Inversion problem for singular integral operators: C*-approach

(C*-algebras/Fredholm theories/Mihlin–Calderon–Zygmund operators/Wiener–Hopf operators/Folland–Stein operators)

ALEXANDER DYNIN[†]

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Communicated by I. M. Singer, July 5, 1978

ABSTRACT The inversion problem is solved for a wide class of singular integral operators, in particular for Wiener–Hopf operators in several variables, Mihlin–Calderon–Zygmund operators on bounded domains, and Folland–Stein operators on compact nondegenerate Cauchy–Riemann manifolds.

Introduction

The *inversion problem* for a bounded linear operator A on a Hilbert space H is twofold: how (i) to recognize that A is invertible and (ii) to construct its inverse.

The problem is regular for Riesz–Schauder operators $1 + T$ with $T \in \mathcal{CL}(H)$, the algebra of compact operators on H . A *regularization* of the inversion problem for arbitrary A means its reduction to some regular cases.

The inversion problem for A as an element of $\mathcal{L}(H)$, the C*-algebra of all bounded linear operators on H , is equivalent to the inversion problem for A as an element of any C*-algebra $\mathfrak{A}$ which contains A and identity. Solvable C*-algebras are chosen. In this framework it is possible to derive a hierarchy of Fredholm properties. I postpone a relevant Index theory for future publications.

Solvable C*-algebras

A separable C*-algebra is defined to be *solvable* if there exists a finite filtration $\mathfrak{A}$

$$\mathfrak{S}_0 = 0 \subset \mathfrak{S}_1 \subset \dots \subset \mathfrak{S}_{N+1} = \mathfrak{A}$$

such that for every j with $0 \leq j \leq N$,

- (i) $\mathfrak{S}_j$ is a closed self-adjoint ideal in $\mathfrak{S}_{j+1}$.
 - (ii) $\mathfrak{S}_{j+1}/\mathfrak{S}_j$ is C*-isomorphic with a C*-algebra $C_0[\Sigma_j, \mathcal{CL}(H_j)]$ of all continuous mappings of a locally compact space Σ_j to the C*-algebra of compact linear bounded operators on a complex Hilbert space H_j .
 - (iii) Let $d_j = \dim H_j$. Then $d_j \geq d_{j+1}$.
- The minimal possible N is called the length of $\mathfrak{A}$ and denoted $\ell(\mathfrak{A})$.

Note that the general C*-theory (cf. ref. 1) implies that all $\mathfrak{S}_j$ are C*-ideals even in $\mathfrak{A}$.

The immediate examples of solvable C*-algebras are commutative C*-algebras (their length is 0), C*-algebras with symbol of Breuer and Cordes (2) (their length is 1), and Gohberg–Krupnik (3) C*-algebras generated by singular integral operators on singular curves (their length can be arbitrary).

Other examples can be provided by tensoring of solvable C*-algebras and by

PROPOSITION. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be solvable C*-subalgebras in a C*-algebra such that the C*-ideals $[\mathfrak{A}_1, \mathfrak{A}_2]$ generated by commutators of elements from $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are solvable C*-algebras. Then the C*-algebra generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is solvable as well.

The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked "advertisement" in accordance with 18 U. S. C. §1734 solely to indicate this fact.

Note that the C*-algebra of a solvable Lie group could not be solvable because a solvable C*-algebra is always a postliminary algebra (1).

The spectrum $\mathfrak{A}^\wedge$ of a solvable C*-algebra $\mathfrak{A}$ has the following properties:

- (iv) The Σ_j are topologically imbedded in $\mathfrak{A}^\wedge$.
- (v) $\mathfrak{A}^\wedge = \cup \Sigma_j$.
- (vi) Closure of Σ_j is $\cup \Sigma_k$ ($k \geq j$).

Symbols and Fredholm hierarchy

Choose *-isomorphisms

$$\rho_j : \mathfrak{S}_{j+1}/\mathfrak{S}_j \rightarrow C_0[\Sigma_j, \mathcal{CL}(H_j)], \quad 0 \leq j \leq N.$$

The general C*-theory (5) defines their unique extensions

$$\tilde{\rho}_j : \mathfrak{A}/\mathfrak{S}_j \rightarrow \text{Map}[\Sigma_j, \mathcal{L}(H_j)].$$

Let $\pi_j : \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{S}_j$ be the mapping of factorization. Then the *j-symbol* σ_j on $\mathfrak{A}$ is defined to be

$$\sigma_j = \tilde{\rho}_j \pi_j : \mathfrak{A} \rightarrow \text{Map}[\Sigma_j, \mathcal{L}(H_j)], \quad 0 \leq j \leq N.$$

PROPOSITION. Ker $\sigma_j = \mathfrak{S}_j$ and for every $A \in \mathfrak{A}$, the evaluation mapping of $\sigma_j(A)$

$$\text{ev}(\sigma_j)(A) : \Sigma_j \times H_j \rightarrow H_j, \text{ev}(\sigma_j)(A)(\xi, u) = \sigma_j(A)(\xi)(u),$$

is continuous.

Remark: In terms of ref. 4, $\sigma_j(A)$ for $A \in \mathfrak{A}$ are continuous families of operators on Σ_j (cf. ref. 5). Note that in all examples below, j -symbols of operators are even continuous with respect to the operator norm in $\mathcal{L}(H_j)$.

Suppose now that $\mathfrak{A}$ contains a unit 1. Then element A of $\mathfrak{A}$ is defined to be *j-Fredholm* if it is invertible modulo $\mathfrak{S}_j$. Therefore, A is *j-Fredholm* if and only if its j -symbol $\sigma_j(A)$ is a continuous family of invertible operators.

The set $\text{Fred}_j(\mathfrak{A})$ of j -Fredholm operators is open in $\mathfrak{A}$ and stable under $\mathfrak{S}_j$ -perturbations. Note that $\text{Fred}_j(\mathfrak{A}) \subset \text{Fred}_k(\mathfrak{A})$ for $j < k$ and $\text{Fred}_0(\mathfrak{A})$ is the group of invertible elements of $\mathfrak{A}$.

An element B is defined to be a *j-parametrix* for A if B is an inverse of A modulo $\mathfrak{S}_j$. We have

$$\sigma_j(A)\sigma_j(B) = \sigma_j(B)\sigma_j(A) = 1 \text{ on } \Sigma_j,$$

$$\sigma_{j-1}(A)\sigma_{j-1}(B) = 1 - T',$$

$$\sigma_{j-1}(B)\sigma_{j-1}(A) = 1 - T'' \text{ on } \Sigma_{j-1},$$

where T' and T'' are compact families of operators on Σ_{j-1} (the latter means local compactness of their evaluation mappings). In terms of ref. 4 (cf. ref. 5) the $\sigma_{j-1}(A)$ is the Fredholm family of operators on Σ_j . In this case we can construct its index in $K_c(\Sigma_{j-1})$ (cf. ref. 6) and define j -index of A as

$$\text{ind}_j(A) = \text{ind}[\sigma_{j-1}(A)].$$

Abbreviation: MCZ-operators, Mihlin–Calderon–Zygmund operators.

[†] Present address: Department of Mathematics, State University of New York, Stony Brook, New York 11794.

Regularization of the inversion problem

I suggest the following algorithm of step-by-step regularization for the inversion problem.

Suppose that A is an invertible element of $\mathfrak{A}$. We have $d_N < \infty$ because of $1 \in \mathfrak{A}$. Therefore, $\sigma_N(A)$ is an invertible continuous family of $(d_N \times d_N)$ -matrices on Σ_N . Let π_N^{-1} be a continuous section of the factorization π_N . Take

$$B_N = \pi_N^{-1} \rho_N^{-1} [\sigma_N(A)^{-1}].$$

Then B_N is an N -parametrix for A . Suppose that we already have a $(j+1)$ -parametrix B_{j+1} for A . Then

$$\text{Ker } \sigma_j(B_{j+1}) = \text{Ker}[\sigma_j(A)\sigma_j(B_{j+1})] = \text{Ker}(1 - T'_j),$$

$$\text{Ker } \sigma_j(B_{j+1})^* = \text{Ker}[\sigma_j(A)^*\sigma_j(B_{j+1})^*] = \text{Ker}(1 - T''_j),$$

where T'_j and T''_j belong to $C_0[\Sigma_j, \mathcal{O}(\mathcal{L}(H_j))]$. Therefore, by solution of the Riesz-Schauder equations we can construct two continuous finite rank projector families P_j and Q_j from $C_0[\Sigma_j, \mathcal{L}_s(H_j)]$ such that $\sigma_j(B_{j+1})$ defines a $1-1$ mapping from $\text{Ker } P_j$ onto $\text{Ker } Q_j$.

Moreover,

$$\text{ind}_j(B_{j+1}) + \text{ind}_j(A) = \text{ind}(1 - T'_j);$$

hence, $\text{ind}_j(B_{j+1}) = 0$. Therefore, we can construct a vector bundle isomorphism $L_j : \text{Im } P_j \rightarrow \text{Im } Q_j$ and define a continuous finite rank operator family

$$F_j = [L_j - \sigma_j(B_{j+1})]P_j.$$

Now the $\sigma_j(B_{j+1}) + F_j$ is a continuous invertible operator family on Σ_j and

$$T_j = \sigma_j(A)[\sigma_j(B_{j+1}) + F_j] - 1 \in C_0[\Sigma_j, \mathcal{O}(\mathcal{L}(H_j))].$$

We see that $1 - T_j$ is an invertible Riesz-Schauder operator family from $C_0[\Sigma_j, \mathcal{L}(H)]$.

Choose a continuous section π_j^{-1} of the canonic mapping $\pi_j : \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{I}_j$. Then the element of $\mathfrak{A}$

$$B_j = B_{j+1}\pi_j^{-1}[1 - \rho_j(T_j)]^{-1}$$

is a j -parametrix of A .

Take $j = N-1, N-2, \dots, 1$. Then $B_0 = A^{-1}$.

Actually we have the following

PROPOSITION. *The regularization above is possible if and only if A is an invertible operator.*

Wiener-Hopf operators in several variables (cf. ref. 7)

Consider Wiener-Hopf operators on $H = L^2(\mathbb{R}_+^n)$, where $\mathbb{R}_+^n = (\mathbb{R}_+)^n$,

$$Au(x) = au(x) + \int_{\mathbb{R}_+^n} b(x-y)u(y)dy, u \in H,$$

where $a = \text{constant}$ and $b(x) \in L^1(\mathbb{R}^n)$. Take the C^* -algebra $\mathfrak{B}^{(n)}$ generated with Wiener-Hopf operators in $\mathcal{L}(H)$.

PROPOSITION. *The C^* -algebra $\mathfrak{B}^{(n)}$ is solvable with length n and $d_j = \infty$ for $j < n$, $d_n = 1$.*

For $j < n$, Σ_j can be realized as the disjoint union of $\binom{n}{j}$ coordinate subspaces $\mathbb{R}^J$ of $\mathbb{R}^n$, $J \subset \{1, \dots, n\}$, $\text{card}(J) = j$. Then $\sigma_j(A)$ is the $\mathfrak{B}^{(n-j)}$ -valued function. Indeed, if $\xi' \in \mathbb{R}^J$, $x'' \in \mathbb{R}_+^{CJ}$, then for $v \in L^2(\mathbb{R}_+^{CJ})$

$$\sigma_j(A)(\xi')v(x'') = av(x'') + \int_{\mathbb{R}_+^{CJ}} \tilde{b}_j(\xi', y'' - x'')v(y'')dy''$$

where

$$\tilde{b}_j(\xi', x'') = \frac{1}{(2\pi)^{n-j}} \int b(x', x'')e^{-ix'\xi'}dx'.$$

Σ_n can be realized as $\mathbb{R}^n \cup \infty$ and

$$\sigma_n(A)(\xi) = a + \frac{1}{(2\pi)^n} \int b(x)e^{-ix\xi}dx, \xi \in \mathbb{R}^n.$$

Mihlin-Calderon-Zygmund operators (MCZ-operators) on manifolds with boundaries (cf. ref. 8)

Let X be a Riemannian compact C^∞ -manifold with boundary bX . Take $H = L^2(X)$. Consider the C^* -algebra $\mathfrak{M}_X$ generated in $\mathcal{L}(H)$ by singular integral MCZ-operators. The Mihlin symbol of a MCZ-operator A we denote $\sigma(A)$. This is a C^∞ -function on the manifold with boundary $S(X)$ of all unit tangent vectors of X .

PROPOSITION. *The C^* -algebra $\mathfrak{M}_X$ is solvable of length 2 with $d_0 = d_1 = \infty = d_2 = 1$.*

We have

$$\Sigma_0 = \text{point}, \quad H_0 = H, \quad \sigma_0 = \text{Id}.$$

$$\Sigma_1 = S(bX), \quad H_1 = L^2(\mathbb{R}^+).$$

if A is a MCZ-operator, then for $\tau \in S(bX)$ the $\sigma_1(A)(\tau)$ is the Wiener-Hopf operator with symbol $\sigma(A)(\tau + s\nu)$, where ν is the inward unit normal and $s \in \mathbb{R}$.

$\Sigma_2 = S(X) \cup \tilde{N}(bX)$, where $\tilde{N}(bX) = \{s\nu : |s| \leq 1\}$ is the manifold of normal vectors to boundary with length ≤ 1 .

$H_2 = \mathbb{C}$ and for a MCZ-operator A we have

$$\begin{aligned} \sigma_2(A)|S(X) &= \sigma(A), \quad \sigma_2(A)(s\nu) \\ &= \frac{1}{2}(s+1)\sigma(\nu) + \frac{1}{2}(s-1)\sigma(-\nu). \end{aligned}$$

Singular integral operators on contact manifolds

In ref. 9 I have defined a class of pseudo-differential operators on contact manifolds. It embraces the Folland-Stein singular integral operators on nondegenerate Cauchy-Riemann manifolds (10).

Let M be a compact contact oriented smooth manifold of dimension $2n+1$, $n > 0$. Let $C(M)$ be the $2n$ -bundle of contact tangent vectors on M , and let $E(M)$ be the dual bundle of $T(M)/C(M)$. We denote by $E_e(M)$ the bundle of unit vectors in this bundle (with respect to some Riemannian metric on M). Let $H = L^2(M)$.

Consider the C^* -algebra $\mathfrak{E}_M$ generated in $\mathcal{L}(H)$ by the zero order pseudo-differential operators of ref. 9. The (σ) -symbol defined for them in ref. 11 can be extended on all $\mathfrak{E}_M$; for every $A \in \mathfrak{E}_M$ the $\sigma(A)$ is a continuous $E_e(M)$ -family of singular integral operators of the type discussed in ref. 12 with phase-spaces $C_m(M)$, $m \in M$.

PROPOSITION. *The $\mathfrak{E}_M$ is a solvable C^* -algebra of length 2 with $d_0 = d_1 = \infty$, $d_2 = 1$.*

Moreover, $\Sigma_0 = \text{point}$, $H_0 = H$, $\sigma_0 = \text{Id}$.

$$\Sigma_1 = E_e(M), \quad H_1 = L^2(\mathbb{R}^n), \quad \sigma_1 = \sigma.$$

$\Sigma_2 = S(M)$, $H_2 = \mathbb{C}$, σ_2 is the restriction of the principal symbol of A (9) on $S(M)$, the manifold of unit cotangent vectors of M .

This research was supported by National Science Foundation Grant MCS77-18723.

1. Dixmier, J. (1964) *Les C^* -algèbres et leurs Représentations* (Gauthier-Villars, Paris).
2. Breuer, M. & Cordes, H. O. (1974) *J. Math. Mech.* **13**, 313–324.
3. Gohberg, I. & Krupnik, N. Ja. (1972) in *Hilbert Space Operator Algebras*, Colloquia on Mathematics, ed. Sz-Nagy, B. (North-Holland, Amsterdam), Vol. 5, pp. 239–264.
4. Dolganova, S. (1972) *Uspekhi* **27**, 211–212 (in Russian).
5. Hörmander, L. (1970) *Matematika* **14**, 78–97 (in Russian).
6. Segal, G. (1970) *Q. J. Math.*, Ser. 2, **21**, 385–402.
7. Douglas, R. G. & Howe, R. (1971) *Trans. Am. Math. Soc.* **158**, 203–217.
8. Boutet de Monvel, L. (1970) *Acta Math.* **126**, 11–51.
9. Dynin, A. (1975) *Sov. Math. (Engl. Transl.)* **16**, 1608–1612.
10. Folland, G. B. & Stein, E. M. (1974) *Commun. Pure Appl. Math.* **27**, 429–522.
11. Dynin, A. (1976) *Sov. Math. (Engl. Transl.)* **17**, 508–512.
12. Grossman, A., Toupias, G. & Stein, E. M. (1969) *Annu. Inst. Fourier* **18**, 343–368.