Model for bubble pulsation in liquid between parallel viscoelastic layers

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A model is presented for a pulsating spherical bubble positioned at a fixed location in a viscous, compressible liquid between parallel viscoelastic layers of finite thickness. The Green's function for particle displacement is found and utilized to derive an expression for the radiation load imposed on the bubble by the layers. Although the radiation load is derived for linear harmonic motion it may be incorporated into an equation for the nonlinear radial dynamics of the bubble. This expression is valid if the strain magnitudes in the viscoelastic layer remain small. Dependence of bubble pulsation on the viscoelastic and geometric parameters of the layers is demonstrated through numerical simulations. © 2012 Acoustical Society of America.

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I. INTRODUCTION

The central role that gas and encapsulated microbubbles often play in biomedical acoustics applications,¹ for example in medical imaging² or targeted drug delivery,^{3,4} has generated increased interest in modeling cavitation in constraining environments.^{5–25} In medical applications the bubble is influenced by the close proximity of the surrounding tissue or vessel walls. Bubble translation, jet formation, and deformation of the confining surfaces have all been observed when bubbles are excited near interfaces,^{23,26–30} and the radial dynamics is also affected.³¹ In the present work a model is derived for the radial dynamics of a pulsating bubble positioned in a channel formed by parallel viscoelastic layers. This geometry is a first-order approximation of a bubble pulsating in a blood vessel.

For large channels the Rayleigh–Plesset equation, which describes the pulsation of a spherical bubble in an unbounded liquid, is often sufficient. However, effects due to the confining surfaces must be taken into account in order to model the conditions encountered by bubbles in narrow channels, e.g., blood vessels. Analytical models for bubble behavior in narrow tubes have been developed under the assumptions of rigid channel surfaces^{7,9–11,32} and linear bubble pulsation.⁹ Models for a bubble (often modeled as a point source) oscillating near fluid,²⁴ elastic,²⁵ or locally reacting planar surfaces have also been presented.^{11,33,34} However, the assumption of a rigid or even a locally reacting surface presupposes that the impedance of the boundary is independent of incidence angle. This condition is rarely met in biological environments, where bubbles are often close to vessel walls and the sound speed in the surrounding tissue may not differ significantly from that in the host liquid.

Several finite and boundary element numerical models, in which assumptions of a rigid channel and linear bubble pulsation are relaxed, have also been developed.^{12,13,15,18–20,22,35}

Although less open to simple physical interpretation, the flexibility of the fully numerical models permits effects due to nonlinear pulsation, bubble jetting and translation, and viscoelastic properties of the surrounding media to be considered. The displacement, strain, and stress fields within the confining media may also be modeled, thus predicting tissue damage resulting from cavitation activity.

Our work is also motivated by laboratory observations of encapsulated microbubble pulsation in the vicinity of a polystyrene membrane.³¹ These measurements show that both the pulsation amplitude and frequency of peak response, or resonance frequency, decrease as the bubble is moved closer to the interface. Modeling the membrane as a rigid interface captures the decrease in resonance frequency, but predicts that the pulsation amplitude should increase, rather than decrease, as the bubble is moved closer to the interface. Recently, an analytic model for contrast agent bubble pulsation near a single elastic layer was developed by Doinikov *et al.*²⁵ in which, contrary to the present work, shear strain is neglected, and viscous and radiation losses are included in an *ad hoc* manner. Simulations performed with this model capture the decrease in oscillation amplitude, but predict that the resonance frequency should increase for bubbles near the membrane (see Fig. 2 in Ref. 25). However, as will be shown in Sec. IV, if the viscosity of the liquid and viscoelastic nature of the layers are taken into account the present model predicts a decrease in both the oscillation amplitude and resonance frequency observed in the measurements reported in Ref. 31.

The present semi-analytical model for the nonlinear dynamics of a bubble in viscous liquid between two parallel layers includes effects due to the viscoelasticity of the layers and nonlinear pulsation of the bubble. The model is valid as long as strain magnitudes in the layers remain small. Expressions for the displacement, stress, and strain fields in the layers have been derived, and bubble translation has been considered, but these enhancements will be reported elsewhere. Here the focus is on the radiation load presented to the bubble by the layers, and a modified Rayleigh–Plesset

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equation which describes the radial dynamics of the bubble is derived. Some limitations of previous analytical models have thus been removed, while the simplicity, straightforward physical interpretation, and computational efficiency characteristic of an analytical approach have been retained.

The present model employs a formalism originally used to derive Green's functions for a source buried in an elastic half-space,³⁶ and is derived by expressing the Green's function for elastic waves radiated by a volume source in terms of its angular spectrum. Coefficients in the angular spectrum are determined by boundary conditions at the interfaces formed by the layers. This approach has also been applied to the geometry of a bubble oscillating in a cylindrical viscoelastic tube, which more closely approximates the geometry of a blood vessel.³⁷ A similar technique has been used to model the displacement of a single elastic layer with traction-free boundary conditions.³⁸

In Sec. II an expression for the pressure field between the layers due to a pulsating bubble, represented by a volume source, is derived. In Sec. III the pressure reflected from the layers is incorporated into models for the radial dynamics of the bubble. The influence of the viscoelastic properties of the layers and the distance between the bubble and the layers on the radial dynamics is demonstrated through simulations in Sec. IV.

II. GREEN'S FUNCTION AND REVERBERANT PRESSURE FIELD

The geometry is illustrated in Fig. 1. Two parallel viscoelastic layers are immersed in a compressible, viscous liquid of density ρ , dilational viscosity ζ , shear viscosity η , and Lamé parameter $\lambda = \rho c^2$, where c is the sound speed in the liquid. A pulsating bubble (volume source) is positioned at an arbitrary fixed position between the layers, and a cylindrical coordinate system is constructed with vertical coordinate z , radial coordinate r , and azimuthal coordinate θ (with unit vectors $\mathbf{e}_z$, $\mathbf{e}_r$, and $\mathbf{e}_\theta$, respectively), such that the bubble is

FIG. 1. Geometry and material parameters for a bubble between parallel viscoelastic layers.

positioned along the z axis at location z_0 , with the layers perpendicular to the z axis. The near face of each layer is located at arbitrary distance $|z_{lj} - z_0|$ from the bubble center. The layers have thickness $|z_{hj} - z_{lj}|$, density ρ_j , dilational viscosity ζ_j , shear viscosity η_j , Lamé parameter λ_j , and shear modulus μ_j for $j = 1, 2$. Gravity is neglected and, according to the orientation of the z axis in Fig. 1, it is assumed that $z_{h2} < z_{l2} < z_0 < z_{l1} < z_{h1}$.

The layers and liquid support transverse and longitudinal wave motion, respectively, represented by the complex amplitudes T and L in Fig. 2. Amplitude superscripts $[(c_j)]$ or $[(l_j)]$ when present indicate the location in the liquid channel or viscoelastic layers and the adjacent arrows indicate the direction of propagation along the z axis. The amplitude subscript signs were chosen to match those of the arguments of the exponentials in Eqs. (13)–(16), presented in Sec. II A.

The momentum equation for the liquid (for $z_{l2} < z < z_{l1}$, $z > z_{h1}$ or $z < z_{h2}$) is

$$\rho \frac{\partial^2 u_n}{\partial t^2} = \left(\lambda + \zeta \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial t} \right) \frac{\partial^2 u_k}{\partial x_n \partial x_k} + \eta \frac{\partial}{\partial t} \left(\frac{\partial^2 u_n}{\partial x_k^2} \right), \quad (1)$$

while for the layers (for $z_{h1} > z > z_{l1}$ or $z_{l2} > z > z_{h2}$) it is

$$\begin{aligned} \rho_j \frac{\partial^2 u_n}{\partial t^2} = & \left(\lambda_j + \zeta_j \frac{\partial}{\partial t} + \mu_j + \eta_j \frac{\partial}{\partial t} \right) \frac{\partial^2 u_k}{\partial x_n \partial x_k} \\ & + \left(\mu_j + \eta_j \frac{\partial}{\partial t} \right) \frac{\partial^2 u_n}{\partial x_k^2} \end{aligned} \quad (2)$$

for $j = 1, 2$, where t is time, and u_n are the components of displacement along the x , y , and z axes (i.e., u_n may take the value u_x , u_y , or u_z). Assuming an $e^{-i\omega t}$ time dependence, where ω is angular frequency, the components of the frequency-domain Green's function $\mathbf{g}$ for a volume source of strength s satisfy

FIG. 2. Wave amplitudes for a bubble between parallel viscoelastic layers.

$$\begin{aligned} & (\hat{\lambda} + \hat{\mu}) \frac{\partial^2 g_k}{\partial x_n \partial x_k} + \hat{\mu} \frac{\partial^2 g_n}{\partial x_k^2} + \rho \omega^2 g_n \\ & = s \delta_{nk} \frac{\partial}{\partial x_k} [\delta(x) \delta(y) \delta(z - z_0)] \end{aligned} \quad (3)$$

for the liquid, and

$$(\hat{\lambda}_j + \hat{\mu}_j) \frac{\partial^2 g_k}{\partial x_n \partial x_k} + \hat{\mu}_j \frac{\partial^2 g_n}{\partial x_k^2} + \rho_j \omega^2 g_n = 0 \quad (4)$$

for the layers, where $\hat{\lambda} = \lambda - i\omega\zeta$, $\hat{\mu} = -i\omega\eta$, $\hat{\lambda}_j = \lambda_j - i\omega\zeta_j$, $\hat{\mu}_j = \mu_j - i\omega\eta_j$, δ_{nk} is the Kronecker delta, and $\delta(\cdot)$ is the Dirac delta function. As shown in Ref. 36, the strength s of the source may be expressed through its volume velocity Q as $s = -(\rho c^2 / i\omega)Q$. The appearance of the gradient acting on the delta function in Eq. (3) is due to the fact that the Green's function satisfies an equation for particle displacement. Particle displacement is proportional to the gradient of the sound pressure, and therefore the gradient of the delta function is the appropriate representation of a volume source in terms of displacement.

Due to symmetry about the z axis, the vector $\mathbf{g}$ is expressed solely in terms of its r and z components in cylindrical coordinates g_r and g_z , respectively. The angular spectrum of the Green's function may be introduced through the transform pairs

$$\begin{aligned} G_z(\kappa, z|z_0) &= \int g_z(r, z|z_0) e^{-i\kappa \cdot \mathbf{r}} d\mathbf{r} \\ &= 2\pi \int_0^\infty g_z(r, z|z_0) J_0(\kappa r) r dr, \end{aligned} \quad (5)$$

$$\begin{aligned} g_z(r, z|z_0) &= \frac{1}{(2\pi)^2} \int G_z(\kappa, z|z_0) e^{i\kappa \cdot \mathbf{r}} d\kappa \\ &= \frac{1}{2\pi} \int_0^\infty G_z(\kappa, z|z_0) J_0(\kappa r) \kappa d\kappa, \end{aligned} \quad (6)$$

and

$$\begin{aligned} G_\kappa(\kappa, z|z_0) &= \int g_r(r, z|z_0) \cos \varphi e^{-i\kappa \cdot \mathbf{r}} d\mathbf{r} \\ &= -i2\pi \int_0^\infty g_r(r, z|z_0) J_1(\kappa r) r dr, \end{aligned} \quad (7)$$

$$\begin{aligned} g_r(r, z|z_0) &= \frac{1}{(2\pi)^2} \int G_\kappa(\kappa, z|z_0) \cos \varphi e^{i\kappa \cdot \mathbf{r}} d\kappa \\ &= -\frac{1}{i2\pi} \int_0^\infty G_\kappa(\kappa, z|z_0) J_1(\kappa r) \kappa d\kappa, \end{aligned} \quad (8)$$

where κ is a two-dimensional vector perpendicular to the z axis with unit vector $\mathbf{e}_\kappa$, G_κ , and G_z are the κ and z components of the angular spectrum, φ is the angle between vectors $\mathbf{r}$ and κ , and $J_n(\cdot)$ are Bessel functions of the first kind. Solutions of Eqs. (3) and (4) are now formulated in terms of their angular spectrum.

A. Solutions of the model equations

Substitution of Eqs. (6) and (8) into the homogeneous form of Eqs. (3) and (4) gives

$$\begin{aligned} & i\kappa(\hat{\lambda} + \hat{\mu}) \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) + \hat{\mu} \left(-\kappa^2 G_\kappa + \frac{d^2 G_\kappa}{dz^2} \right) \\ & + \rho \omega^2 G_\kappa = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} & (\hat{\lambda} + \hat{\mu}) \frac{d}{dz} \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) + \hat{\mu} \left(-\kappa^2 G_z + \frac{d^2 G_z}{dz^2} \right) \\ & + \rho \omega^2 G_z = 0 \end{aligned} \quad (10)$$

for the liquid and

$$\begin{aligned} & i\kappa(\hat{\lambda}_j + \hat{\mu}_j) \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) + \hat{\mu}_j \left(-\kappa^2 G_\kappa + \frac{d^2 G_\kappa}{dz^2} \right) \\ & + \rho_j \omega^2 G_\kappa = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} & (\hat{\lambda}_j + \hat{\mu}_j) \frac{d}{dz} \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) + \hat{\mu}_j \left(-\kappa^2 G_z + \frac{d^2 G_z}{dz^2} \right) \\ & + \rho_j \omega^2 G_z = 0 \end{aligned} \quad (12)$$

for the layers. The general form of the solution satisfying Eqs. (9) and (10) for the liquid channel between the layers ($z_{l1} > z > z_0$ or $z_0 > z > z_{l2}$) is³⁹

$$\begin{aligned} G_\kappa &= i \left[\kappa_t \left(T_+^{(c_j)} e^{\kappa_t z} - T_-^{(c_j)} e^{-\kappa_t z} \right) \right. \\ & \left. + \kappa \left(L_+^{(c_j)} e^{\kappa_l z} + L_-^{(c_j)} e^{-\kappa_l z} \right) \right], \end{aligned} \quad (13)$$

$$\begin{aligned} G_z &= \kappa \left(T_+^{(c_j)} e^{\kappa_t z} + T_-^{(c_j)} e^{-\kappa_t z} \right) \\ & + \kappa_l \left(L_+^{(c_j)} e^{\kappa_l z} - L_-^{(c_j)} e^{-\kappa_l z} \right), \end{aligned} \quad (14)$$

and for the liquid surrounding the layers ($z > z_{h1}$ or $z < z_{h2}$) the solution is

$$G_\kappa = i \left(\pm \kappa_t T_\pm e^{\pm \kappa_t z} + \kappa L_\pm e^{\pm \kappa_l z} \right), \quad (15)$$

$$G_z = \kappa T_\pm e^{\pm \kappa_t z} \pm \kappa_l L_\pm e^{\pm \kappa_l z}. \quad (16)$$

From Figs. 1 and 2 it is apparent that in Eqs. (15) and (16) the minus sign is selected for the region $z > z_{h1}$ and the plus sign is selected for $z < z_{h2}$. Quantities κ_t and κ_l are defined as

$$\kappa_t = \sqrt{\kappa^2 - k_t^2}, \quad \kappa_l = \sqrt{\kappa^2 - k_l^2}, \quad (17)$$

where

$$k_t = \omega \sqrt{\frac{\rho}{\hat{\mu}}}, \quad k_l = \omega \sqrt{\frac{\rho}{\hat{\lambda} + 2\hat{\mu}}} \quad (18)$$

are the associated complex wavenumbers. For complex κ_t or κ_l the principal branch of the square root in their definitions is taken to ensure that G_κ and G_z decay in the direction of propagation. The notation is such that, in the absence of losses, real values of κ_t and κ_l in Eqs. (13)–(16) correspond to exponentially damped motion (evanescent waves), imaginary values to propagating waves. As demonstrated in

Sec. IV A 3, the influence of evanescent waves on bubble pulsation is dominant when the bubble is located less than a few radii from a layer. The general form of the solutions satisfying Eqs. (11) and (12) for the layers ($z_{h1} > z > z_{l1}$ or $z_{h2} < z < z_{l2}$) is

$$G_\kappa = i \left[\kappa_t^{(lj)} \left(T_+^{(lj)} e^{\kappa_t^{(lj)} z} - T_-^{(lj)} e^{-\kappa_t^{(lj)} z} \right) + \kappa \left(L_+^{(lj)} e^{\kappa_l^{(lj)} z} + L_-^{(lj)} e^{-\kappa_l^{(lj)} z} \right) \right], \quad (19)$$

$$G_z = \kappa \left(T_+^{(lj)} e^{\kappa_t^{(lj)} z} + T_-^{(lj)} e^{-\kappa_t^{(lj)} z} \right) + \kappa_l^{(lj)} \left(L_+^{(lj)} e^{\kappa_l^{(lj)} z} - L_-^{(lj)} e^{-\kappa_l^{(lj)} z} \right) \quad (20)$$

for $j = 1, 2$, where

$$\kappa_t^{(lj)} = \sqrt{\kappa^2 - \left(k_t^{(lj)}\right)^2}, \quad \kappa_l^{(lj)} = \sqrt{\kappa^2 - \left(k_l^{(lj)}\right)^2}$$

and

$$k_t^{(lj)} = \omega \sqrt{\frac{\rho_j}{\mu_j}}, \quad k_l^{(lj)} = \omega \sqrt{\frac{\rho_j}{\lambda_j + 2\mu_j}}.$$

B. Boundary conditions

The twenty complex amplitudes $T_\pm$, $L_\pm$, $T_\pm^{(c_j)}$, $L_\pm^{(c_j)}$, $T_\pm^{(lj)}$, and $L_\pm^{(lj)}$ illustrated in Fig. 2 and appearing in Eqs. (13)–(16), (19), and (20) must be specified in order to evaluate the Green's function. These amplitudes are determined by satisfying the boundary conditions in the plane $z = z_0$ as well as on the layer interfaces at $z = z_{lj}$ and $z = z_{hj}$.

1. Conditions at $z = z_0$

The first condition in the plane containing the source is continuity of displacement, which requires that Eqs. (13) and (14) be equal, i.e.,

$$\begin{aligned} \kappa_t \left(\tilde{T}_+^{(c1)} - \tilde{T}_-^{(c1)} \right) + \kappa \left(\tilde{L}_+^{(c1)} + \tilde{L}_-^{(c1)} \right) \\ = \kappa_t \left(\tilde{T}_+^{(c2)} - \tilde{T}_-^{(c2)} \right) + \kappa \left(\tilde{L}_+^{(c2)} + \tilde{L}_-^{(c2)} \right), \end{aligned} \quad (21)$$

$$\begin{aligned} \kappa \left(\tilde{T}_+^{(c1)} + \tilde{T}_-^{(c1)} \right) + \kappa_l \left(\tilde{L}_+^{(c1)} - \tilde{L}_-^{(c1)} \right) \\ = \kappa \left(\tilde{T}_+^{(c2)} + \tilde{T}_-^{(c2)} \right) + \kappa_l \left(\tilde{L}_+^{(c2)} - \tilde{L}_-^{(c2)} \right), \end{aligned} \quad (22)$$

where $\tilde{T}_\pm^{(c_j)} = T_\pm^{(c_j)} e^{\pm \kappa_l z_0}$ and $\tilde{L}_\pm^{(c_j)} = L_\pm^{(c_j)} e^{\pm \kappa_l z_0}$ have been introduced for notational convenience.

A second set of conditions can be found by integrating Eq. (3) along the z axis from $z_0 - \varepsilon$ to $z_0 + \varepsilon$ and taking the limit as ε approaches zero. It is convenient to investigate two cases separately: (a) $n = k = x$ or $n = k = y$, and (b) $n = k = z$. Since the problem is linear, the results from cases (a) and (b) may be summed to obtain the solution for a volume source.

For case (a), substitution of Eqs. (6) and (8) into Eq. (3) gives

$$\begin{aligned} i\kappa \left(\hat{\lambda} + \hat{\mu} \right) \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) \\ + \hat{\mu} \left(-\kappa^2 G_\kappa + \frac{d^2 G_\kappa}{dz^2} \right) + \rho\omega^2 G_\kappa = i\kappa\delta(z - z_0), \end{aligned} \quad (23)$$

$$\begin{aligned} \left(\hat{\lambda} + \hat{\mu} \right) \frac{d}{dz} \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) + \hat{\mu} \left(-\kappa^2 G_z + \frac{d^2 G_z}{dz^2} \right) \\ + \rho\omega^2 G_z = 0. \end{aligned} \quad (24)$$

Since $\mathbf{G}$ is continuous across the plane $z = z_0$, the result of integrating terms in Eqs. (23) and (24) proportional to the Green's function or its first derivative with respect to z along the z axis is zero in the limit $\varepsilon \rightarrow 0$, leaving

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{dG_\kappa}{dz} \Big|_{z=z_0+\varepsilon} - \frac{dG_\kappa}{dz} \Big|_{z=z_0-\varepsilon} \right) = \frac{i\kappa}{\hat{\mu}}, \quad (25)$$

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{dG_z}{dz} \Big|_{z=z_0+\varepsilon} - \frac{dG_z}{dz} \Big|_{z=z_0-\varepsilon} \right) = 0. \quad (26)$$

Substitution of Eqs. (13) and (14) for G_κ and G_z in Eqs. (25) and (26) yields

$$\begin{aligned} \kappa_t^2 \left(\tilde{T}_+^{(c1)} + \tilde{T}_-^{(c1)} - \tilde{T}_+^{(c2)} - \tilde{T}_-^{(c2)} \right) \\ + \kappa\kappa_l \left(\tilde{L}_+^{(c1)} - \tilde{L}_-^{(c1)} - \tilde{L}_+^{(c2)} + \tilde{L}_-^{(c2)} \right) = \frac{s\kappa k_t^2}{\rho\omega^2}, \end{aligned} \quad (27)$$

$$\begin{aligned} \kappa\kappa_l \left(\tilde{T}_+^{(c1)} - \tilde{T}_-^{(c1)} - \tilde{T}_+^{(c2)} + \tilde{T}_-^{(c2)} \right) \\ + \kappa_l^2 \left(\tilde{L}_+^{(c1)} + \tilde{L}_-^{(c1)} - \tilde{L}_+^{(c2)} - \tilde{L}_-^{(c2)} \right) = 0. \end{aligned} \quad (28)$$

Equations (21), (22), (27), and (28) may be rearranged to obtain

$$T_-^{(c1)} - T_-^{(c2)} = -\frac{s\kappa e^{\kappa_l z_0}}{2\rho\omega^2}, \quad (29)$$

$$T_+^{(c2)} - T_+^{(c1)} = \frac{s\kappa e^{-\kappa_l z_0}}{2\rho\omega^2}, \quad (30)$$

$$L_-^{(c1)} - L_-^{(c2)} = -\frac{s\kappa^2 e^{\kappa_l z_0}}{2\rho\omega^2 \kappa_l}, \quad (31)$$

$$L_+^{(c2)} - L_+^{(c1)} = -\frac{s\kappa^2 e^{-\kappa_l z_0}}{2\rho\omega^2 \kappa_l}. \quad (32)$$

For case (b), when $n = k = z$, the integration may be simplified by temporarily replacing the volume source with a force source acting along the z axis. To do this the right-hand side of Eq. (3) is changed to $s\delta(x)\delta(y)\delta(z - z_0)$. Substitution of Eqs. (6) and (8) into Eq. (3) shows that the angular spectrum components must satisfy

$$\begin{aligned} i\kappa \left(\hat{\lambda} + \hat{\mu} \right) \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) + \hat{\mu} \left(-\kappa^2 G_\kappa + \frac{d^2 G_\kappa}{dz^2} \right) \\ + \rho\omega^2 G_\kappa = 0, \end{aligned} \quad (33)$$

$$\begin{aligned} & \left(\hat{\lambda} + \hat{\mu} \right) \frac{d}{dz} \left(i\kappa G_\kappa + \frac{dG_z}{dz} \right) + \hat{\mu} \left(-\kappa^2 G_z + \frac{d^2 G_z}{dz^2} \right) \\ & + \rho\omega^2 G_z = s\delta(z - z_0). \end{aligned} \quad (34)$$

Integration of Eqs. (33) and (34) along the z axis from $z_0 - \varepsilon$ to $z_0 + \varepsilon$, following substitution of Eqs. (13) and (14) for G_κ and G_z , yields, in the limit $\varepsilon \rightarrow 0$,

$$\begin{aligned} & \kappa_t^2 \left(\tilde{T}_+^{(c_1)} + \tilde{T}_-^{(c_1)} - \tilde{T}_+^{(c_2)} - \tilde{T}_-^{(c_2)} \right) \\ & + \kappa\kappa_l \left(\tilde{L}_+^{(c_1)} - \tilde{L}_-^{(c_1)} - \tilde{L}_+^{(c_2)} + \tilde{L}_-^{(c_2)} \right) = 0, \end{aligned} \quad (35)$$

$$\begin{aligned} & \kappa\kappa_l \left(\tilde{T}_+^{(c_1)} - \tilde{T}_-^{(c_1)} - \tilde{T}_+^{(c_2)} + \tilde{T}_-^{(c_2)} \right) \\ & + \kappa_l^2 \left(\tilde{L}_+^{(c_1)} + \tilde{L}_-^{(c_1)} - \tilde{L}_+^{(c_2)} - \tilde{L}_-^{(c_2)} \right) = \frac{sk_l^2}{\rho\omega^2}. \end{aligned} \quad (36)$$

Equations (21), (22), (35), and (36) may be rearranged to obtain

$$T_-^{(c_1)} - T_-^{(c_2)} = -\frac{s\kappa e^{\kappa_l z_0}}{2\rho\omega^2 \kappa_t}, \quad (37)$$

$$T_+^{(c_2)} - T_+^{(c_1)} = -\frac{s\kappa e^{-\kappa_l z_0}}{2\rho\omega^2 \kappa_t}, \quad (38)$$

$$L_-^{(c_1)} - L_-^{(c_2)} = -\frac{se^{\kappa_l z_0}}{2\rho\omega^2}, \quad (39)$$

$$L_+^{(c_2)} - L_+^{(c_1)} = \frac{se^{-\kappa_l z_0}}{2\rho\omega^2}. \quad (40)$$

Comparing the right-hand side of Eq. (3) to our temporary replacement $s\delta(x)\delta(y)\delta(z - z_0)$, it is apparent that the solution for case (b) may be derived from Eqs. (37)–(40) by differentiating with respect to z_0 , and reversing the sign, i.e.,

$$T_-^{(c_1)} - T_-^{(c_2)} = \frac{s\kappa e^{\kappa_l z_0}}{2\rho\omega^2}, \quad (41)$$

$$T_+^{(c_2)} - T_+^{(c_1)} = -\frac{s\kappa e^{-\kappa_l z_0}}{2\rho\omega^2}, \quad (42)$$

$$L_-^{(c_1)} - L_-^{(c_2)} = \frac{s\kappa_l e^{\kappa_l z_0}}{2\rho\omega^2}, \quad (43)$$

$$L_+^{(c_2)} - L_+^{(c_1)} = \frac{s\kappa_l e^{-\kappa_l z_0}}{2\rho\omega^2}. \quad (44)$$

The right-hand sides of Eqs. (29)–(32) may now be added to those of Eqs. (41)–(44) to find the following amplitude relationships for a volume source:

$$T_-^{(c_1)} - T_-^{(c_2)} = 0, \quad (45)$$

$$T_+^{(c_2)} - T_+^{(c_1)} = 0, \quad (46)$$

$$L_-^{(c_1)} - L_-^{(c_2)} = -\frac{sk_l^2}{2\rho\omega^2 \kappa_l} e^{\kappa_l z_0}, \quad (47)$$

$$L_+^{(c_2)} - L_+^{(c_1)} = -\frac{sk_l^2}{2\rho\omega^2 \kappa_l} e^{-\kappa_l z_0}. \quad (48)$$

Without loss of generality the source location may now be assumed to be $z_0 = 0$. Equations (45)–(48) may be solved for wave amplitudes traveling away from the source and written more compactly in matrix form as

$$\mathbf{X}_-^{(c_1)} = \mathbf{X}_-^{(c_2)} + \mathbf{S} \quad (49)$$

$$= \mathbf{K}^{(c_2)} \mathbf{X}_+^{(c_2)} + \mathbf{S}, \quad (50)$$

$$\mathbf{X}_+^{(c_2)} = \mathbf{X}_+^{(c_1)} + \mathbf{S} \quad (51)$$

$$= \mathbf{K}^{(c_1)} \mathbf{X}_-^{(c_1)} + \mathbf{S}, \quad (52)$$

where the column vectors $\mathbf{S}$ and $\mathbf{X}_\pm^{(c_j)}$ are

$$\mathbf{S} = -\frac{sk_l^2}{2\rho\omega^2 \kappa_l} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{X}_\pm^{(c_j)} = \begin{pmatrix} T_\pm^{(c_j)} \\ L_\pm^{(c_j)} \end{pmatrix}, \quad (53)$$

where the $\mathbf{K}^{(c_j)}$ are 2×2 reflection coefficient matrices defined by $\mathbf{X}_-^{(c_2)} = \mathbf{K}^{(c_2)} \mathbf{X}_+^{(c_2)}$ and $\mathbf{X}_+^{(c_1)} = \mathbf{K}^{(c_1)} \mathbf{X}_-^{(c_1)}$ for the top and bottom layers respectively. Equations (50) and (52) may be solved to obtain

$$\mathbf{X}_-^{(c_1)} = \mathbf{F}_1 \mathbf{S}, \quad (54)$$

$$\mathbf{X}_+^{(c_2)} = \mathbf{F}_2 \mathbf{S}, \quad (55)$$

where

$$\mathbf{F}_1 = \left[\mathbf{I} - \mathbf{K}^{(c_2)} \mathbf{K}^{(c_1)} \right]^{-1} \left[\mathbf{K}^{(c_2)} + \mathbf{I} \right], \quad (56)$$

$$\mathbf{F}_2 = \left[\mathbf{I} - \mathbf{K}^{(c_1)} \mathbf{K}^{(c_2)} \right]^{-1} \left[\mathbf{K}^{(c_1)} + \mathbf{I} \right], \quad (57)$$

and $\mathbf{I}$ is the 2×2 identity matrix.

2. Conditions at $\mathbf{z} = \mathbf{z}_l$

The matrices $\mathbf{K}^{(c_j)}$ relate the incident and reflected wave amplitudes from either layer at the location of the source and must be determined in order to calculate the amplitudes given by Eqs. (54) and (55). Explicit analytical expressions for $\mathbf{K}^{(c_j)}$ may be found by considering the boundary conditions on the interfaces. These conditions are continuity of displacement and continuity of normal and shear stresses. Strain and stress are taken in the linear approximation as

$$\varepsilon_{nk} = \frac{1}{2} \left(\frac{\partial u_n}{\partial x_k} + \frac{\partial u_k}{\partial x_n} \right), \quad (58)$$

$$\sigma_{nk} = 2\hat{\mu}\varepsilon_{nk} + \hat{\lambda}\varepsilon_{ll}\delta_{nk}, \quad (59)$$

for the liquid, and by similar expressions for the layers with $\hat{\mu}$ and $\hat{\lambda}$ replaced by $\hat{\mu}_j$ and $\hat{\lambda}_j$, respectively.

At the layer interfaces closest to the source ($z = z_{l_j}$) continuity of displacement requires that Eq. (13) equal Eq. (19), and Eq. (14) equal Eq. (20), or

$$\begin{aligned} \kappa_t \left(\bar{T}_+^{(c_j)} - \bar{T}_-^{(c_j)} \right) + \kappa \left(\bar{L}_+^{(c_j)} + \bar{L}_-^{(c_j)} \right) \\ = \kappa_t^{(l_j)} \left(\bar{T}_+^{(l_j)} - \bar{T}_-^{(l_j)} \right) + \kappa \left(\bar{L}_+^{(l_j)} + \bar{L}_-^{(l_j)} \right), \end{aligned} \quad (60)$$

$$\begin{aligned} \kappa \left(\bar{T}_+^{(c_j)} + \bar{T}_-^{(c_j)} \right) + \kappa_l \left(\bar{L}_+^{(c_j)} - \bar{L}_-^{(c_j)} \right) \\ = \kappa \left(\bar{T}_+^{(l_j)} + \bar{T}_-^{(l_j)} \right) + \kappa_l^{(l_j)} \left(\bar{L}_+^{(l_j)} - \bar{L}_-^{(l_j)} \right), \end{aligned} \quad (61)$$

and the conditions on normal and shear stress require that the angular spectrum coefficients satisfy

$$\begin{aligned} \hat{\mu} \left[2\kappa\kappa_t \left(\bar{T}_+^{(c_j)} - \bar{T}_-^{(c_j)} \right) + (\kappa^2 + \kappa_t^2) \left(\bar{L}_+^{(c_j)} + \bar{L}_-^{(c_j)} \right) \right] \\ = \hat{\mu}_j \left[2\kappa\kappa_t^{(l_j)} \left(\bar{T}_+^{(l_j)} - \bar{T}_-^{(l_j)} \right) \right. \\ \left. + \left(\kappa^2 + (\kappa_t^{(l_j)})^2 \right) \left(\bar{L}_+^{(l_j)} + \bar{L}_-^{(l_j)} \right) \right], \end{aligned} \quad (62)$$

$$\begin{aligned} \hat{\mu} \left[(\kappa^2 + \kappa_t^2) \left(\bar{T}_+^{(c_j)} + \bar{T}_-^{(c_j)} \right) + 2\kappa\kappa_l \left(\bar{L}_+^{(c_j)} - \bar{L}_-^{(c_j)} \right) \right] \\ = \hat{\mu}_j \left[\left(\kappa^2 + (\kappa_t^{(l_j)})^2 \right) \left(\bar{T}_+^{(l_j)} + \bar{T}_-^{(l_j)} \right) \right. \\ \left. + 2\kappa\kappa_l^{(l_j)} \left(\bar{L}_+^{(l_j)} - \bar{L}_-^{(l_j)} \right) \right], \end{aligned} \quad (63)$$

where $\bar{T}_\pm^{(c_j)} = T_\pm^{(c_j)} e^{\pm \kappa_l z_{l_j}}$, $\bar{L}_\pm^{(c_j)} = L_\pm^{(c_j)} e^{\pm \kappa_l z_{l_j}}$, $\bar{T}_\pm^{(l_j)} = T_\pm^{(l_j)} \exp(\pm \kappa_t^{(l_j)} z_{l_j})$, and $\bar{L}_\pm^{(l_j)} = L_\pm^{(l_j)} \exp(\pm \kappa_l^{(l_j)} z_{l_j})$. Equations (60)–(63) written in matrix form are

$$\mathbf{C}_{d+}^{(f)} \bar{\mathbf{X}}_+^{(c_j)} + \mathbf{C}_{d-}^{(f)} \bar{\mathbf{X}}_-^{(c_j)} = \mathbf{C}_{d+}^{(l_j)} \bar{\mathbf{X}}_+^{(l_j)} + \mathbf{C}_{d-}^{(l_j)} \bar{\mathbf{X}}_-^{(l_j)}, \quad (64)$$

$$\hat{\mu} \left(\mathbf{C}_{s+}^{(f)} \bar{\mathbf{X}}_+^{(c_j)} + \mathbf{C}_{s-}^{(f)} \bar{\mathbf{X}}_-^{(c_j)} \right) = \hat{\mu}_j \left(\mathbf{C}_{s+}^{(l_j)} \bar{\mathbf{X}}_+^{(l_j)} + \mathbf{C}_{s-}^{(l_j)} \bar{\mathbf{X}}_-^{(l_j)} \right). \quad (65)$$

In Eqs. (64) and (65) the column vectors $\bar{\mathbf{X}}_\pm^{(c_j)}$ and $\bar{\mathbf{X}}_\pm^{(l_j)}$ are defined to be $\bar{\mathbf{X}}_\pm^{(c_j)} = \left(\bar{T}_\pm^{(c_j)} \bar{L}_\pm^{(c_j)} \right)^T$ and $\bar{\mathbf{X}}_\pm^{(l_j)} = \left(\bar{T}_\pm^{(l_j)} \bar{L}_\pm^{(l_j)} \right)^T$, where $(\cdot)^T$ denotes transposition. In addition, the matrices

$$\begin{aligned} \mathbf{C}_{d\pm}^{(f)} &= \begin{pmatrix} \pm \kappa_t & \kappa \\ \kappa & \pm \kappa_l \end{pmatrix}, \\ \mathbf{C}_{s\pm}^{(f)} &= \begin{pmatrix} \pm 2\kappa\kappa_t & \kappa^2 + \kappa_t^2 \\ \kappa^2 + \kappa_t^2 & \pm 2\kappa\kappa_l \end{pmatrix}, \end{aligned}$$

as well as corresponding matrices $\mathbf{C}_{d\pm}^{(l_j)}$ and $\mathbf{C}_{s\pm}^{(l_j)}$ with κ_t and κ_l replaced by $\kappa_t^{(l_j)}$ and $\kappa_l^{(l_j)}$, respectively, have been introduced.

3. Conditions at $z = z_{h_j}$

At the layer interfaces farthest from the source ($z = z_{h_j}$) continuity of displacement requires that Eq. (15) equal Eq. (19), and Eq. (16) equal Eq. (20), or

$$\begin{aligned} \kappa_t^{(l_j)} \left(\hat{T}_+^{(l_j)} - \hat{T}_-^{(l_j)} \right) + \kappa \left(\hat{L}_+^{(l_j)} + \hat{L}_-^{(l_j)} \right) \\ = \pm \kappa_t \hat{T}_\pm + \kappa \hat{L}_\pm, \end{aligned} \quad (66)$$

$$\kappa \left(\hat{T}_+^{(l_j)} + \hat{T}_-^{(l_j)} \right) + \kappa_l^{(l_j)} \left(\hat{L}_+^{(l_j)} - \hat{L}_-^{(l_j)} \right) = \kappa \hat{T}_\pm \pm \kappa_l \hat{L}_\pm, \quad (67)$$

and the conditions on normal and shear stress require

$$\begin{aligned} \hat{\mu}_j \left[2\kappa\kappa_t^{(l_j)} \left(\hat{T}_+^{(l_j)} - \hat{T}_-^{(l_j)} \right) \right. \\ \left. + \left(\kappa^2 + (\kappa_t^{(l_j)})^2 \right) \left(\hat{L}_+^{(l_j)} + \hat{L}_-^{(l_j)} \right) \right] \\ = \hat{\mu} \left[\pm 2\kappa\kappa_t \hat{T}_\pm + (\kappa^2 + \kappa_t^2) \hat{L}_\pm \right], \end{aligned} \quad (68)$$

$$\begin{aligned} \hat{\mu}_j \left[\left(\kappa^2 + (\kappa_t^{(l_j)})^2 \right) \left(\hat{T}_+^{(l_j)} + \hat{T}_-^{(l_j)} \right) + 2\kappa\kappa_l^{(l_j)} \left(\hat{L}_+^{(l_j)} - \hat{L}_-^{(l_j)} \right) \right] \\ = \hat{\mu} \left[(\kappa^2 + \kappa_t^2) \hat{T}_\pm \pm 2\kappa\kappa_l \hat{L}_\pm \right], \end{aligned} \quad (69)$$

where $\hat{T}_\pm^{(l_j)} = T_\pm^{(l_j)} \exp(\pm \kappa_t^{(l_j)} z_{h_j})$, $\hat{L}_\pm^{(l_j)} = L_\pm^{(l_j)} \exp(\pm \kappa_l^{(l_j)} z_{h_j})$, $\hat{T}_\pm = T_\pm e^{\pm \kappa_t z_{h_j}}$, and $\hat{L}_\pm = L_\pm e^{\pm \kappa_l z_{h_j}}$. According to Fig. 2, the minus sign in Eqs. (66)–(69) must be selected when $j=1$ and the plus sign must be selected when $j=2$. Equations (66)–(69) in matrix form are

$$\mathbf{C}_{d+}^{(l_j)} \hat{\mathbf{X}}_+^{(l_j)} + \mathbf{C}_{d-}^{(l_j)} \hat{\mathbf{X}}_-^{(l_j)} = \mathbf{C}_{d\pm}^{(f)} \hat{\mathbf{X}}_\pm, \quad (70)$$

$$\hat{\mu}_j \left[\mathbf{C}_{s+}^{(l_j)} \hat{\mathbf{X}}_+^{(l_j)} + \mathbf{C}_{s-}^{(l_j)} \hat{\mathbf{X}}_-^{(l_j)} \right] = \hat{\mu} \mathbf{C}_{s\pm}^{(f)} \hat{\mathbf{X}}_\pm, \quad (71)$$

where $\hat{\mathbf{X}}_\pm^{(l_j)} = \left(\hat{T}_\pm^{(l_j)} \hat{L}_\pm^{(l_j)} \right)^T$ and $\hat{\mathbf{X}}_\pm = \left(\hat{T}_\pm \hat{L}_\pm \right)^T$.

C. Reflection coefficient matrices

The reflection coefficient matrices $\mathbf{K}^{(c_j)}$ relating the incident and reflected amplitudes from the layer interfaces nearest to the source are now found. Let $\bar{\mathbf{K}}^{(l_1)}$ relate wave amplitudes in the top layer at $z = z_{l_1}$, i.e., $\bar{\mathbf{X}}_+^{(l_1)} = \bar{\mathbf{K}}^{(l_1)} \bar{\mathbf{X}}_-^{(l_1)}$. An expression for $\bar{\mathbf{K}}^{(l_1)}$ may be derived from the boundary conditions at $z = z_{h_1}$ [Eqs. (70) and (71) with $j = 1$ and selecting the minus sign on the right-hand side]

$$\mathbf{C}_{d+}^{(l_1)} \hat{\mathbf{X}}_+^{(l_1)} + \mathbf{C}_{d-}^{(l_1)} \hat{\mathbf{X}}_-^{(l_1)} = \mathbf{C}_{d-}^{(f)} \hat{\mathbf{X}}_-, \quad (72)$$

$$\hat{\mu}_1 \left(\mathbf{C}_{s+}^{(l_1)} \hat{\mathbf{X}}_+^{(l_1)} + \mathbf{C}_{s-}^{(l_1)} \hat{\mathbf{X}}_-^{(l_1)} \right) = \hat{\mu} \mathbf{C}_{s-}^{(f)} \hat{\mathbf{X}}_-. \quad (73)$$

Eliminating the wave amplitude $\hat{\mathbf{X}}_-$ from Eqs. (72) and (73) gives $\hat{\mathbf{X}}_+^{(l_1)} = \hat{\mathbf{K}}^{(l_1)} \hat{\mathbf{X}}_-^{(l_1)}$, where

$$\begin{aligned} \hat{\mathbf{K}}^{(l_1)} &= - \left[\hat{\mu} \left(\mathbf{C}_{d-}^{(f)} \right)^{-1} \mathbf{C}_{d+}^{(l_1)} - \hat{\mu}_1 \left(\mathbf{C}_{s-}^{(f)} \right)^{-1} \mathbf{C}_{s+}^{(l_1)} \right]^{-1} \\ &\quad \times \left[\hat{\mu} \left(\mathbf{C}_{d-}^{(f)} \right)^{-1} \mathbf{C}_{d-}^{(l_1)} - \hat{\mu}_1 \left(\mathbf{C}_{s-}^{(f)} \right)^{-1} \mathbf{C}_{s-}^{(l_1)} \right] \end{aligned} \quad (74)$$

relates the amplitudes in the layer at $z = z_{h_1}$. Matrix $\bar{\mathbf{K}}^{(l_1)}$ is related to $\hat{\mathbf{K}}^{(l_1)}$ by

$$\bar{\mathbf{K}}^{(l_1)} = \mathbf{N}^{(l_1)} \hat{\mathbf{K}}^{(l_1)} \mathbf{N}^{(l_1)}, \quad (75)$$

where the diagonal matrix $\mathbf{N}^{(l_1)}$ is

$$\mathbf{N}^{(l_1)} = \begin{pmatrix} \exp(-\kappa_l^{(l_1)} |z_{h_1} - z_{l_1}|) & 0 \\ 0 & \exp(-\kappa_l^{(l_1)} |z_{h_1} - z_{l_1}|) \end{pmatrix}. \quad (76)$$

Boundary conditions at $z = z_{l_1}$ are given by Eqs. (64) and (65) with $j = 1$:

$$\mathbf{C}_{d+}^{(f)} \bar{\mathbf{X}}_+^{(c_1)} + \mathbf{C}_{d-}^{(f)} \bar{\mathbf{X}}_-^{(c_1)} = \mathbf{C}_{d+}^{(l_1)} \bar{\mathbf{X}}_+^{(l_1)} + \mathbf{C}_{d-}^{(l_1)} \bar{\mathbf{X}}_-^{(l_1)}, \quad (77)$$

$$\begin{aligned} \hat{\mu} \left(\mathbf{C}_{s+}^{(f)} \bar{\mathbf{X}}_+^{(c_1)} + \mathbf{C}_{s-}^{(f)} \bar{\mathbf{X}}_-^{(c_1)} \right) \\ = \hat{\mu} \left(\mathbf{C}_{s+}^{(l_1)} \bar{\mathbf{X}}_+^{(l_1)} + \mathbf{C}_{s-}^{(l_1)} \bar{\mathbf{X}}_-^{(l_1)} \right). \end{aligned} \quad (78)$$

Vectors $\bar{\mathbf{X}}_+^{(l_1)}$ and $\bar{\mathbf{X}}_-^{(l_1)}$ may be eliminated from Eqs. (77) and (78) by using the definition of $\bar{\mathbf{K}}^{(l_1)}$

$$\begin{aligned} \hat{\mu}_1 \left(\mathbf{C}_{d+}^{(l_1)} \bar{\mathbf{K}}^{(l_1)} + \mathbf{C}_{d-}^{(l_1)} \right)^{-1} \left(\mathbf{C}_{d+}^{(f)} \bar{\mathbf{X}}_+^{(c_1)} + \mathbf{C}_{d-}^{(f)} \bar{\mathbf{X}}_-^{(c_1)} \right) \\ = \hat{\mu} \left(\mathbf{C}_{s+}^{(l_1)} \bar{\mathbf{K}}^{(l_1)} + \mathbf{C}_{s-}^{(l_1)} \right)^{-1} \left(\mathbf{C}_{s+}^{(f)} \bar{\mathbf{X}}_+^{(c_1)} + \mathbf{C}_{s-}^{(f)} \bar{\mathbf{X}}_-^{(c_1)} \right). \end{aligned} \quad (79)$$

Matrix $\bar{\mathbf{K}}^{(c_1)}$, defined by $\bar{\mathbf{X}}_+^{(c_1)} = \bar{\mathbf{K}}^{(c_1)} \bar{\mathbf{X}}_-^{(c_1)}$, relates the wave amplitudes at $z = z_{l_1}$ in the liquid. This gives

$$\begin{aligned} \bar{\mathbf{K}}^{(c_1)} = - \left(\hat{\mu}_1 \mathbf{L}_d^{(l_1)} \mathbf{C}_{d+}^{(f)} - \hat{\mu} \mathbf{L}_s^{(l_1)} \mathbf{C}_{s+}^{(f)} \right)^{-1} \\ \times \left(\hat{\mu}_1 \mathbf{L}_d^{(l_1)} \mathbf{C}_{d-}^{(f)} - \hat{\mu} \mathbf{L}_s^{(l_1)} \mathbf{C}_{s-}^{(f)} \right), \end{aligned} \quad (80)$$

where

$$\mathbf{L}_d^{(l_1)} = \left(\mathbf{C}_{d+}^{(l_1)} \bar{\mathbf{K}}^{(l_1)} + \mathbf{C}_{d-}^{(l_1)} \right)^{-1}, \quad (81)$$

$$\mathbf{L}_s^{(l_1)} = \left(\mathbf{C}_{s+}^{(l_1)} \bar{\mathbf{K}}^{(l_1)} + \mathbf{C}_{s-}^{(l_1)} \right)^{-1}. \quad (82)$$

Finally, $\mathbf{K}^{(c_1)}$ is related to $\bar{\mathbf{K}}^{(c_1)}$ as

$$\mathbf{K}^{(c_1)} = \mathbf{N}^{(c_1)} \bar{\mathbf{K}}^{(c_1)} \mathbf{N}^{(c_1)}, \quad (83)$$

where

$$\mathbf{N}^{(c_1)} = \begin{pmatrix} e^{-\kappa_l |z_{l_1}|} & 0 \\ 0 & e^{-\kappa_l |z_{l_1}|} \end{pmatrix}. \quad (84)$$

The reflection coefficient matrix for the second layer $\mathbf{K}^{(c_2)}$ may be found in a similar fashion. If the bottom layer has identical material properties, then the matrix $\mathbf{K}^{(c_2)}$ is related to $\mathbf{K}^{(c_1)}$ by

$$\mathbf{K}^{(c_2)} = \begin{pmatrix} K_{TT}^{(c_1)} & -K_{TL}^{(c_1)} \\ -K_{LT}^{(c_1)} & K_{LL}^{(c_1)} \end{pmatrix}. \quad (85)$$

A general form of the reflection coefficient matrix is therefore

$$\mathbf{K}^{(c_j)} = \begin{pmatrix} K_{TT}^{(c_j^*)} & \text{sgn}(z_{l_j}) K_{TL}^{(c_j^*)} \\ \text{sgn}(z_{l_j}) K_{LT}^{(c_j^*)} & K_{LL}^{(c_j^*)} \end{pmatrix}, \quad (86)$$

where $\mathbf{K}^{(c_j^*)}$ is given by Eq. (83) with (c_1) replaced by (c_j) .

At this point, particle displacement due to the source in the liquid channel may be calculated by evaluating Eqs. (6) and (8) after substituting the expressions for $T_{\pm}^{(c_j)}$ and $L_{\pm}^{(c_j)}$ into Eqs. (13) and (14). Through the reflection coefficient matrices $\mathbf{K}^{(c_j)}$, amplitudes $T_{+}^{(c_1)}$, $L_{+}^{(c_1)}$, $T_{-}^{(c_2)}$, and $L_{-}^{(c_2)}$ are expressed in terms of $T_{+}^{(c_2)}$, $L_{+}^{(c_2)}$, $T_{-}^{(c_1)}$, and $L_{-}^{(c_1)}$, which are given explicitly by Eqs. (54) and (55).

While not discussed here in detail, a similar procedure yields expressions for transmission coefficient matrices which relate the amplitudes $T_{\pm}^{(l_j)}$ and $L_{\pm}^{(l_j)}$ to the amplitudes given by Eqs. (54) and (55). This allows evaluation of the Green's function within the layers and therefore yields estimates of the displacement, strain, and stress fields in the layers.

D. Reverberant pressure field in the channel

The reverberant pressure at the location of the source $(z, r) = (0, 0)$ is

$$P_{\text{rev}}(\omega) = -\frac{\rho \omega^2}{2\pi} \int_0^\infty \Psi_{\text{rev}}(\kappa) J_0(0) \kappa d\kappa, \quad (87)$$

where

$$\Psi_{\text{rev}}(\kappa) = \left(\mathbf{X}_{-}^{(c_1)} + \mathbf{X}_{+}^{(c_1)} - \mathbf{S} \right) \cdot (0 \ 1)^T \quad (88)$$

$$= -\frac{sk_l^2}{2\rho\omega^2\kappa_l} \tilde{\Psi}_{\text{rev}}(\kappa) \quad (89)$$

is the angular spectrum of the longitudinal component of the displacement potential ψ , defined by $\mathbf{g} = \nabla\psi$, corresponding to the reverberant field. Transverse (shear) contributions are omitted because they do not affect volumetric pulsation. In Eq. (88) the sum $\mathbf{X}_{-}^{(c_1)} + \mathbf{X}_{+}^{(c_1)}$ represents the total potential (free plus reverberant fields), and the free-field potential $\mathbf{S}$ is subtracted from the total potential to obtain the reverberant portion. Equations (51), (54), and (55) are applied to Eq. (88) to obtain an expression for $\tilde{\Psi}_{\text{rev}}(\kappa)$. This yields

$$\tilde{\Psi}_{\text{rev}}(\kappa) = (\mathbf{F}_1 + \mathbf{F}_2 - 2\mathbf{I})(0 \ 1)^T \cdot (0 \ 1)^T, \quad (90)$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are given by Eqs. (56) and (57), and $\mathbf{I}$ is the identity matrix.

If the point volume source is replaced by a pulsating bubble with finite equilibrium radius R_0 then the pressure at the bubble wall (e.g., at $r = 0$, $|z| = R_0$) in a free field is

$$\begin{aligned} P_{\text{free}}(\omega) &= -\frac{\rho \omega^2}{2\pi} \int_0^\infty \mathbf{S} \cdot (0 \ 1)^T e^{-\kappa_l R_0} J_0(0) \kappa d\kappa \\ &= \frac{sk_l^2}{4\pi R_0} e^{ik_l R_0}, \end{aligned} \quad (91)$$

which for $k_l R_0 \ll 1$ is approximately $P_{\text{free}}(\omega) \cong sk_l^2/4\pi R_0$. The pressure incident on the bubble due to the reverberant field may therefore be expressed as

$$P_{\text{rev}}(\omega) = \frac{sk_l^2}{4\pi} \int_0^\infty \tilde{\Psi}_{\text{rev}}(\kappa) \frac{\kappa}{\kappa_l} d\kappa = P_{\text{free}}(\omega) H(\omega) \quad (92)$$

for $k_l R_0 \ll 1$, where $H(\omega)$ is the frequency response of the environment. For the parallel layer geometry Eq. (92) shows that the frequency response is

$$H_{\text{parallel layers}}(\omega) = R_0 \int_0^\infty \tilde{\Psi}_{\text{rev}}(\kappa) \frac{\kappa}{\kappa_l} d\kappa. \quad (93)$$

If there is only one layer present such that $\mathbf{K}^{(c_2)} \rightarrow \mathbf{0}$ then Eq. (89) simplifies to $\tilde{\Psi}_{\text{rev}}(\kappa) = K_{LL}^{(c_1)}$, where $K_{LL}^{(c_1)} = L_+^{(c_1)}/L_-^{(c_1)}$ is the (2, 2) entry in $\mathbf{K}^{(c_1)}$. In this case the frequency response is

$$H_{\text{single layer}}(\omega) = R_0 \int_0^\infty K_{LL}^{(c_1)} \frac{\kappa}{\kappa_l} d\kappa. \quad (94)$$

The integration indicated in Eqs. (93) and (94) must be performed numerically.

III. RADIAL DYNAMICS OF THE BUBBLE

Although the above expressions for $H(\omega)$ are valid only for linear strains, a nonlinear expression for the free field pressure P_{free} can be substituted into Eq. (92), with the restriction that $|\varepsilon_{nk}| \ll 1$ in the layers. In a free field the time-domain nonlinear pulsation of a gas bubble is described by the Rayleigh–Plesset equation⁴⁰

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{P_{\text{bub}}}{\rho}, \quad (95)$$

where R is the time-dependent bubble radius, overdots indicate differentiation with respect to time and

$$P_{\text{bub}} = \left(P_0 + \frac{2\sigma}{R_0}\right) \left(\frac{R}{R_0}\right)^{-3\gamma} - \frac{2\sigma}{R} - \frac{4\eta\dot{R}}{R} - P_0 - P_{\text{ac}} \quad (96)$$

is the pressure in the liquid at the bubble surface, where γ is the polytropic exponent, σ is surface tension, and P_{ac} is pressure from an external acoustic source.⁴¹ The pressure due to the reverberant field is represented by the expression

$$P_{\text{free}}(t) * h(t) = \frac{\rho \dot{V}}{4\pi R_0} * h(t) = \frac{\rho R}{R_0} (R\ddot{R} + 2\dot{R}^2) * h(t), \quad (97)$$

where $V = \frac{4}{3}\pi R^3$ is the bubble volume, $h(t)$ is the inverse Fourier transform of $H(\omega)$, and the asterisk denotes convolution. The nonlinear time-domain model for the radial dynamics of the bubble is then

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{R}{R_0} (R\ddot{R} + 2\dot{R}^2) * h(t) = \frac{P_{\text{bub}}}{\rho}. \quad (98)$$

A model for the radial dynamics of coated ultrasound contrast agent bubbles is readily obtained by substituting an appropriate expression for P_{bub} accounting for the bubble shell into Eq. (98).^{42–44}

Substituting

$$R = R_0 + \xi(t), \quad |\xi| \ll R_0 \quad (99)$$

into Eq. (98) and neglecting nonlinear terms in ξ yields a model for linear bubble pulsation:

$$[\delta(t) + h(t)] * \ddot{\xi} + \omega_0 \delta_d \dot{\xi} + \omega_0^2 \xi = -\frac{P_{\text{ac}}}{R_0 \rho}, \quad (100)$$

where

$$\omega_0 = \frac{1}{R_0} \sqrt{\frac{3\gamma P_0}{\rho} + \frac{2\sigma(3\gamma - 1)}{\rho R_0}} \quad (101)$$

is the natural frequency of the bubble in a free field and

$$\delta_d = \frac{4\eta}{\rho R_0^2 \omega_0} \quad (102)$$

is the damping coefficient. Substituting $P_{\text{ac}} = p_0 e^{-i\omega t}$ and $\xi(t) = \Xi(\omega) e^{-i\omega t}$ into Eq. (100) yields the frequency response

$$\frac{\Xi(\omega)}{\Xi(0)} = \left\{ 1 - i\delta_d \left(\frac{\omega}{\omega_0}\right) - [1 + H(\omega)] \left(\frac{\omega}{\omega_0}\right)^2 \right\}^{-1}, \quad (103)$$

where $\Xi(0) = -p_0/R_0\rho\omega_0^2$ is the static radial displacement.

IV. MODEL SIMULATIONS

In the following simulations the equilibrium radius of the bubble is always taken to be $R_0 = 1.5 \mu\text{m}$, a size appropriate for biomedical acoustics applications, and the polytropic exponent is assumed to be $\gamma = 1.4$. The parameters for the surrounding liquid are $\rho = 1000 \text{ kg/m}^3$, $c = 1500 \text{ m/s}$, $\eta = 1 \text{ mPa s}$, $\zeta = 3.09 \text{ mPa s}$,⁴⁵ $\sigma = 0.073 \text{ N/m}$, and for the ambient pressure $P_0 = 101.3 \text{ kPa}$. These parameters correspond to an air bubble in water at one standard atmosphere.

Results for viscoelastic layers made of plastic and vascular tissue as well as for a lossless rigid interface are presented in order to demonstrate dependence of bubble pulsation on the material properties of the layers. The material parameters of the plastic layer were assumed to be $E_j = 3.3 \text{ GPa}$ (Young's modulus), $\mu_j = 1.23 \text{ GPa}$, and $\rho_j = 1050 \text{ kg/m}^3$. These properties are similar to those of polystyrene, which was used to confine ultrasound contrast agent bubbles in Ref. 31. For the vascular tissue layers, it is assumed that $E_j = 90 \text{ kPa}$, $\mu_j = 30.1 \text{ kPa}$, $\eta_j = 2.5 \text{ Pa s}$, and $\rho_j = 1103 \text{ kg/m}^3$.^{46,47} Lamé parameter λ_j expressed in terms

of these quantities is $\lambda_j = \mu_j(E_j - 2\mu_j)/(3\mu_j - E_j)$. The dilational viscosities ζ_j and the shear viscosity of polystyrene were set to zero due to lack of information. In the case of a bubble near a single lossless rigid interface, $H(\omega)$ is

$$H_{\text{rigid}}(\omega) = \frac{R_0}{2D} e^{2i\omega D/c}, \quad (104)$$

or equivalently

$$h_{\text{rigid}}(t) = \frac{R_0}{2D} \delta(t - 2D/c), \quad (105)$$

where D is the distance between the bubble center and the interface. Note that when Eq. (105) is substituted into Eq. (100) to obtain the pulsation amplitude, viscous losses are partially accounted for via Eq. (102), but losses due to viscous stresses along the interface are ignored. This is identical to the approach taken in Ref. 48. For the lossless rigid interface, attenuation of the reflected pressure is due solely to spherical spreading over the round trip distance between the bubble and layer ($2D$), and the phase is shifted accordingly. For the plastic and tissue viscoelastic layers, losses due to viscous stress near the interface are included. The expression for $H(\omega)$, given by Eqs. (93) or (94), must be integrated numerically and in nondimensional form to ensure numerical stability. Lengths and wavenumbers are scaled according to the real part of the longitudinal wavenumber in the liquid $\text{Re}(k_l) = \omega/c$, and the upper limit of integration is changed from ∞ to $\kappa/\text{Re}(k_l) = 10^3$.

To demonstrate dependence of the radial dynamics on the geometry of the confining environment, simulations for bubbles oscillating near a single layer and between parallel layers for various values of the bubble-layer offset distance $|z_0 - z_l|$ are presented. The layers are assumed to be 5 mm thick, at which value their influence on the bubble is similar to that due to viscoelastic half spaces. This is because, at 5 mm, the layer thickness is large compared to the wavelength of the surface (Scholte) wave (approximately 1 mm for polystyrene). The bubble dynamics do change for layers with thicknesses approaching or less than the wavelength of the surface wave, but these cases are not considered here.

Simulations of low amplitude pulsation near a single layer are presented first in order to eliminate complications due to nonlinear pulsation and focus instead on material and geometric parameters. It is assumed that the bubble is driven by a time-harmonic external acoustic pressure source of sufficiently low amplitude (here, $p_0 = 1$ kPa) such that the radial dynamics of the bubble are linear. In this case the pulsation amplitude scales with p_0 , there is no harmonic generation, and Eq. (103) gives the normalized pulsation amplitude. After the relative importance of evanescent and propagating modes at various bubble-layer offset distances are briefly examined, comparisons between the single and parallel layer models are presented to demonstrate effects due to the presence of a second layer.

Finally, weakly nonlinear bubble pulsation near a single layer is illustrated. At drive pressures of higher amplitude (here, 20 kPa) nonlinearities are significant and therefore Eq. (98) is used to obtain the pulsation amplitude instead of

Eq. (103). An inverse fast Fourier transform routine is used to find $h(t)$ after $H(\omega)$ is calculated. Equation (98) is then integrated numerically with a standard backward differentiation routine⁴⁹ to obtain the bubble radius. It is computationally more efficient to perform the convolution indicated in Eq. (97) directly instead of using a fast Fourier transform technique. This is due to the fact that the result of the convolution is only needed at the current time, and therefore computational complexity scales as $O(N)$, where N is the number of samples, instead of as $O(N \log N)$ for convolution performed via fast Fourier transform. The normalized pulsation amplitude is taken to be $(R_+ - R_-)/2R_0$, where R_+ is the peak excursion of the bubble radius above its equilibrium radius R_0 , and R_- is the peak excursion below R_0 , after transients have disappeared. Repeating this process at multiple forcing frequencies allows a resonance curve to be generated. At low drive pressure amplitudes $(R_+ - R_-)/2R_0$ is equal to $|\Xi(\omega)|/R_0$ obtained from Eq. (103). Strain in the layers, calculated according to the brief description at the end of Sec. II C, is monitored to ensure that strain magnitudes remain small.

A. Low amplitude pulsation near a single layer

The single layer geometry is obtained from Fig. 1 by removing the bottom layer. Recall that in this case $H(\omega)$ is given by Eq. (94) for the viscoelastic layers and by Eq. (104) for the lossless rigid surface. In this section the dependence of bubble pulsation amplitude on layer material properties, as well as on the distance $D = |z_l - z_0|$ between the center of the bubble and the near face of the layer, is explored.

1. Viscoelastic layer vs lossless rigid surface

Figure 3 shows the normalized amplitude of the bubble wall displacement obtained from Eq. (103) for the bubble pulsating at normalized offset distances of $D/R_0 = 1, 2, 10$, and 200 away from a viscoelastic polystyrene layer [Fig. 3(a)], a viscoelastic vascular tissue layer [Fig. 3(b)], and a lossless rigid surface [Fig. 3(c)]. In each case the bubble responds as if it were in a free field at $D/R_0 = 200$.

In general, as the bubble is moved toward a viscoelastic layer [Figs. 3(a) and 3(b)], both the peak pulsation amplitude and the resonance frequency of the bubble decrease. This behavior is in qualitative agreement with laboratory measurements of bubbles pulsating near a polystyrene layer.³¹ In comparison to the polystyrene layer, interaction with the vascular tissue layer causes a greater decrease in pulsation amplitude at small offset distances and an upward shift in resonance frequency. Also note that at $D/R_0 = 10$ in Fig. 3(b), the natural frequency of the bubble near the tissue layer is slightly higher than its value in a free field.

Interaction with the rigid surface [Fig. 3(c)] also causes a downward shift in resonance frequency as the bubble is moved toward the surface. However, in contrast with its behavior near the viscoelastic layers, bubble pulsation amplitude increases near a lossless rigid surface. A physical understanding may be gained by examining Eqs. (102) and (103). The resonance frequency of the bubble decreases as the bubble approaches the surface, which means that the magnitude

FIG. 3. Normalized amplitude of the bubble wall displacement obtained from Eq. (103) for a gas bubble of equilibrium radius $R_0 = 1.5 \mu\text{m}$ driven by a low amplitude time-harmonic acoustic pressure source. The bubble is positioned a distance D away from single 5 mm thick viscoelastic (a) polystyrene or (b) vascular tissue layers, or (c) a rigid surface.

of viscous damping (proportional to ω) decreases, resulting in an increase in pulsation amplitude. However, as will be shown in the following section, losses due to viscous stress along the layer interface affect the pulsation amplitude only if the bubble is very close to the interface, and effects due to the layer stiffness are dominant when the bubble is more than a few equilibrium radii from the interface.

2. Effect of viscosity and layer stiffness

One might expect that the decrease in pulsation amplitude near the viscoelastic layer, illustrated in Fig. 3, is due to viscous losses at the layer surface. However, the viscous boundary layer thickness in the liquid is approximately

$$\delta = \sqrt{\frac{2\eta}{\rho\omega}} \cong \frac{0.6}{\sqrt{\omega/\omega_0}} \mu\text{m},$$

and therefore for our example, damping due to viscous losses along the layer surface should be confined to distances less than a few microns. As Fig. 3 illustrates, the decrease in pulsation amplitude persists at much greater distances. To investigate the influence of losses due to shear viscosity at the layer surface Young's modulus is held constant at 1 GPa while varying the shear viscosity in the liquid. Both the peak pulsation amplitude and resonance frequency were evaluated for bubble-layer distances ranging from 1 to 1000 equilibrium radii. Figure 4 shows the dependence of the peak pulsation amplitude [Fig. 4(a)] and resonance frequency [Fig. 4(b)] on the offset distance for shear viscosities of 0.1, 1, and 10 mPa s. As Fig. 4(a) shows, the influence of viscosity on the peak pulsation amplitude is significant close to the layer (for approximately $D/R_0 < 5$) but is negligible at larger distances. Note that, in agreement with the results for

FIG. 4. Peak bubble wall displacement (a) and resonance frequency (b) obtained from Eq. (103) for a gas bubble of equilibrium radius $R_0 = 1.5 \mu\text{m}$ driven by a low amplitude time-harmonic acoustic pressure source. The bubble is positioned at a range of distances D away from a single 5 mm thick polystyrene layer for liquid shear viscosities of 0.1, 1, and 10 mPa s.

the lossless rigid surface presented in Fig. 3(c), for small viscosity (e.g., 0.1 mPa s) the peak pulsation amplitude increases as the bubble approaches the interface. Dependence of the resonance frequency on offset distance is illustrated in Fig. 4(b), where it is shown that viscosity has a negligible influence on the resonance frequency of the bubble.

To investigate dependence on the stiffness of the layer, the shear viscosity in the liquid is held at its nominal value while the Young's modulus of the layer is varied. Figure 5 shows the dependence of the peak pulsation amplitude [Fig.

FIG. 5. Peak bubble (a) wall displacement and (b) resonance frequency obtained from Eq. (103) for a gas bubble of equilibrium radius $R_0 = 1.5 \mu\text{m}$ driven by a low amplitude time-harmonic acoustic pressure source. The bubble is positioned at a range of distances D away from single 5 mm thick layers with Young's modulus values of 0.1, 1, and 10 GPa. Other material properties of the layer are identical to that of polystyrene.

5(a)] and resonance frequency [Fig. 5(b)] on the bubble-layer offset distance for layers with Young's modulus of 0.1, 1, and 10 GPa. As Fig. 5(a) shows, layer stiffness affects bubble pulsation amplitude at much larger distances (up to approximately $D/R_0 = 100$). In addition, as Fig. 5(b) illustrates, in comparison to shear viscosity layer stiffness has a more dominant effect on the resonance frequency of the bubble.

3. Relative importance of evanescent and propagating modes

According to Eq. (17), values of κ for which $\kappa/\text{Re}(k_l) < 1$ correspond to propagating modes in the liquid while values of κ for which $\kappa/\text{Re}(k_l) > 1$ correspond to evanescent modes. The relative importance of evanescent and propagating modes is investigated in Fig. 6, in which the absolute value of the integrand in Eq. (94) (i.e., $|K_{LL}^{(c1)} \kappa/\kappa_l|$) is plotted as a function of $\kappa/\text{Re}(k_l)$. Simulations for both polystyrene [Fig. 6(a)] and tissue [Fig. 6(b)] layers of thickness 5 mm are shown for a bubble with equilibrium radius $1.5 \mu\text{m}$ positioned at normalized distances D/R_0 of 1, 10, and 200 from the layers. In each case the angular frequency ω was chosen to be the frequency at which the bubble achieved its maximum pulsation amplitude. This ranged from $\omega/\omega_0 \cong 0.8$ at $D/R_0 = 1$ to $\omega/\omega_0 \cong 1$ at $D/R_0 = 200$.

As Fig. 6 shows, when the bubble is close to the interface ($D/R_0 = 1$) evanescent modes are dominant, but at large distances ($D/R_0 = 200$) their contribution is negligible. This is readily explained by the fact that evanescent modes decay rapidly with distance and, therefore, only interact with the bubble when it is close to the interface. Figure 7 focuses on the transition between propagating and evanescent modes occurring at $\kappa/\text{Re}(k_l) = 1$. For the polystyrene

FIG. 6. Absolute value of the integrand in Eq. (94) for a gas bubble of equilibrium radius $R_0 = 1.5 \mu\text{m}$ positioned various distances D away from single 5 mm thick viscoelastic (a) polystyrene or (b) tissue layers. Values of $\kappa/\text{Re}(k_l) < 1$ represent propagating modes and $\kappa/\text{Re}(k_l) > 1$ represent evanescent modes.

FIG. 7. Absolute value of the integrand in Eq. (94), focusing on the transition from propagating [$\kappa/\text{Re}(k_l) < 1$] to evanescent [$\kappa/\text{Re}(k_l) > 1$] modes, for a gas bubble of equilibrium radius $R_0 = 1.5 \mu\text{m}$ positioned various distances D away from single 5 mm thick viscoelastic (a) polystyrene or (b) tissue layers.

layer [Fig. 7(a)], the peak near $\kappa/\text{Re}(k_l) = 1.4$ corresponds to the wavenumber of the Scholte wave.

B. Low amplitude pulsation between parallel layers

In this section simulation results for the parallel layer and single layer geometries will be compared. In the case of the parallel layer geometry, the expression for $H(\omega)$ is given by Eq. (93). Bubble pulsation between two nonidentical layers, and the effect of varying the position of the bubble in the channel between the layers, will also be explored.

1. Single vs parallel layers

Figure 8 shows the absolute value of Eq. (103) for a bubble pulsating near a single tissue layer (solid lines) and between two parallel tissue layers (dotted lines). In the case of the parallel layers the bubble is positioned in the center of the channel, a normalized distance D/R_0 of 1 [Fig. 8(a)], 5 [Fig. 8(b)], and 150 [Fig. 8(c)] from either layer so that the normalized distance separating the layers is $H = |z_{l1} - z_{l2}| = 2D$. As Fig. 8 illustrates, the presence of the second layer changes the dynamics dramatically for sufficiently small bubble-layer offset distances.

2. Bubble location between nonidentical layers

The influence of the relative position of the bubble between two nonidentical layers is now explored. Figure 9 shows the radial response of a bubble between polystyrene and tissue layers separated by distance $H = z_{l1} - z_{l2} = 10R_0$. In this case, D denotes the distance from the polystyrene layer, and thus $D/H < 0.5$ indicates that the bubble is closer to the polystyrene layer, and

FIG. 8. Normalized amplitude of the bubble wall displacement obtained from Eq. (103) for a gas bubble of equilibrium radius $R_0 = 1.5 \mu\text{m}$ driven by a low amplitude time-harmonic acoustic pressure source. The bubble is positioned at normalized distances D/R_0 of (a) 150, (b) 5, and (c) 1 away from a single tissue layer of thickness 5 mm (solid lines) or between two parallel tissue layers (dotted lines).

$D/H > 0.5$ indicates that the bubble is closer to the tissue layer.

In all cases the bubble pulsation amplitude in the channel is less than the free field pulsation amplitude (see results for $D/R_0 = 200$ in Fig. 3) but the pulsation amplitude and resonance frequency are highly dependent on the position of the bubble within the channel and the material parameters of the layers. For example, note that the largest pulsation amplitude is not achieved with the bubble in the center of the channel.

C. High amplitude pulsations

Figure 10 shows the normalized bubble pulsation amplitude $(R_+ - R_-)/2R_0$ for a bubble near a single polystyrene layer driven by a sinusoidal pressure source with amplitude

FIG. 9. Normalized amplitude of the bubble wall displacement obtained from Eq. (103) for a gas bubble of equilibrium radius $R_0 = 1.5 \mu\text{m}$ driven by a time-harmonic acoustic pressure source of amplitude 1 kPa. The bubble is located at various positions between a tissue layer and a polystyrene layer separated by $H = 10R_0$. The distance from the bubble to the polystyrene layer is denoted D .

FIG. 10. Steady state pulsation amplitude for a gas bubble with equilibrium radius $R_0 = 1.5 \mu\text{m}$ driven by a time-harmonic acoustic pressure source of amplitude 20 kPa. The pulsation amplitude was obtained via numerical integration of Eq. (98), and the bubble is located at various distances D away from a polystyrene layer of thickness 5 mm.

20 kPa and normalized angular frequency ω/ω_0 , where ω_0 is given by the square root of Eq. (101). Values of R_+ and R_- were obtained by numerically integrating Eq. (98) through time until transients decayed. At this pressure amplitude the nonlinear terms in Eq. (98) are significant and the bubble pulsation exhibits skewing toward lower frequencies as compared to Fig. 3, as well as the harmonic generation and saturation that is characteristic of nonlinear bubble response in a free field (see, for example, Fig. 4 in Ref. 50).

Figure 11 shows the time dependence of the normalized radius R/R_0 when the bubble is driven by a sinusoidal acoustic pressure source with amplitude 20 kPa and normalized angular frequency $\omega/\omega_0 = 1$. Simulations are shown for small ($D/R_0 = 1$) and large ($D/R_0 = 200$) separation distances between the bubble and a polystyrene layer of thickness 5 mm. The acoustic source is turned off at $t = 2T_0$, where $T_0 = 2\pi/\omega_0$, after which the pulsation amplitude begins to decay. Note that the bubble initially responds at its resonance frequency which, as examination of Fig. 10 shows, is lower when it is positioned close to the layer.

FIG. 11. Pulsation amplitude of a gas bubble with equilibrium radius $R_0 = 1.5 \mu\text{m}$ driven by two cycles of a time-harmonic acoustic pressure source with amplitude 20 kPa and normalized angular frequency $\omega/\omega_0 = 1$. The acoustic source is turned off at $t = 2T_0$. The pulsation amplitude was obtained via numerical integration of Eq. (98), and the bubble is located at normalized distances D/R_0 of 1 or 200 from a polystyrene layer of thickness 5 mm.

V. SUMMARY

Harmonic analysis and angular spectrum decomposition were employed to derive the Green's function for particle displacement for a spherical bubble pulsating in a viscous compressible liquid between two parallel viscoelastic layers, or near a single layer. The Green's function was used to derive an expression for the pressure due to reflections from the layers, which was incorporated into linear and nonlinear models describing the pulsation of the bubble. Both evanescent and propagating modes were included, and evanescent modes were shown to be dominant when the bubble is close to either layer. Dependence of the bubble pulsation amplitude on the viscoelastic parameters of the layers, as well as several geometric parameters such as bubble-layer offset distance, and relative position of the bubble between the layers, were explored. Simulations show that shear viscosity has a dominant influence on the dynamics close to the layers, but effects due to layer stiffness continue to affect bubble pulsation at much larger distances.

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