Counterpropagating Rossby waves in confined plane wakes

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In the present work, we revisit the temporal and the spatio-temporal stability of confined plane wakes under the perspective of the counterpropagating Rossby waves (CRWs). Within the context of broken line velocity profiles, each vorticity discontinuity can be associated to a counterpropagating Rossby wave. In the case of a wake modeled by a broken line profile, the interaction of two CRWs is shown to originate in a shear instability. Following this description, we first recover the stability results obtained by Juniper [J. Fluid Mech. **590**, 163–185 (2007)] and Biancofiore and Gallaire [Phys. Fluids **23**, 034103 (2011)] by means of the classical normal mode analysis. In this manner, we propose an explanation of the stabilizing influence of the confinement on the temporal stability properties. The CRW description further allows us to propose a new interpretation of the counterintuitive spatio-temporal destabilization in wake flows at moderate confinement noticed by Juniper [J. Fluid Mech. **565**, 171–195 (2006)]: it is well predicted by the mean group velocity of the uncoupled CRWs. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4729617>]

I. INTRODUCTION

With the aim to understand why adding a recess to a coaxial injector of the rockets engine was improving the mixing quality, Juniper and Candel¹ have applied the concept of absolute and convective instability to confined wake flows. Broadly speaking, according to Briggs² and Bers,³ a flow is absolutely unstable if a disturbance of finite spatial extent grows at every fixed point, whereas if such disturbance decays eventually at every fixed point in space, the flow is deemed convectively unstable. Using as crude wake model a top-hat profile, Juniper and co-workers^{1,4} have found that the presence of a moderate confinement enhances the absolute character of the instability in wake flows. This was explained as the result of an interaction of the modes of the inner and outer streams, which have a similar thickness in a moderate confinement case. This destabilizing effect was confirmed adding a finite shear layer thickness to the model,^{5,6} thereby confirming the accuracy of broken-line velocity profiles for modeling the temporal and spatio-temporal instabilities of wakes.⁶ The influence of the presence of confining walls on the spatio-temporal development of the varicose mode of jets and wakes was furthermore recovered by Healey,⁷ owing to the geometric similarity of such mode with asymmetric confined mixing layers.

The concept of counterpropagating Rossby waves (CRWs) was developed originally by Bretherton⁸ to explain the baroclinic instability mechanism for cyclone growth in the atmosphere. He showed that the instability in flows featuring two distinct potential-vorticity gradients can be interpreted in terms of the interaction of two distinct Rossby edge waves. Such waves were named counterpropagating by Hoskins *et al.*⁹ since their direction is opposite to the mean flow. It is important to note that, although each wave in itself is neutral and propagates via the Rossby mechanism, the mutual interaction yields an instability through a phase-locking mechanism. Depending on their phase difference, each CRW modifies the propagation rate of the other CRW and can also promote its growth. Growing normal modes are obtained if a suitable phase difference exists where interaction makes the CRW phase speeds equal so that they phase lock which results in mutual growth.

More recently the interaction of these counterpropagating Rossby waves was used by Heifetz *et al.*¹⁰ to describe an unconfined inviscid mixing layer of finite shear layer thickness, thereby importing this CRW concept from the geophysics community into the fluid mechanics. The Kelvin-Helmholtz instability could also be interpreted in the CRW perspective as a limiting case of the mixing layer with vanishing shear layer thickness.¹² Furthermore, instabilities in stratified shear flows (i.e., Holmboe modes) have been interpreted as the interaction of two Rossby-gravity waves.^{13–15} Afterwards Bakas and Ioannou¹⁶ have revisited with this physical description instabilities in free surface shear flows. The only attempt to connect the CRWs and spatio-temporal instability is due to Bishop and Heifetz,¹⁷ analyzing the baroclinic Eady's model.

By explicitly exhibiting the interaction coefficient, this approach constitutes an extension of the reinterpretation proposed by Caulfield¹⁸ for the stratified mixing layer model of Taylor¹⁹ and Holmboe.²⁰ Using the concepts of Cairns,²¹ according to which the stability of a flow with discontinuities in the background density, velocity, or vorticity fields may be qualitatively analyzed by an investigation of the waves that could occur on each of the interfaces if they were totally isolated from each other, he identified the different type of unstable modes with interactions of waves on the various interfaces of either vorticity or density. It was shown that the large wavenumber limit corresponds to considering each interface in isolation. In the case of homogeneous density, i.e., in the unstratified case, the isolated waves were called Rayleigh waves. The dispersive behavior of the isolated waves was seen to be crucial for a destabilizing interaction to take place, but the spatio-temporal propagation properties were not analyzed.

Our purpose is to reinterpret both the temporal and the spatio-temporal stability of confined wakes as the interaction of CRWs. In particular, we suggest that the dispersive behavior of the underlying pure counterpropagating Rossby waves, taken uncoupled, can both explain the stabilizing influence of confinement onto the temporal instability⁶ and its counterintuitive spatio-temporal destabilizing influence noticed by Juniper⁴ and confirmed in several further studies.^{5,6,22} The outline of the paper is as follows: in Sec. II we describe the model and we obtain the dynamical system which describes the interacting counterpropagating Rossby waves, while in Sec. III we reinterpret the temporal stability in the light of the CRW perspective. The spatio-temporal behavior of the flows is interpreted in Sec. IV, followed by conclusions in Sec. V.

II. MODEL AND DYNAMICAL SYSTEM

A. Model

The model flow consists in a two-dimensional wake in a confined channel (see Figure 1). The channel has infinite extent in the longitudinal x -direction and confining walls are located at $y = \pm(h_1 + h_2)$. The fluid is incompressible and homogeneous of uniform density, and its viscosity is assumed to be negligible, so that free slip conditions are imposed at the channel walls. As a consequence of the parallel and inviscid flow assumption, the base flow is unidirectional with an arbitrary longitudinal velocity distribution $\mathbf{U} = U(y)\mathbf{e}_x$ and constant pressure, where (x, y) are the streamwise and cross-stream Cartesian coordinates. The basic velocity profile of the model flow is

$$U(y) = \begin{cases} U_2 & \text{for } h_1 + \frac{\delta_w}{2} \leq |y| \leq h_1 + h_2 \\ \frac{U_1+U_2}{2} - \frac{(U_1-U_2)}{\delta_w}(|y| - h_1) & \text{for } h_1 - \frac{\delta_w}{2} < |y| < h_1 + \frac{\delta_w}{2} \\ U_1 & \text{for } |y| \leq h_1 - \frac{\delta_w}{2}. \end{cases} \quad (1)$$

Such a piecewise velocity profile has been recently analyzed by Juniper⁵ and Biancofiore and Gallaire⁶ and consists of the adaptation of Taylor's¹⁹ broken line model to wake flows. Since the velocity is taken piecewise linear, the background vorticity field is piecewise constant.

Using as reference length scale h_1 and as reference velocity scale $U_m = (U_1 + U_2)/2$ the non-dimensional parameters can be built as follows: $h^* = h_2/h_1$ is the confinement parameter, $\delta_w^* = \delta_w/h_1$ is the non-dimensional shear layer thickness, $\Lambda^* = (U_1 - U_2)/(U_1 + U_2)$ is the velocity ratio. Using this set of scales but omitting the asterisks from now on, the non-dimensional velocity profile

FIG. 1. The wake model in the confined channel.

becomes

$$U(y) = \begin{cases} 1 - \Lambda & \text{for } 1 + \frac{\delta_w}{2} \leq |y| \leq 1 + h \\ 1 + \frac{2\Lambda}{\delta_w}(|y| - 1) & \text{for } 1 - \frac{\delta_w}{2} < |y| < 1 + \frac{\delta_w}{2} \\ 1 + \Lambda & \text{for } |y| \leq 1 - \frac{\delta_w}{2}, \end{cases} \quad (2)$$

while the non-dimensional basic vorticity profile $Q = -dU/dy$ reads

$$Q(y) = \begin{cases} 0 & \text{for } 1 + \frac{\delta_w}{2} \leq |y| \leq 1 + h \\ \frac{2\Lambda}{\delta_w} & \text{for } 1 - \frac{\delta_w}{2} < y < 1 + \frac{\delta_w}{2} \\ -\frac{2\Lambda}{\delta_w} & \text{for } -1 - \frac{\delta_w}{2} < y < -1 + \frac{\delta_w}{2} \\ 0 & \text{for } |y| \leq 1 - \frac{\delta_w}{2}. \end{cases} \quad (3)$$

Note that if $\Lambda < 0$, this velocity field corresponds to a wake profile, while if $\Lambda > 0$, the flow is a jet. Although the present paper is devoted to the analysis of wake flows, a similar study could be performed on jet flows, by freely varying the parameter Λ .

B. Dynamical system: Sinuous and varicose perturbations

Since the model is symmetric with respect to the x -axis, two different perturbations can be distinguished: sinuous perturbations, where the shear layers move parallel to each other (Figure 2(a)) and varicose perturbations, where the shear layers move as mirror images of each other (Figure 2(b)). Taking advantage of these symmetries, we may consider just one half of the domain for both kinds of perturbations.

Starting from the velocity profile described by Eq. (2), it is possible to obtain the dispersion relations^{5,6} for both sinuous and varicose perturbations with the classic normal mode decomposition²³ $\exp[i(kx - \omega t)]$. These dispersion relations are algebraic equations which associate to each wavenumber k , a pair of complex conjugate angular frequencies ω .

Under such normal mode decomposition, we need to take k real to analyze the temporal stability, while for the spatio-temporal stability, which distinguishes convectively from absolutely unstable flows, the wavenumber is taken complex. The reader is referred to Huerre and Rossi²⁴ for more details on these distinct approaches and to Juniper⁵ and Biancofiore and Gallaire⁶ for a detailed account on the spatio-temporal behavior of the model analyzed in the present study.

On the other hand, the velocity $\mathbf{v}$ and the vorticity q can be linearized with respect to the basic state: $\mathbf{v} = (U + u', v')$ and $q = Q + q'$, where the capital letters and the primes indicate the basic state (Eqs. (2) and (3)) and perturbation, respectively. Then, the linearized vorticity equation $\frac{Dq}{Dt} = 0$

FIG. 2. Sinuous (a) and varicose (b) perturbations.

gives

$$\left[\frac{\partial}{\partial t} + U(y) \frac{\partial}{\partial x} \right] q' = -v' \frac{\partial Q(y)}{\partial y} = v' m \left[\delta(y - 1 + \frac{\delta_w}{2}) - \delta(y - 1 - \frac{\delta_w}{2}) \right], \quad (4)$$

where $m = -\frac{2\Lambda}{\delta_w}$ and δ symbolizes the Dirac delta function.

Applying the CRW perspective,¹⁰ we look for a vorticity perturbation field that is concentrated on the discontinuities of the base flow vorticity, i.e., on the two edges of the velocity profile

$$\hat{q}' = \left(q_1(k, t) \delta[y - 1 + \frac{\delta_w}{2}] + q_2(k, t) \delta[y - 1 - \frac{\delta_w}{2}] \right). \quad (5)$$

Classically, $\hat{q}' = -L\hat{\psi}'$, with $L = [d^2/dy^2 - k^2]$, since the vorticity equals the opposite of the Laplacian of the streamfunction. The Green function of $-L$ for the varicose mode, which imposes Dirichlet conditions $\partial_x \psi' = ik\psi' = 0$ at the boundaries $y = 0$ and $y = 1 + h$, is given by

$$G(y, \tilde{y}) = \frac{1}{k \sinh[k(1+h)]} \begin{cases} \sinh(k\tilde{y}) \sinh[k(1+h-y)] & \text{for } \tilde{y} \leq y \leq 1+h \\ \sinh[k(1+h-\tilde{y})] \sinh(ky) & \text{for } 0 \leq y \leq \tilde{y}, \end{cases} \quad (6)$$

while for the sinuous mode, for which a Neumann condition $\partial_{x,y} \psi' = 0$ on the axis $y = 0$ and a Dirichlet one at the wall $y = 1 + h$ are imposed,

$$G(y, \tilde{y}) = \frac{1}{k \cosh[k(1+h)]} \begin{cases} \cosh(k\tilde{y}) \sinh[k(1+h-y)] & \text{for } \tilde{y} \leq y \leq 1+h \\ \sinh[k(1+h-\tilde{y})] \cosh(ky) & \text{for } 0 \leq y \leq \tilde{y}. \end{cases} \quad (7)$$

Because wakes are known to be destabilized preferentially by sinuous modes, only the sinuous perturbation is analyzed from here on. For the varicose one, the reasoning is identical, with the only difference that Eq. (6) should be used; furthermore, the varicose mode is equivalent to study a mixing layer analyzed by Heifetz *et al.*¹⁰ and Heifetz and Methven,¹¹ however adding asymmetric confinement. Thus, by means of the Green function (7), we can obtain $\hat{\psi}'$:

- for $y \geq 1 + \frac{\delta_w}{2}$

$$\begin{aligned} \hat{\psi}'(y) = & -q_1(k, t) \frac{\cosh[k(1-\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & -q_2(k, t) \frac{\cosh[k(1+\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)], \end{aligned}$$

- for $1 - \frac{\delta_w}{2} \leq y \leq 1 + \frac{\delta_w}{2}$

$$\begin{aligned}\hat{\psi}'(y) = & -q_1(k, t) \frac{\cosh[k(1 - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & -q_2(k, t) \frac{\sinh[k(h - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky),\end{aligned}\quad (8)$$

- for $y \leq 1 - \frac{\delta_w}{2}$

$$\begin{aligned}\hat{\psi}'(y) = & -q_1(k, t) \frac{\sinh[k(h + \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky) + \\ & -q_2(k, t) \frac{\sinh[k(h - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky),\end{aligned}$$

where ψ' satisfies $u' = \frac{\partial \psi'}{\partial y}$, $v' = -\frac{\partial \psi'}{\partial x}$, i.e., in Fourier space $\hat{u}' = \partial_y \hat{\psi}'$ and $\hat{v}' = ik \hat{\psi}'$, and $q_1(k, t)$, $q_2(k, t)$ represent the counterpropagating Rossby waves at $y = 1 \pm \frac{\delta_w}{2}$. Substituting Eqs. (5) and (8) into Eq. (4), one is left with

$$\begin{aligned}& \dot{q}_1 \delta(y - 1 + \frac{\delta_w}{2}) + \dot{q}_2 \delta(y - 1 - \frac{\delta_w}{2}) + ikU(y)[q_1 \delta(y - 1 + \frac{\delta_w}{2}) + q_2 \delta(y - 1 - \frac{\delta_w}{2})] \\ & = [-imq_1 \frac{\cosh[k(1 - \frac{\delta_w}{2})] \sinh[k(1+h-y)]}{\cosh[k(1+h)]} - imq_2 \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[ky]}{\cosh[k(1+h)]}] \\ & \quad \cdot [\delta(y - 1 + \frac{\delta_w}{2}) - \delta(y - 1 - \frac{\delta_w}{2})],\end{aligned}\quad (9)$$

where the notation $\dot{q}_i$ denotes dq_i/dt .

Calculating Eq. (9) at the edges of CRW ($1 \pm \frac{\delta_w}{2}$), the evolution of vorticity perturbation is obtained:

$$\dot{q} = Mq, \quad \text{where} \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} -ikc_1 & -i\gamma \\ i\gamma & -ikc_2 \end{bmatrix}, \quad (10)$$

with

$$\gamma = -\frac{2\Lambda}{\delta_w} \cdot \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 - \frac{\delta_w}{2})]}{\cosh[k(1+h)]}, \quad (11)$$

$$c_1 = \frac{1-\Lambda}{\delta_w} \cdot \frac{\cosh[k(1 - \frac{\delta_w}{2})] \sinh[k(h + \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} + (1+\Lambda)(1 - \frac{\cosh[k(1 - \frac{\delta_w}{2})] \sinh[k(h + \frac{\delta_w}{2})]}{k\delta_w \cosh[k(1+h)]}), \quad \text{and} \quad (12)$$

$$c_2 = \frac{1+\Lambda}{\delta_w} \cdot \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 + \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} + (1-\Lambda)(1 - \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 + \frac{\delta_w}{2})]}{k\delta_w \cosh[k(1+h)]}). \quad (13)$$

III. REINTERPRETING THE TEMPORAL STABILITY BY MEANS OF CRW PERSPECTIVE

In this section we reinterpret the temporal stability examined by Biancofiore and Gallaire⁶ of the confined wake model described in Sec. II A.

Without surprise, the eigenvalues of M (see Eq. (10))

$$-i\omega_{1,2} = \lambda_{1,2} = \frac{-ik(c_1 + c_2)}{2} \pm \sqrt{k^2 c_1 c_2 + \gamma^2 - \frac{k^2(c_1 + c_2)^2}{4}} \quad (14)$$

correspond to the dispersion relations found by Juniper⁵ and Biancofiore and Gallaire⁶ using classical normal mode analysis. However, the present method proposes a physical interpretation in terms of the interaction of counterpropagating Rossby waves.

Figure 3 shows the growth rate (a) and the frequency (b) of the most unstable of the two modes, given by the real and the imaginary parts of the eigenvalues of Eq. (14), respectively, for $\delta_w = 0.535$, $\Lambda = -0.75$, and different confinement parameters h . The maximum growth rate as well as the cut-off wavenumber decrease when increasing the confinement, pointing to a systematic stabilizing

FIG. 3. Growth rate (a) and frequency (b) of coupled CRWs in the system represented by Eq. (10) for several confinements, $\delta_w = 0.535$ and $\Lambda = -0.75$. The continuous line depicts $h = 9$, the dashed one $h = 1$, and the dash-dotted one $h = 0.5$.

influence of confinement on the temporal instability of wake flows. In contrast, the frequency seems not to be significantly affected by variations on the confinement parameter.

The vorticity perturbation can be regarded as two vorticity edge waves $q_1 = Q_1(k, t)e^{i\epsilon_1(k, t)}$, $q_2 = Q_2(k, t)e^{i\epsilon_2(k, t)}$, where Q_1, Q_2 are the amplitudes and ϵ_1, ϵ_2 are the phases. Replacing these relationships into Eq. (10) the evolution of counterpropagating Rossby waves is obtained:

$$\dot{Q}_1 = \gamma Q_2 \sin \epsilon, \quad \dot{Q}_2 = \gamma Q_1 \sin \epsilon, \quad (15)$$

$$\dot{\epsilon}_1 = -kc_1 - \gamma \frac{Q_2}{Q_1} \cos \epsilon, \quad \text{and} \quad \dot{\epsilon}_2 = -kc_2 + \gamma \frac{Q_1}{Q_2} \cos \epsilon. \quad (16)$$

The interaction between the CRWs is governed by coefficient γ , while $\epsilon = \epsilon_2 - \epsilon_1$ is the CRW phase difference. If $\gamma = 0$, the waves would propagate with their intrinsic phase speeds c_1 and c_2 given by Eqs. (12) and (13). In the locked configuration, the CRW growth rates (15), have to be equal, $\dot{Q}_1 = \dot{Q}_2$, thus, if $\sin \epsilon \neq 0$, their amplitudes Q_1 and Q_2 should also be equal. Concerning the phase difference ϵ , the locked configuration occurs when $\dot{\epsilon}_1 = \dot{\epsilon}_2$, which yields from Eq. (16)

$$\cos \epsilon = \frac{k(c_2 - c_1)}{2\gamma} \quad \text{and} \quad \text{therefore} \quad |\sin \epsilon| = \sqrt{1 - \frac{k^2(c_2 - c_1)^2}{4\gamma^2}}. \quad (17)$$

Considering Eq. (15), one could imagine that the maximum of the growth rate is reached for $|\sin(\epsilon)| = 1$, i.e., $\epsilon = \pi/2$, however this is not the case as explained by Heifetz and Methven.¹¹ They have shown that, although the maximum instantaneous growth rate is obtained for $\epsilon = \pi/2$, phase locking can only be obtained with a higher phase-shift between the two waves.

In order to confirm this fact, we plot in Figure 4(a) the isocontours of growth rate (black lines) and the comparison between the locus of the maximum growth rate for each confinement and the isocontour $\sin(\epsilon) = 1$, i.e., $\epsilon = \pi/2$, depicted by gray continuous and dashed lines, respectively, for $\delta_w = 0.535$ and $\Lambda = -0.75$. It is clear that the two lines do not coincide, however, they are close for strong confinements. Furthermore, the behaviors of the maximum of the growth rate (gray continuous line) and the corresponding phase-shift ϵ (black dashed line) are illustrated in Figure 4(b) with respect to the confinement h for the same set of parameters. One can note that the maximum growth rate $\max(\dot{Q}_{1,2}) = 0.592$ is attained for weak confinements $h > 2$ which corresponds to an optimal phase-shift $\epsilon \approx 0.67\pi$, which is remarkably close to the value obtained by Heifetz *et al.*¹⁰ for free shear flows. For stronger confinements, the phase-shift decreases and moves away from optimal. Therefore, the maximum growth rate also decreases for stronger confinements. Note that

FIG. 4. (a) Contours of growth rate in the plane (k, h) for $\delta_w = 0.535$ and $\Lambda = -0.75$. The gray bold line depicts the locus of the maximum growth rate for each confinement, while the gray dashed one depicts the locus of (k, h) -points where $\sin(\epsilon) = 1$, i.e., $\epsilon = \pi/2$. The isocontours are equispaced by 0.025 and their maximal and minimal values are indicated. (b) The maximum growth rate (gray continuous line) and the corresponding phase-shift ϵ (dashed black line) versus the confinement h . For $h \approx 0.45$ only is the phase-shift $\epsilon = \pi/2$.

for $h \approx 0.45$, the maximum growth rate is reached for a phase-shift equal to $\epsilon = \pi/2$, yielding however suboptimal growth. In summary, the presence of confinement inhibits the two CRWs to phase lock with an optimal phase-shift which maximizes the self-propagation of both waves. In this manner, the temporal stabilizing influence of the confinement can be interpreted owing to the CRW perspective.

The CRW approach is based on the destabilizing cooperation and interaction of two waves, that can however also be individually analyzed at first by means of Eq. (10). Neglecting the interaction term γ , the two CRWs can be analyzed freely to evolve, according to

$$M_0 = \begin{bmatrix} -ikc_1 & 0 \\ 0 & -ikc_2 \end{bmatrix}. \quad (18)$$

The eigenvalues $\lambda_{1,2}^0 = \{-ikc_1, -ikc_2\}$ are purely imaginary, and $c_i = \lambda_i^0/k$ represents the phase speed of the i th-CRW. Figure 5(a) displays the frequencies λ_i^0 of the two counterpropagating Rossby waves for the same co-flow parameter, shear layer thickness as Figure 2 and three different values of the confinement parameter. It is seen that an increasing confinement reduces the dispersive nature of the long waves (low wavenumber regime), while it has less influence on the short waves (high wavenumber regime).

The uncoupled phase velocities are presented in Figure 5(b) for the same parameters. The phase velocity of the upper edge wave in $1 + \delta_w/2$, referred to as the CRW2, is seen to monotonously increase towards the value of the velocity of the upper layer $1 - \Lambda$, while the lower edge wave, called CRW1, located at $1 - \delta_w/2$, is seen to monotonously decrease towards $1 + \Lambda$. This figure readily makes clear that each edge wave is, respectively, accelerated or decelerated in comparison to its associated vorticity sheet. The counterpropagating nature of the edge waves should however be evaluated by comparison to the mean velocity, which equals 1 in nondimensional terms. It is seen that in absence of confinement, both edge waves are strictly speaking counterpropagating for some finite low wavenumber interval. The influence of confinement is to reduce this wavenumber range to a point that CRW1 loses its counterpropagating property when the confinement becomes strong enough. As the counterpropagating nature of one CRW is lost, the physical mechanism enabling mutual growth is damped. For a symmetrically confined mixing layer, the damping effect of the confinement on counterpropagation was noticed by Heifetz *et al.*,²⁵ who have interpreted confinement-induced damping as a reduction of the counterpropagating character of CRWs in link with the pseudoenergy conservation.

FIG. 5. Frequencies (a) and phase velocities (b) of uncoupled CRWs in the system represented by Eq. (18) for several confinements, $\delta_w = 0.535$ and $\Lambda = -0.75$. The continuous line depicts $h = 9$, the dashed one $h = 1$, and the dash-dotted one $h = 0.5$. Black bold lines represent the first CRW, while gray thin lines the second one.

IV. IMPLICATIONS FOR THE SPATIO-TEMPORAL STABILITY

In Sec. III we have discussed how the temporal stability of the confined piecewise broken line profile could be interpreted as the interaction of two counterpropagating Rossby waves (CRWs). The objective of the present section is to establish a connection between the CRWs perspective and the spatio-temporal analysis, particularly with the reinforcement of the absolutely unstable character of the flow at a medium confinement for the sinuous mode, noticed by Juniper⁴ for the top-hat wake profile and confirmed when a finite shear layer thickness is included.^{6,22}

A. Explaining the effect of the confinement by means of CRWs

The CRW approach is based on the destabilizing cooperation and interaction of two edge waves, that can however also be analyzed in isolation by means of Eq. (18). Of particular interest for spatio-temporal propagation are the group velocities of the two uncoupled CRWs $v_{1,2}^g = d(kc_i)/dk$, which are depicted in Figure 6(a). The counterpropagating nature of the waves is readily visible, as well as their asymptotic trend at $k \rightarrow \infty$ towards the velocity of both interfaces $1 - \Lambda$ and $1 + \Lambda$. The dispersive character of the waves is particularly pronounced at weak confinement (here $h = 9$), where the evolution of the group velocity towards the interface velocities is not monotonous, starting with a large group velocities in absolute value at low wavenumbers ($CRW1 > 0$, $CRW2 < 0$), before going through an extremum. At medium to strong confinement, the evolution is monotonous. The influence of confinement smoothes the variations and levels out the group velocities towards their asymptotic value. In other words, the influence of confinement largely depends on the wavenumber regime and is much stronger at small wavenumbers, hindering counterpropagation at strong confinement.

The behavior of the average group velocity of the CRWs $\bar{v}_{CRW} = (v_1^g + v_2^g)/2$ is illustrated in Figure 6(b) with respect to the wavenumber for different confinement configurations, $h = 9$, $h = 1$, and $h = 0.5$ for $\delta_w = 0.535$ and $\Lambda = -0.75$. Note that this mean group velocity both carries a physical sense and appears naturally in the imaginary part of the coupled eigenvalues in Eq. (14) in the unstable range of wavenumbers. Each curve shows a minimum, both value and position of which vary as function of the confinement. Note that for a symmetrically confined mixing layer²⁵ the average group velocity does not depend on the degree of the confinement, but is simply equal to the mean flow in between the interfaces. This contrasts with the present study where the wall position is independent from the wake thickness. The minimum $\min(\bar{v}_{CRW})$ can tentatively be associated to the slowest velocity of a virtually compound wavepacket. Note that there exists a minimum minimum

FIG. 6. Group velocities (a) of uncoupled CRWs in the system represented by Eq. (18) and their average $\bar{v}_{CRW}$ (b), for several confinements, $\delta_w = 0.535$ and $\Lambda = -0.75$. The continuous line depicts $h = 9$, the dashed one $h = 1$, and the dash-dotted one $h = 0.5$.

attained at $h \sim 1$. This shows that an order one confinement is associated with slower mean group CRW velocities. At this point, it is worthwhile noticing that among the three confinements chosen in Figure 6(b), only the $h = 1$ case, which has the lowest mean CRW group velocity, is absolutely unstable (see Biancofiore and Gallaire⁶). This fact can be also discerned in Figure 7 where the spatio-temporal diagram of the log-vorticity perturbation $\log_{10}(q'(x, t))$ in the plane (x, t) for $\Lambda = -0.75$, $\delta_w = 0.535$, and (a) $h = 9$ and (b) $h = 1$. These diagrams are obtained by means of a Fourier inverse transform of the matrix M computed with Matlab using the command IFFT. The analyzed k -domain is $0.01 \leq k \leq 32$ and the values of the wavenumber are equispaced with a k -step

FIG. 7. Spatio-temporal diagram of the log-vorticity perturbation $\log_{10}(q'(x, t))$ in the plane (x, t) for $\Lambda = -0.75$, $\delta_w = 0.535$, and (a) $h = 9$, (b) $h = 1$, and (c) $h = 0.7$. In (a) and (c) the unstable wavepacket is advected away, while in (b) it grows everywhere.

FIG. 8. Contours of the minimal group velocity $\min(\bar{v}_{CRW})$ compared to the C/A transition obtained by Biancofiore and Gallaire⁶ in the plane (Λ^{-1}, h) for different values of the shear layer thickness: (a) $\delta_w = 0.01$, (b) $\delta_w = 0.160$, (c) $\delta_w = 0.321$, and (d) $\delta_w = 0.535$. The black bold line depicts the C/A transition, while the colored lines represent the isocontours of the minimal group velocity.

$\Delta k = 0.05$, while the time step is $\Delta t = 0.1$. It is clear that for $h = 9$ and $h = 0.5$, the flow is convectively unstable, while for $h = 1$ it is absolutely unstable.

The tentative link between minimal group velocity of the uncoupled CRWs and the convective (C)/absolute (A) transition is readily evidenced in Figure 8, which illustrates the comparison between the C/A limit and the isocontours of $\min(\bar{v}_{CRW})$ in the plane (Λ^{-1}, h) for different values of the shear layer thickness (a) $\delta_w = 0.01$, (b) $\delta_w = 0.160$, (c) $\delta_w = 0.321$, and (d) $\delta_w = 0.535$. The contours present a clear minimum at moderate confinement for each value of the shear layer thickness. Note that specially when $\delta_w \rightarrow 0$, the behavior of the contours fits very well the C/A transition at medium confinement.

Therefore, we can conclude that the maximum of the instability at moderate confinement noticed by Juniper,⁴ and confirmed in his more recent analyses^{5,22} and by Biancofiore & Gallaire⁶ can be attributed principally to the influence of the confinement on the group velocities of the uncoupled CRWs. It remains however impossible to further attribute this phenomenon to one edge wave in particular.

B. Discussion

In this section we try to physically justify the results obtained in Sec. IV A. In particular, we ask why the behavior of the CRWs taken alone can explain the destabilizing influence of the confinement.

Considering the dispersion relation (14) resulting from the coupled CRW, of the form $\omega(k) = f(k) \pm i\sqrt{g(k)}$, it is in general impossible (with the noticeable exception of non-dispersive waves where f is proportional to k) to separate the advection of the wavepacket by the term f from the spreading due to the pure exponential growth of the wavepacket. In such a case, the spatio-temporal dynamics can only be understood by an analysis of the dispersion relation in the complex k plane. In the present case, it is however tempting, as a consequence of the physical interpretation associated

FIG. 9. Spatio-temporal diagram of the logarithm of the inverse Fourier transform of the spreading term due to pure growth in the (x, t) plane. Same parameters as in Fig. 7: $\Lambda = -0.75$, $\delta_w = 0.535$, and (a) $h = 9$, (b) $h = 1$, and (c) $h = 0.5$. The difference between $h = 9$ and $h = 1$ is hardly visible.

with the two CRWs, to approximate the velocity of the edge of the wavepacket as the sum of the mean group velocity of the pure waves and a second contribution associated to the spreading due to the pure growth of the wavepacket. In mathematical term, this conjecture suggests to approximate the minimal velocity of the wavepacket as

$$v^- \sim \min(\bar{v}_{CRW}) + v_s, \quad (19)$$

where v_s is the symmetric spreading of the pure growth. This approximation is consistent with the fact that the average frequency of the CRW, whether coupled or uncoupled, does not depend on the interaction coefficient γ .

Recall that for $h \geq 1$ the growth rate curve does almost not depend on h (once rescaled properly), v_s can be assumed to be almost independent of h and the dependance of v^- as a function of h dominated by $\bar{v}_{CRW}$. This fact can be observed in Figure 9, where the spatio-temporal diagram of the Fourier inverse transform of $i\sqrt{g(k)}$, computed with Matlab using the same discretization, is illustrated for different confinement (a) $h = 9$, (b) $h = 1$, and (c) $h = 0.5$ prescribing the other parameters. Such quantity could give an idea of the spreading term v_s . Only for $h = 0.5$ is the spreading slightly slowed down in comparison to the other two cases, whereas the difference in the spatio-temporal behavior between $h = 9$ and $h = 1$ is hardly visible. The fact that the spreading rate decreases for $h = 0.5$ could further contribute to the enhanced convective instability for $h < 1$ in comparison to $h > 1$ (see Figure 8).

V. CONCLUSIONS

In the present paper, we have revisited both temporal and spatio-temporal stability properties of the confined piecewise broken-line wake profile studied by Juniper⁵ and Biancofiore and Gallaire⁶ in the light of the counterpropagating Rossby waves (CRW) perspective,^{8,9} distinguishing sinuous perturbations from varicose ones. We interpret the stabilizing effect of confinement on temporal stability as follows: an increased confinement damps the counterpropagating nature of the Rossby waves, thereby inhibiting an optimal phase lock between the two waves.

Furthermore, we have tried to explain the enhancement of the absolutely unstable nature of the flow noticed at moderate confinement.^{4–6} Prescribing the shear layer thickness and the velocity ratio, we have examined the behavior of the average group velocity of the two CRWs and noticed the presence of a minimum for each value of the confinement parameter. The minimum minimum is attained for $h = 1$ which approximately also corresponds to the most absolute unstable configurations.⁶ The remarkable correspondence between absolute/convective transition and the average group velocity (Figure 8), suggests that the destabilization triggered by the confinement, is mainly due to the dependence on the confinement of the group velocity of the CRWs taken freely to propagate, without interacting one with each other. Moreover, the relative insensitivity to confinement of the spreading of the growing wavepacket caused by the growth rate alone also supports this interpretation. In general, confinement is seen to modify the behavior of both CRWs, precluding to attribute the average retardation to one edge wave in particular. These results confirm the versatility of the CRW perspective for the interpretation of several kinds of instabilities.^{11,13,16}

However the destabilizing influence of the confinement on wakes illustrated in the present paper was confirmed also for round geometry²⁶ and including a surface tension on the model.^{27,28} For the former problem, it will be interesting to attempt to generalize the CRW perspective also for round jets and wakes. For the latter one, coupling a surface tension to CRWs will be useful to better understand the role of this parameter at least in the local approach. In such manner also the counterintuitive destabilizing influence of the surface tension noticed by Tammisola *et al.*²⁹ in their numerical simulations may be explained by means of the CRW perspective, as preliminary attempted by Biancofiore.³⁰

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