) for which $-\lambda_0 < \xi < \lambda_0$ intersects the curve $U(\xi)$ in two and only two points at $\xi = \pm \gamma$.

First, we differentiate (2.8) and recall (2.5) to obtain

$$\frac{dU(\xi)}{d\xi} = \Sigma(\lambda_0 + \xi) - \Sigma(\lambda_0 - \xi), \quad (4.2)$$

from which we confirm the equilibrium condition (2.14) at $\xi = 0$. Consider an arbitrary symmetry point $\xi = \delta$ and observe from (4.2) that, as expected,

$$\left. \frac{dU(\xi)}{d\xi} \right|_{\xi=-\delta} = - \left. \frac{dU(\xi)}{d\xi} \right|_{\xi=\delta} \quad (4.3)$$

for all $\lambda_0 \in (0, \infty)$. It thus follows from (4.2) and in accord with (4.3) that the general potential energy function for any given static stretch $\lambda_0 \in (0, \infty)$ is a concave upward parabolic-like function, possibly having wavy sides, symmetric about its global minimum at $\xi = 0$, if and only if $dU(\xi)/d\xi|_{\xi>0} > 0$ for all $\lambda_0 \in (0, \infty)$; and therefore, by (4.2), when and only when the uniaxial engineering stress $\Sigma(\lambda)$ is a monotone increasing function of the stretch λ , and hence if and only if $\dot{W}_2(\lambda) > 0$ for all $\lambda \in (0, 2\lambda_0)$ with $\lambda_0 \in (0, \infty)$. Moreover, $U(\xi)$ grows indefinitely toward vertical asymptotes at $\xi = \pm \lambda_0$ provided that the strain energy function in (2.8) satisfies $\lim_{\lambda \rightarrow 0} \dot{W}(\lambda) \rightarrow \infty$. In consequence, under these conditions and for arbitrary initial data and for any assigned initial static stretch, there exists at most two points $\xi = \pm \gamma$ at which the line $U(\xi) = \mathcal{E} > 2 \dot{W}(\lambda_0)$ intersects the parabolic-like potential energy curve; and hence the motion of the load is periodic and symmetric about the deformed state of the springs at $\xi = 0$. In geometrical terms, the phase plane map of $\pm v(\xi(t))$ in (2.15) is a symmetric closed curve about the equilibrium point $\xi = 0, v = 0$. Obviously, then, for sufficiently small amplitudes we recover our earlier result for the infinitesimal stability of the corresponding small free vibration of the load about an arbitrary uniaxial static stretch λ_0 when and only when the engineering stress is a monotone increasing function of λ_0 . It should be mentioned that our periodicity theorem for all finite motions was first proved by Young [10, pp. 66–74] using an altogether different analytical argument.

It is clear that exact analytical solution of the finite amplitude vibration problem is most difficult, if not impossible for various kinds of complex models, including the Gent [17] and Ogden [18] models, among others. Our periodicity result, however, is much easier to test. It can be shown, for example, that both the 3-term empirical Ogden and Mooney–Rivlin models, and hence the neo-Hookean model as well, are characterized in a uniaxial deformation by a continuous concave upward strain energy function $\dot{W}(\lambda)$ that grows indefinitely as $\lambda \rightarrow 0$ and to ∞ , with its minimum at the undeformed state where it vanishes. Moreover, the uniaxial engineering stress for both models is monotone increasing with the stretch, growing from $-\infty$ at $\lambda = 0$ to $+\infty$ as $\lambda \rightarrow \infty$. It thus follows that the motion of a load supported symmetrically with either Mooney–Rivlin or Ogden material springs is periodic, and therefore stable and symmetric about $\xi = 0$ for all amplitudes $\gamma \in (-\lambda_0, \lambda_0)$. Similar results hold for the Gent material. In this case, however, a minor adjustment in the argument is needed because the energy limits occur at finite, positive limited extensibility stretches in compression and extension. Although exact analysis for the Gent and Ogden materials may be formidable, the oscillatory nature of the solution for these models is clear. On the other hand, the exact solution for the Mooney–Rivlin material is known [10]. It is instructive, however, to focus first on the simple neo-Hookean model, still a bit complicated, and reserve study of the inclusive Mooney–Rivlin case for another time.

5. The symmetric neo-Hookean oscillator

Let us consider the initial value problem described by (2.15) for the molecular based neo-Hookean rubberlike material [15] whose strain energy function in a homogeneous uniaxial stretch is given in (3.6). It is evident that the conditions on $\dot{W}(\lambda)$ and $\dot{\Sigma}(\lambda)$ in (3.7) for periodicity of a finite amplitude motion about the equilibrium state with stretch λ_0 are satisfied. So, there exists turning points $\gamma = \pm \alpha$ of the symmetric periodic motion of the load about $\xi = 0$ given by (2.15). Substitution of (2.8) and (3.6) into (2.15) evaluated at a turning point $v_0 = 0, \xi_0 = \alpha$, we find

$$v_0^2 - \frac{\kappa}{\mu} \left(\frac{(\alpha^2 - \xi_0^2)(\eta_0^2 - \alpha^2)}{(\lambda_0^2 - \alpha^2)} \right) = 0, \quad (5.1)$$

wherein κ is the equivalent spring constant (3.5)₂ and $\eta_0^2 = N(\xi_0)$ in which, by definition,

$$N(\xi) \equiv \lambda_0^2 + \frac{2\lambda_0}{\lambda_0^2 - \xi^2}. \quad (5.2)$$

So, (5.1) determines $\alpha(\xi_0, v_0)$ in terms of the initial data. Similarly, the uniaxial velocity of the load in (2.15) may be written as

$$v(\xi) = \frac{d\xi(t)}{dt} = \pm \omega^* \sqrt{\frac{(\alpha^2 - \xi^2)(\eta^2 - \xi^2)}{(\lambda_0^2 - \xi^2)}}, \quad (5.3)$$

wherein we recall (3.5)₁ and, by (5.2), write

$$\eta^2 \equiv N(\alpha) = \lambda_0^2 + \frac{2\lambda_0}{\lambda_0^2 - \alpha^2}. \quad (5.4)$$

Finally, integration of (5.3) delivers the time of travel of the load from its initial position ξ_0 to its current position ξ in accordance with (2.16);

$$t = \pm \frac{\tau^2}{2\pi} [\hat{I}(\xi) - \hat{I}(\xi_0)], \quad (5.5)$$

in which $\hat{I}(\xi)$ is the hyperelliptic integral

$$\hat{I}(\xi) \equiv \int_0^\xi \sqrt{\frac{\lambda_0^2 - \xi^2}{(\alpha^2 - \xi^2)(\eta^2 - \xi^2)}} d\xi. \quad (5.6)$$

We note from (2.3) and (5.2) that the parameters α , η , and λ_0 are ordered as

$$-\lambda_0 < -\alpha \leq \xi \leq \alpha < \lambda_0 < \eta. \quad (5.7)$$

This completes the formal solution of the problem for the neo-Hookean model in general terms. The exact closed form solution is developed next.

5.1. Reduction to standard form

First, we reduce $\hat{I}(\xi)$ to a standard Legendre form by use of the transformation

$$\xi^2 = \frac{\alpha^2 \eta^2 \sin^2 \phi}{\eta^2 - \alpha^2 \cos^2 \phi}, \quad (5.8)$$

$$\phi = \sin^{-1} \sqrt{\frac{(\eta^2 - \alpha^2)\xi^2}{\alpha^2(\eta^2 - \xi^2)}}. \quad (5.9)$$

Then (5.6) becomes

$$\hat{I}(\xi) = I(\phi) = \frac{\lambda_0}{\sqrt{\eta^2 - \alpha^2}} \int_0^\phi \frac{\Delta \phi \, d\phi}{1 + n' \sin^2 \phi}, \quad (5.10)$$

wherein, as usual, $\Delta \phi = \sqrt{1 - k^2 \sin^2 \phi}$, the modulus k is given by

$$k^2 \equiv \frac{\alpha^2(\eta^2 - \lambda_0^2)}{\lambda_0^2(\eta^2 - \alpha^2)}, \quad (5.11)$$

and

$$n' \equiv \frac{\alpha^2}{\eta^2 - \alpha^2}. \quad (5.12)$$

Clearly, since $0 < \alpha < \lambda_0$, $k \in (0, 1)$. The integral (5.10) is now easily identified as

$$\hat{I}(\xi) = I(\phi) = \sqrt{\frac{\lambda_0^2}{\eta^2 - \alpha^2}} \left[\frac{\eta^2}{\lambda_0^2} \Pi(\phi; n', k) - \frac{\eta^2 - \lambda_0^2}{\lambda_0^2} F(\phi; k) \right], \quad (5.13)$$

wherein here and below $F(\phi; k)$ is the elliptic integral of the first kind with modulus k and $\Pi(\phi; n', k)$ is the elliptic integral of the third kind [12]. This yields the travel time (5.5) explicitly. The period of the oscillation, with (5.4), (5.9) and (5.13), is then given by $\tau = (4\tau^*/2\pi)\hat{I}(\alpha) = (2\tau^*/\pi)\hat{I}(\pi/2)$ in terms of complete elliptic integrals $K(k) = F(\pi/2; k)$ and $\Pi(n', k) = \Pi(\pi/2; n', k)$ of the first and third kinds, respectively [12]. Still, as shown similarly in [1], a further reduction that leads to a simple vibrational period-amplitude relation is possible. Let us continue with this in mind.

We observe from (5.11) and (5.12) that in (5.13)

$$\frac{\eta^2 - \lambda_0^2}{\lambda_0^2} = \frac{k^2}{n'}, \quad (5.14)$$

and

$$\sqrt{\frac{\lambda_0^2}{\eta^2 - \alpha^2}} = \sqrt{\frac{1 + n'}{1 + k^2/n'}}. \quad (5.15)$$

We next recall the beta function $B(\phi; n', k)$ defined by

$$B(\phi; n', k) \equiv \frac{3}{\pi} \sqrt{\frac{1 + n'}{1 + k^2/n'}} \left[\left(1 + \frac{k^2}{n'} \right) \Pi(\phi; n', k) - \frac{k^2}{n'} F(\phi; k) \right], \quad (5.16)$$

as shown in [1, Eq. (6.17)]. Consequently, $I(\phi)$ in (5.13) is precisely a simple multiple of the B -function:

$$I(\phi) = \frac{\pi}{2} B(\phi; n', k). \quad (5.17)$$

Recalling (5.5) in which $\hat{I}(\xi) = I(\phi)$, we may now write the general solution of our initial value problem in terms of the B -function:

$$t = \pm \frac{\tau^*}{4} [B(\phi; n', k) - B(\phi_0; n', k)] \quad (5.18)$$

for arbitrary initial data with $\phi_0 = \phi(\xi_0)$ in (5.9). The period is now determined succinctly as shown next.

5.2. Period-amplitude equation

We recall that the beta function $B(\phi; n', k)$ is the complement to Heuman's [11] lambda function $\Lambda(\phi; n, k)$. It is shown in [1, Eq. (7.4)] that for the same angle ϕ these functions are related by

$$B(\phi; n', k) = \Lambda(\phi; n, k) + \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{-nn'} \sin \phi \cos \phi}{\Delta \phi} \right), \quad (5.19)$$

where n is defined by

$$n \equiv -n' \left(\frac{1+k^2/n'}{1+n'} \right) = -\frac{\alpha^2}{\lambda_0^2}. \quad (5.20)$$

Consequently, by (5.18), the general solution of the initial value problem also may be formulated in terms of Heuman's Λ -function. We omit this representation and observe from (5.16) and (5.19) that $\Lambda(0; n, k) = B(0; n', k) = 0$ and $B(\pi/2; n', k) = \Lambda(\pi/2; n, k) \equiv \Lambda_0(\phi; k)$ is the complete Heuman lambda function defined by

$$\Lambda_0(\varphi; k) \equiv \frac{2}{\pi} \{ [E(k) - K(k)] F(\varphi; k') + K(k) E(\varphi; k') \}, \quad (5.21)$$

which is readily computed [11, 12]. Herein $E(\varphi; k')$ denotes the usual elliptic integral of the second kind with the complementary modulus $k' \equiv \sqrt{1-k^2}$,

$$\varphi \equiv \sin^{-1} \sqrt{\frac{1+k^2/n}{1-k^2}} = \sin^{-1} \sqrt{\frac{1}{1+k^2/n'}}, \quad (5.22)$$

and $E(k)$ is the corresponding complete integral. Explicitly, with (5.4), (5.11) and (5.14),

$$k^2 = \frac{2\alpha^2}{\lambda_0 [(\lambda_0^2 - \alpha^2)^2 + 2\lambda_0]}, \quad (5.23)$$

$$\varphi = \sin^{-1} \frac{\lambda_0}{\eta} = \sin^{-1} \sqrt{\frac{\lambda_0(\lambda_0^2 - \alpha^2)}{\lambda_0(\lambda_0^2 - \alpha^2) + 2}}. \quad (5.24)$$

Finally, it is seen from (5.9) that as ξ goes from 0 to α , representing one quarter of the period τ of the oscillation, ϕ varies from $\phi_0 = 0$ to $\pi/2$. Therefore, from (5.18), $\tau = \tau^* B(\pi/2; n, k)$ and hence the period τ of the finite amplitude oscillation is given exactly and most simply by

$$\tau = \tau^* \Lambda_0(\varphi; k). \quad (5.25)$$

Properties and special values of $\Lambda_0(\varphi; k)$ are provided in [12, pp. 35–37] with graphs illustrating clearly that all values of $\Lambda_0(\varphi; k)$ for $k \in [0, 1]$ fall within the area enclosed by the $\sin \varphi$ curve and its chord $2\varphi/\pi$ for $\varphi \in [0, \pi/2]$. Hence, all values of $\Lambda_0(\varphi; k)$ for the physical variables λ_0 and $\alpha < \lambda_0$ in (5.23) and (5.25) fall within this bounded region for $0 < k < 1$ and $0 < \varphi < \pi/2$, and may fall upon the boundary only as limit values $k = 0$ for $\alpha \rightarrow 0$ and $k = 1$ for $\alpha \rightarrow \lambda_0$. That is, the oscillational period τ and hence the frequency ω of the load given precisely by (5.25) for all amplitudes $0 < \alpha < \lambda_0$ are bounded as follows:

$$0 \leq \frac{2\varphi}{\pi} \bigg|_{k=1} < \frac{\tau}{\tau^*} = \frac{\omega^*}{\omega} < \sin \varphi \big|_{k=0} \leq 1. \quad (5.26)$$

Therefore, the finite amplitude, free vibrational frequency ω (period τ) of a symmetrically supported neo-Hookean oscillator of effective mass μ is always greater (smaller) than the frequency ω^* (period τ^*) of a linear spring oscillator having the same mass and constant stiffness κ . Recall that the result includes the mass of the springs in (3.5)₁. We conclude with some details regarding the period, hence the frequency, as a function of the amplitude.

6. Concluding remarks on the period vs. amplitude relation

Of course, our earlier result for the small amplitude frequency and period follow readily from (5.24) to (5.26) in the limit as the amplitude $\alpha \rightarrow 0$ with $k \rightarrow 0$ to obtain

$$\Lambda_0(\varphi; 0) = \sin \varphi = \sqrt{\frac{\lambda_0^3}{\lambda_0^3 + 2}}. \quad (6.1)$$

Thus, the period approaches this upper bound in (5.26), and (5.25) returns the small amplitude frequency rule (3.8). Moreover, since $\Lambda_0(0, k) = 0$ for all $k \in [0, 1]$ and by (5.24) $\varphi \rightarrow 0$ as the amplitude $\alpha \rightarrow \lambda_0$, we have $\tau/\tau^* = 0$ by (5.25). Indeed, by (5.23), $\alpha \rightarrow \{0, \lambda_0\}$ if and only if $k \rightarrow \{0, 1\}$, respectively. Finally, because $\Lambda_0(\varphi; k)$ is a monotone function decreasing with respect to increasing $k \rightarrow (0, 1)$ and decreasing with respect to decreasing φ , we conclude by (5.25) that the vibrational period of the load is a monotone decreasing function of the amplitude α , varying from its greatest value in (6.1) for small α to its least value $\tau = 0$ at its limit $\alpha = \lambda_0$. Then, of course, the vibrational frequency of the load increases monotonically with the amplitude. In analytical terms, the reader will find that the aforementioned monotonicity properties as functions of α may be established independently from $d \sin^2 \varphi / d(\alpha^2) < 0$ and $dk^2 / d(\alpha^2) > 0$ which hold for all $\alpha \in [0, \lambda_0]$.

To illustrate this general result for the period versus amplitude dependence at arbitrary amplitudes $\alpha < \lambda_0$, we turn to (5.21) for computation of the Heuman lambda function in

which we recall (5.23) and (5.24). Then for the set of static stretch values $\lambda_0 = \{\frac{1}{2}, 1, 2, 3, 4\}$, the normalized period of the oscillations described by (5.25) for $\alpha < \lambda_0$ is mapped in Fig. 2 as $\tilde{\tau} = \tau/\tau^*$ versus α . The corresponding normalized frequency ratio is $\tilde{\omega} = \omega/\omega^* = 1/\tilde{\tau}$. As shown generally above, it is seen that for each value of the primary stretch λ_0 the normalized period (frequency) is a monotone decreasing (increasing) function of the amplitude $\alpha \in (0, \lambda_0)$. Several numerical values of $\tilde{\tau}$ for the small amplitude oscillations about the deformed static state obtained from (6.1) are recorded in the diagram. Notice that for a large primary stretch $\lambda_0 = 4$, say, the curve is fairly flat for $\alpha \ll 3$, so $\tilde{\tau}$ is equal to roughly 98% of its ultimate limit value $\tilde{\tau} = 1/\tilde{\omega} = 1$ for which $\lambda_0 \rightarrow \infty$.

The relationship between the period and the underlying static stretch is explored next. We see from (5.23) and (5.24) that as $\lambda_0 \rightarrow \{\alpha, \infty\}$, $k \rightarrow \{1, 0\}$ and $\phi \rightarrow \{0, \pi/2\}$, respectively. And it is known that $\Lambda_0(0; k) = 0$ and $\Lambda_0(\pi/2; k) = 1$ for all k . Therefore, from the aforementioned dependence of $\Lambda_0(\phi; k)$ on its parameters, we conclude that for any fixed amplitude α , the normalized period of oscillations of the load is a monotone increasing function that grows from $\tilde{\tau} = 0$ to 1 (asymptotically) as the static stretch advances from $\lambda_0 = \alpha$ to ∞ . The aforementioned monotonicity properties may be confirmed independently by our noting that $d\sin^2\phi/d\lambda_0 > 0$ and $dk^2/d\lambda_0 < 0$ hold for all $\lambda_0 \in [\alpha, \infty]$.

For the set of fixed amplitudes $\alpha = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$ used in (5.21), the normalized period relation (5.25) plotted in Fig. 3 shows that for all α the normalized period $\tilde{\tau}$ increases rapidly and monotonically toward its limit $\tilde{\tau} = 1$ as the underlying static stretch grows indefinitely great. So, for a sufficiently large but finite stretch the period is virtually independent of the amplitude.

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Fig. 1. Horizontal motion of a load M symmetrically supported by isotropic hyperelastic springs S_1 and S_2 of mass m .

Fig. 2. Normalized period τ/τ^* vs. the amplitude α of the oscillation of a load superimposed on a specified static stretch λ_0 of a symmetric neo-Hookean oscillator.

Fig. 3. Normalized period τ/τ^* vs. the static stretch λ_0 for a specified value of the amplitude α of the oscillation of a load with symmetric neo-Hookean springs.