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Null-Space Function Estimation for the Interior Problem

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Abstract

In single photon emission computed tomography (SPECT), projection data can be truncated when the collimator's field-of-view (FOV) is smaller than the object to be imaged. Using truncated projections to reconstruct a region-of-interest (ROI) is a reality we must face if small detectors are used. The truncated data result in an underdetermined system of imaging equations, which may lead to non-unique solutions. Data sampling and photon attenuation may also affect the solution uniqueness and stability. The uniqueness of the system solutions in the ROI can be investigated by studying the null-space functions in the ROI. This paper uses an iterative algorithm to estimate the null-space image, to determine the sampling conditions under which a stable ROI reconstruction is possible with truncated data, and to investigate whether the attenuation can influence the ROI reconstruction bias. This iterative algorithm is validated by the singular value decomposition (SVD) method. We show that if the ROI is sufficiently sampled, the null-space image is close to zero inside the ROI, and this small almost-zero offset is insignificant in SPECT, because the noise is a much more dominating degradation factor.

Index Terms

Interior problem; data truncation; iterative reconstruction; medical imaging

1. INTRODUCTION

SPECT (Single Photon Emission Computed Tomography) is a popular, effective, functional imaging technique with relatively low cost to the patient. It is one of the most important imaging modalities for displaying blood flow, perfusion, and metabolic activity of the heart and other vital organs (Kastis et al 1998, MacDonald et al 2001, Shaw et al 2005, Patton et al 2007). It unequivocally adds value in enhancing patient care and is effective in guiding patient management decisions. New dedicated SPECT systems (Herzog et al 2010, Patton et al 2006, Zaman 2010) can provide high sensitivity images of the target organ with convergent or small field-of-view (FOV) cameras. Imaging time thus can be shortened. The motivation of writing this paper is the attempt to develop tools for designing stationary cardiac SPECT systems that only have a limited number of views and each view only has a small detector which measures truncated projection data. We investigate the conditions under which the designed ROI imaging system can provide a unique cardiac image.

Small field-of-view (FOV) cameras can truncate the projection data of the background. Mathematicians refer to ROI reconstruction using truncated projections as the interior problem, which is not mathematically solvable (Natterer 1986). This mathematical conclusion is, in general, correct; however, many researchers are seeking solutions to the interior problem if some additional information or constraints about the object can be enforced in the ROI image reconstruction (Kudo *et al* 2008).

The singular value decomposition (SVD) method has been used to study the 2D interior problem of the Radon transform (Maass 1992). Maass found that the FOV reconstruction with truncated data is almost correct except for a smooth unknown bias function which is approximately constant with very low frequency components. Maass derived closed-form SVD expressions for the null-space functions, while others use numerical evaluation of the SVD to study the performance of medical imaging systems (Gullberg and Zeng 1990, Gullberg *et al* 1996, Zeng and Gullberg 1997, Jorgensen and Zeng 2008, Kyprianou *et al* 2008, Clarkson *et al* 2010).

Kudo *et al* (2008) pointed out that if the image value is known in a very small sub-region of the ROI, then the interior problem is solvable via a POCS (projection onto convex sets) algorithm. The derivation of Kudo's algorithm is based on the concept of differentiated backprojection (Noo *et al* 2004, Pan *et al* 2005, Defrise *et al* 2006) and finite Hilbert transform inversion (Sidky and Pan 2005, Tricomi 1957).

It has been proposed that if the ROI image is piecewise constant, the total variation minimization technique can converge to the true solution of the interior problem (Yu *et al* 2009, Yu and Wang 2009). The total variation minimization technique has recently drawn a lot of attention, especially in the area of image reconstruction with limited data (which is sometimes referred to as compressed sensing) (Sidky *et al* 2006, Sidky and Pan 2008, Chen *et al* 2008). When projection data are insufficient, strong assumptions about the ROI image can significantly improve the solvability of the image reconstruction problem.

Following the footsteps of the work mentioned above, our group also published papers on exact ROI SPECT reconstruction using truncated projections (Zeng and Gullberg 2009a, Zeng and Gullberg 2009b, Zeng *et al* 2010). To date the majority of the research work on interior tomography problems is for Radon transforms without attenuation. Little has been done on interior problems with attenuated Radon transforms with a non-uniform attenuator. It is the goal of this paper to continue our investigation of the SPECT interior problem considering sampling, small known-region, uniform attenuation, and non-uniform attenuation effects.

In this paper, we will use the concept of null-space to investigate the stability of an ROI solution. The null-space investigation of a medical imaging system was first suggested by Wilson and Barrett (1998), who proposed an iterative method to estimate the null-space component of an image. Maass performed a singular value decomposition (SVD) study of the projector and found that even though the truncated operator is not invertible, the singular values have a modest (meaning slow) decay (Maass 1992). This means that the generalized inverse can be computed in a fairly stable way. For the 2D Radon transform (i.e., parallel line integrals without attenuation), many mathematicians have studied the properties of the null-space functions when the projections are truncated (Natterer 1986, Maass 1992, Louis and Rieder 1989, Leahy *et al* 1979). All of these studies were based on sufficient sampling of the ROI, while the focus of this paper is on the effect of the sampling.

In SPECT, the projection data are always attenuated by the patient body. We investigate the properties of the ROI of the null space functions for attenuated projectors. Due to attenuation, the behavior of the null space functions may be different. The attenuation effect

as well as the truncation effect is studied via computer simulations under different sampling conditions.

In summary, it has been mathematically proven that if the ROI is fully measured with continuous angular sampling and the measurements are not attenuated, then the ROI reconstruction is almost exact except for a bias. This bias can be removed if a small sub-region within the ROI is known. It is reasonable to investigate further: What if the number of views is finite? What if the data are attenuated? How large is the bias? How much this small sub-region can help in solving an interior tomography problem? The purpose of this paper is to answer these questions numerically, and this paper is focused only on these specific questions. Of course, there are many other important questions related to the interior problem, such as inadequate system modeling, noise characteristics and the effects of the number of iterations when an iterative algorithm is used for image reconstruction, which are not in the scope of this paper.

2. Methods

Incomplete data image reconstruction can be classified into three categories (Natterer 1986):

- a. Limited angle (or limited view) problems, where angular sampling is not sufficient.
- b. Exterior problems, where some projection data are missing at the center of the detector.
- c. Interior problems, where only data close to the center of rotation are fully measured.

In this paper, we only consider the interior imaging problem. When studying the interior problem, people normally assume that the ROI sampling is sufficient. The extreme case is that every ray passing through the ROI is measured. We will investigate whether a unique ROI solution can be obtained with truncated attenuated projections under different sampling situations.

A. Iterative algorithm and SVD algorithm

We can describe the imaging problem with a system of linear equations:

$$AX=P \quad (1)$$

where P is the projection data stored in a column array, X is an image whose pixels are re-arranged in a column array, and A is the projection matrix. An ROI is a subset of pixels in X . The ROI is normally a spherical region with its center at the axis of rotation of the detector, and it is the common field-of-view (FOV) of the imaging system.

If X_{Null} satisfies

$$AX_{Null} = \vec{0} \quad (2)$$

then X_{Null} is a vector in the null-space, $N(A)$, of the projector A . It is worth to point out that the matrix A and the matrix $A^T A$ have the same null space (Laub 2005). If X is a solution of (1) and X_{Null} is any vector (i.e., an image) in $N(A)$, then $(X + X_{Null})$ is also a solution of (1), that is,

$$A(X + X_{Null}) = P. \quad (3)$$

In this case, the solution of (1) is not unique if $N(A)$ contains vectors other than the zero vector. The solutions of the truncated imaging problem (1) have a general form of $(X + X_{Null})$, with $AX_{Null} = \vec{0}$. Previous work suggests that X_{Null} should be almost a constant within the ROI and may have some singularities at the boundary of the ROI if the ROI is continuously sampled (Maass 1992).

As indicated in (2), the null-space functions do not depend on the projections P nor on the noise that corrupts P , they only depend on the projector A , that is, only depend on the imaging geometry and the attenuator. The normal approach to study the null-space functions is to evaluate the SVD of the projector A . The challenges are that the closed-form SVD is not available yet for the attenuated Radon transform and that a numerical solution for the SVD is not practical for a realistic 3D medical imaging problem with today's computer technology.

Our strategy is to numerically solve the system of homogeneous equations (2) using a modified Landweber iterative algorithm:

$$X^{n+1} = X^n + sA^T(0 - AX^n) = (I - sA^TA)X^n \quad (4)$$

where the parameter s in (4) is the step size of the iterative algorithm. To guarantee convergence, the step size s should be chosen in the range of $0 < s < 2/\sigma_1$, where σ_1 is the largest singular value of A^TA . If the step size is too large, the algorithm will diverge. If the step size is too small, it will take too long for the algorithm to converge. In our computer simulations, the step size s is empirically determined. We first set the step size s as 0.1. If this step size makes the algorithm diverge, s is reduced to 0.01. If it still diverges, s is further reduced to 0.001, and so on until a proper s value is achieved. It usually takes fewer than 10 trials before a satisfactory value is selected. Equation (4) is essentially equation (11) in (Wilson and Barrett 1998).

The null-space functions are obtained by using an initial image X^0 (referred to as an exemplary phantom image), which in the computer simulations in this paper is a heart/torso NCAT phantom (see Fig. 1). If the null-space component X_{Null} associated with X^0 has a zero-valued (or almost zero-valued) sub-region, then the image is uniquely determined in this particular sub-region by the given imaging system, even though the entire image cannot be uniquely reconstructed. The answer to the question of whether the ROI subset of the null space (note: not the entire null space) has a unique solution is determined by the sampling geometry/density and is not affected by the object to be imaged. Therefore evaluating the null image in the ROI of an exemplary heart/torso NCAT image is informative. If the ROI cannot be accurately reconstructed, the null-space image will contain the shape of the test image. If the null-space image is almost zero in the ROI, the image can be uniquely reconstructed within the ROI.

By restricting a region in X_{Null} and forcing it to be zero for the entire iteration procedure is equivalent to knowing the solution X_{Recon} in that region, which is identical to knowing the true image in that region.

The purpose of the iterative algorithm (4) is to estimate the null-space component of the given exemplary image X^0 . An image X^0 can be decomposed into two parts: $X^0 = X_{Recon} + X_{Null}$, where X_{Null} is in the null space of the projector A , i.e., $AX_{Null} = \vec{0}$. This decomposition in general is not unique. Theoretically, a unique decomposition via SVD is possible if we further require that X_{Recon} can only contain the zero vector in the null space.

The SVD method is a powerful tool to investigate this. Using the SVD, one can obtain a generalized pseudo-inverse $A^\dagger$ of A . Let X^0 be a typical image (say, a heart/torso phantom). Then $X_{Recon} = A^\dagger A X^0$ is a reconstructed image, and $X_{Null} = (I - A^\dagger A)X^0$ is the associated null-space image. In this paper the null-space component X_{Null} of the image X^0 is obtained via an iterative algorithm (4). The null-space image X_{Null} provides some important information about the imaging system A . If X_{Null} contains a region consisting of zero-valued (or almost zero-valued) image pixels, then this region can be accurately reconstructed. The system should be designed so that X_{Null} in the ROI contains almost all zero-valued image pixels for a well-designed imaging system.

We must point out that the iterative ML-EM or the OS-EM algorithm cannot be used in investigating the null-space functions, because these two algorithms enforce non-negativity constraints upon the solutions. A null-space function of a projector A must have negative values, otherwise there is only the trivial solution of zero value everywhere. The purpose of the null-space component evaluation is for system design. Once a stable imaging system is designed and built, any image reconstruction algorithm can be used to reconstruct the image. Even though the ML-EM algorithm cannot be used to find the null-space components, using the ML-EM algorithm to reconstruct the ROI image is encouraged, because it enforces the nonnegativity constraint. When projection data are limited, constraints always help to select a desired image. At least the ML-EM algorithm can provide a clean background where the true values are zero. The nonnegativity constraint in the ML-EM algorithm may not be effective to remove the null-space component in regions where the true image values are not zero.

In the next section, some 2D parallel-beam results of null-space functions are presented. The algorithm presented in (4) is used in all simulations.

B. Computer Simulation Setup

The projector A and backprojector A^T used the model of line-length weighted ray-sum of the pixilated image, i.e., a ray-driven projector/backprojector pair was used. The attenuation effect was modeled as the line integral through a pixel with constant intensity and constant attenuator. Collimator blurring and scattering were not modeled.

The iterative algorithm given by (4) was implemented in our computer simulations. The initial image X^0 was a heart/torso NCAT phantom. In order to consider the case as the iteration number approaches infinity, a very large iteration number of 10,000,000 was used. The resultant image X was the null-space component of the test image X^0 . The SVD method was used to validate the iterative algorithm approach. In computer simulations, the image array size was 64×64 . The detector size was 37.5% of that of the image array size, simulating 62.5% data truncation. We defined the unit of length to be the same as the length of the side of the image pixel. The ROI was a circular region of radius = 12 units and its center coincided with the center of rotation of the detector.

Three attenuation cases were implemented. The first case had no attenuation. The second case used a uniform torso as an attenuator; the attenuation coefficient was 0.1 per unit. In the third case, the non-uniform attenuator was the NCAT phantom. The emission phantom, the uniform attenuator, and the non-uniform attenuator are displayed in Fig. 1, respectively. Each image is displayed from its minimum value to its maximum value.

The computer simulation cases are listed in Table 1. It is indicated in Table 1 when the solution image was enforced to have a zero sub-region. A zero-valued ROI null-space image indicates that the ROI solution is unique. In the computer simulations that used a known sub-region within the ROI, the null space image values in that sub-region were forced to be

zero. We kept this sub-region to be zero for all iterations. This forced the null-space image X_{Null} to be zero in that particular small sub-region. For the SVD method, this enforcement was implemented by adding more equations to assign some pixel values to their known values. This sub-region was an off-center 6×6 square, in which all pixels in the null-space estimate were forced to be zero; the location of this sub-region is shown in Figure 1(a). In cases (a) – (f), the number of views was 120 over 180° . There were 48 detector bins and each bin was half the size of the image pixels (Huesman 1977). The detector bin size is denoted as Δ and its unit is the image pixel size. In cases (g) – (j) the sampling situation was changed, where either the number of views was reduced over 180° or the detector bin size was increased. Finally, a non-truncation situation with sparse sampling was considered in case (k). The details of the changes are listed in Table 1.

Some SVD examples are also presented in this paper to check whether the SVD algorithm results are consistent with the iterative algorithm results. In the SVD method, the projector/backprojector operation matrix $A^T A$ was first decomposed into the expansion:

$$A^T A = V^T S V \quad (5)$$

where $S = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_r, 0, \dots, 0\}$ is a diagonal matrix with singular values $\sigma_1 \ \sigma_2 \ \dots \ \sigma_r > 0$ and V was an orthogonal matrix with each row of V being a singular vector of A . There existed an index r such that $\sigma_r > 0$ and $\sigma_{r+1} = 0$. For an exemplary image X^0 , its “reconstructed” image was

$$X_{Recon} = V^T I_r V X^0 \quad (6)$$

with $I_r = \text{diag}\{1, 1, \dots, 1, 0, \dots, 0\}$ and the number of 1's being r , and the null-space was

$$X_{Null} = X^0 - X_{Recon}. \quad (7)$$

In computer simulations, it is difficult to determine the cut-off index r due to numerical errors, because a zero can be represented by a very small value. In our computer simulations, if $\sigma_{r+1} < 5 \times 10^{-5}$, σ_{r+1} was considered to be zero. Some SVD cases considered in this paper are listed in Table 2. The SVD was calculated using MATLAB's built-in function.

If the image solution in the ROI is unique, the null-space image X_{Null} is zero in the ROI. In numeric calculation of the null-space image values X_{Null} in the ROI, the mean value of the ROI and STD of the ROI null-space image were used to characterize whether the ROI null-space image was almost constant and whether this constant was close to zero. A very small STD value implies a uniform ROI null-space image. If the mean value is also very small, the ROI null-space image is almost zero. The ROI null-space image was also evaluated by calculating its maximum ROI value and minimum ROI value. If these two values are close to zero, the ROI is approximately zero.

3. Results

The computer simulation null-space function results are reported in the Tables 2–6 below. All images in the tables are displayed from its minimum value to its maximum value with the linear gray-scale bar shown in Fig. 2.

Table 2 shows the results of the iterative algorithm using dense sampling (120 views over 180° and the detector bin size being half of the image pixel size, i.e., $\Delta=0.5$), and compares the effects of uniform attenuation as well as non-uniform attenuation. It also shows whether

it helps to make the null-space component inside the ROI to be zero by knowing the true image values in a small sub-region of the ROI. Table 2' has the same layout as Table 2, showing the mean value, standard deviation value, minimum value and maximum value of the ROI for each null space image in Table 2. The results in Table 2 and Table 2' indicate that as long as the ROI sampling is sufficient, the amplitudes of the null-space images are small and the errors in the ROI of the reconstructed images are small, regardless of whether a small known-region is provided and whether the attenuator is uniform or non-uniform. In all tables in this section, the "Recon Image" indicates the image that was calculated as the true image minus the null-space image.

Table 3 shows the results of the iterative algorithm using a non-uniform attenuator and knowing the true image values in a small sub-region within the ROI. Table 3 compares the effects of different sampling densities (120, 60, 30 and 15 views over 180° and $\Delta=0.5$; 120 views over 180° and $\Delta=0.5$). Table 3' has the same layout as Table 3, showing the mean value, standard deviation value, minimum value and maximum value of the ROI for each null space image in Table 3. The results in Table 3 and Table 3' indicate that the ROI sampling is most important. When sampling is not sufficient, the image cannot be correctly reconstructed because a likeness of the emission object appears in the null-space image [see, for example, case (j)]. When the ROI of the null-space image is significantly greater than zero, the boundary of the small square known-region [see cases (i) and (j)] can be visualized in the null-space images. The ROI of the null-space images is supposed to be a constant zero if the ROI image can be correctly reconstructed. The case (k) in Table 3 shows that if the projection data are not truncated, the sampling does not need to be as dense as in the case of using truncated data.

Table 4 compares the iterative algorithm results with the results of the SVD method for all cases (a-k). Table 4' shows the mean value, standard deviation value, minimum value and maximum value of the ROI for each null space image obtained by the SVD method in Table 4.

Cases (a) - (f) result in almost zero null-space functions within the ROI. Cases (g) - (j) are not sampled as finely, and the null-space image within the ROI is significantly greater than zero. Using the iterative or the SVD method, the data sampling density plays a dominating role when projection data are severely truncated. When number of angles is sufficiently large over 180° and the detector bin size is half of the image pixel size, the null-space image in the ROI is very close to zero. Otherwise, the resultant image is not unique even when a sub-region is known. A large number of iterations were used for each example, and a typical run time took more than one week to finish (using an AMD 270 core processor with the Linux operating system). The reconstructed image was calculated as the true image minus the null-space image.

These computer simulations show that the by knowing the true image value in a small region does not help the ROI reconstruction very much. The uniform attenuation and non-uniform attenuation also do not make any noticeable differences in ROI reconstruction.

4. DISCUSSION

One lesson we learn from these studies is that when projections are truncated, we must have sufficient angular sampling and the detector bin size should be smaller than the image pixel size. Using fine angular sampling and detector sampling was suggested by Huesman (1977) a long time ago; it turns out to be critically important when the data are truncated. Enforcing a small sub region in the null-space function to be zero does not seem to be effective to drive the entire null-space function within the ROI to zero. The knowledge of the true image value

in a small sub-region in the ROI may have some influence on the reconstruction of its neighborhood, but it may not be able to transform an unsolvable problem into a solvable problem.

The iterative approach to calculate the null-space component is equivalent to finding the difference between the true image and the reconstructed image. However, this reconstruction would require a set of exact projection data. In null-space component decomposition, only the projection matrix and the true image are required. The exact projection data can be generated by applying the projection matrix on the true image without adding noise and errors of any sort.

The iterative algorithm used in this paper is to find the null-space component of a particular image. In principle, the results depend on the image under study. Notice that the results are obtained by a linear iterative algorithm or the linear SVD algorithm, if the null-space component is almost zero, it will be almost image independent; if the null-space component is relatively large, the null-space component will propagate into the reconstruction and the artifacts in the reconstruction is image dependent.

The noise amplification in the reconstructed images can be determined by the singular value spectrum $\{\sigma_1, \sigma_2, \dots\}$, especially by the condition number which is the ratio of largest singular value over the smallest singular value. The condition number gives the noise sensitivity of the entire reconstructed image, not restricted to the ROI. In addition, this spectrum is difficult to estimate via an iterative algorithm. A practical method to evaluate the noise sensitivity will be left for future investigation.

In simulations, a large number (say, 120) of views over 180° together with a detector bin size which is half of the image pixel size is used to illustrate that if sufficient sampling is achieved the ROI reconstruction is unique and the attenuation does not affect the result. If the sampling is not enough, when fewer than 60 views are used over 180° , the solution is not unique. Sampling is the key issue and the attenuation in the projection has an insignificant effect. If the projection data are not truncated, as shown in case (k), where 60 views over 180° are used and the detector size is the same as the image pixel size, the null-space image is almost zero. The condition number for this non-truncation sparse-sampling case is $\sigma_1/\sigma_{4096} = 1.7082 \times 10^9$. Thus the reconstruction is reasonably stable. One can see the trade-off or a sort of “data sampling conservation law”: in order to obtain a stable reconstruction, one either uses sparse sampling without truncation or uses dense sampling with truncation.

We need to point out that the condition number calculated via SVD can only indicate whether the reconstruction of the *entire* image is stable, and provides no information about whether an ROI can be reconstructed. The null-space image method provides a tool to investigate whether an ROI can be reconstructed uniquely.

We observe that the iterative algorithm and the SVD algorithm do not give the same null-space image result. In computer simulations, none of the singular values are exactly zero. In the SVD method, we truncate the spectrum when the singular value is 1×10^{-6} times smaller than the largest singular value, and the singular values and vectors are discarded thereafter during SVD reconstruction. Therefore only an approximate null-space image can be obtained. The iterative method, on the other hand, can only approximately give a rough estimate of a null-space image after a fixed number of iterations. However, both methods provide the same useful information of the structure of the ROI in the null-space image and the potential bias of the reconstructed image.

The suggested null-space function estimation technique requires a large number of iterations, which may seem impractical in practice. It must be pointed out that this technique

is not to be used for daily image reconstruction; it is only used in designing imaging systems and determines whether the designed system is able to provide a unique ROI image.

In reality, the discrete projector A that is used in an iterative reconstruction algorithm is only an approximation of the actual projector $(A+E)$, where E is the modeling error and is difficult to estimate. The discrepancy of the continuous-to-discrete model and the discrete-to-discrete model is the model mismatch error E . The actual projection data are acquired with $(A+E)$ with the true object X^0 . An iterative algorithm attempts to solve

$$A^T A X = A^T (A+E) X^0 \quad (8)$$

using a pseudo-inverse $(A^T A)^\dagger$:

$$X_{Recon} = (A^T A)^\dagger [A^T (A+E)] X^0 = (A^T A)^\dagger (A^T A) X^0 + (A^T A)^\dagger (A^T E) X^0. \quad (9)$$

In (9), $(A^T A)^\dagger (A^T E) X^0$ is the model mismatch error, not discussed in the above null-space image analysis. This error depends on the modeling error E , the condition of $(A^T A)$, and the object X^0 . Unfortunately, this error may be larger than the null-space error in magnitude, especially when the data are truncated which leads to an ill-conditioned $(A^T A)$. Therefore, a stable ROI reconstruction with truncated data is possible only when the modeling error is very small.

Truncated data imaging is an underdetermined imaging problem, which gives an extremely ill-conditioned projection matrix A . Increasing the sampling density both linearly and angularly can increase more information about the image and make the projection matrix A less ill-conditioned. In a real imaging system, both collimator blurring and patient scattering significantly reduces information content. If the blurring effect is not modeled in the reconstruction algorithm, blurring will make the model mismatch error E larger as mentioned above. A larger error E results in larger errors in the reconstruction. If the blurring is modeled in the projection matrix A , the projection matrix A will become even more ill-conditioned. In both cases, denser data sampling becomes more important in order to reduce the reconstruction errors.

5. CONCLUSIONS

Wilson and Barrett (1998) suggested to use null-space to characterize a medical imaging system and proposed an iterative algorithm to calculate the null-space of a digitized image. This method has been applied to investigate the uniqueness of the ROI reconstruction problem with truncated data. If the null-space function (image) is always zero in a region, then the reconstructed image in this region is uniquely determined by the imaging configuration. If the null-space function (image) is a constant value in a region, then the reconstructed image in that region can suffer from a constant bias but there are no structural artifacts. If the ROI is sufficiently sampled, the attenuation and small-known region effects are insignificant — much smaller than the noise level in a typical clinical SPECT scan. The bad news is that if the ROI angular and linear sampling is not sufficient, some residue of the emission image will propagate into the null space and the ROI reconstruction is distorted.

It is observed that forcing the null-space function (image) in a very small sub-region of the ROI to be zero does not significantly improve solution uniqueness, especially when the sampling is sparse. Besides, it is difficult in practice to obtain a known sub-region, unless the ROI contains a sub-region that is outside the patient body.

The iterative algorithm that is used to find the null-space component of an image was validated by the SVD method.

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Figure 1. Computer generated NCAT phantom used in 2D computer simulations. The phantom is positioned so that the heart is at the center of rotation. (a) Emission phantom [The broken-line circle indicates the FOV, and the solid-line square indicates the known sub-region]. (b) Uniform attenuation map. (c) Non-uniform attenuation map.

Figure 2.
The linear gray-scale bar that is used to display all images.

Table 1

Iterative and SVD simulation cases (for the iterative algorithm: # of iterations = 10,000,000, $s = 0.0005$ with attenuation, $s=0.00005$ without attenuation)

Case	Attenuator	Enforce a zero sub-region?
(a) Dense sampling (120 views over 180° , $\Delta=0.5$)	None	No
(b) Dense sampling (120 views over 180° , $\Delta=0.5$)	None	Yes
(c) Dense sampling (120 views over 180° , $\Delta=0.5$)	Uniform	No
(d) Dense sampling (120 views over 180° , $\Delta=0.5$)	Uniform	Yes
(e) Dense sampling (120 views over 180° , $\Delta=0.5$)	Non-uniform	No
(f) Dense sampling (120 views over 180° , $\Delta=0.5$)	Non-uniform	Yes
(g) Sparse sampling (120 views over 180° , $\Delta=1$)	Non-uniform	Yes
(h) Sparse sampling (60 views over 180° , $\Delta=0.5$)	Non-uniform	Yes
(i) Sparse sampling (30 views over 180° , $\Delta=0.5$)	Non-uniform	Yes
(j) Sparse sampling (15 views over 180° , $\Delta=0.5$)	Non-uniform	Yes
(k) No truncation, Sparse (60 views over 180° , $\Delta=1$)	None	No

Table 2

Iterative simulation results with dense sampling (120 views over 180° , $\Delta=0.5$). “Recon Image” is calculated as the true image minus the null-space image.

	Do not enforce a zero sub-region		Enforce a zero sub-region	
	Recon Image	Null-space Image	Recon Image	Null-space Image
No Attenuation	(a)	(a)	(b)	(b)
Uniform Attenuation	(c)	(c)	(d)	(d)
Non-uniform Attenuation	(e)	(e)	(f)	(f)

Table 2'

Iterative simulation null-space ROI image results with dense sampling (120 views over 180° , $\Delta=0.5$)

	Do not enforce a zero sub-region		Enforce a zero sub-region	
	ROI mean $\pm$ STD	ROI min & max	ROI mean $\pm$ STD	ROI min & max
No Attenuation	0.0034 $\pm$ 0.00025	0.0029 & 0.0048	0.00066 $\pm$ 0.00035	0.0 & 0.0015
Uniform Attenuation	0.0060 $\pm$ 0.00080	0.0040 & 0.0104	0.00136 $\pm$ 0.00061	0.0 & 0.0037
Non-uniform Attenuation	0.0069 $\pm$ 0.00102	0.0043 & 0.0115	0.00039 $\pm$ 0.00042	-0.00085 & 0.0014

Table 3

Iterative simulation results with non-uniform attenuation and enforcement of a zero sub-region. “Recon Image” is calculated as the true image minus the null-space image.

Sampling	Recon Image	Null-space Image
(f) 120 views over 180°, $\Delta=0.5$
(g) 120 views over 180°, $\Delta=1$
(h) 60 views over 180°, $\Delta=0.5$
(i) 30 views over 180°, $\Delta=0.5$
(j) 15 views over 180°, $\Delta=0.5$
(k) No truncation, Sparse (60 views over 180°, $\Delta=1$)

Table 3'

Iterative simulation null-space ROI image results with non-uniform attenuation and enforcement of a zero sub-region

Sampling	ROI mean $\pm$ STD	ROI min & max
(f) 120 views over 180°, $\Delta=0.5$	0.00039 $\pm$ 0.00042	−0.00085 & 0.0014
(g) 120 views over 180°, $\Delta=1$	0.0304 $\pm$ 0.0276	−0.1016 & 0.1530
(h) 60 views over 180°, $\Delta=0.5$	0.0253 $\pm$ 0.0167	−0.0425 & 0.0812
(i) 30 views over 180°, $\Delta=0.5$	0.0977 $\pm$ 0.0518	−0.0092 & 0.3510
(j) 15 views over 180°, $\Delta=0.5$	0.1679 $\pm$ 0.1651	−0.3090 & 0.5480
(k) No truncation, Sparse (60 views over 180°, $\Delta=1$)	−0.000030 $\pm$ 0.00745	−0.0397 & 0.0399

Table 4

Validating the iterative method with SVD results. (NA – no attenuation, UA – Uniform attenuation, NUA – Non uniform attenuation, No – Do not enforce a zero sub-region, Yes - Enforce a zero sub-region). “Recon Image” is calculated as the true image minus the null-space image.

	Iterative Method		SVD Method	
	Recon Image	Null-space Image	Recon Image	Null-space Image
(a) Dense sampling (120 views over 180°, $\Delta=0.5$) NA, No
(b) Dense sampling (120 views over 180°, $\Delta=0.5$) NA, Yes
(c) Dense sampling (120 views over 180°, $\Delta=0.5$) UA, No
(d) Dense sampling (120 views over 180°, $\Delta=0.5$) UA, Yes
(e) Dense sampling (120 views over 180°, $\Delta=0.5$) NUA, No
(f) Dense sampling (120 views over 180°, $\Delta=0.5$) NUA, Yes
(g) Sparse sampling (120 views over 180°, $\Delta=1$) NUA, Yes
(h) Sparse sampling (60 views over 180°, $\Delta=0.5$) NUA, Yes
(i) Sparse sampling (30 views over 180°, $\Delta=0.5$) NUA, Yes

	Iterative Method		SVD Method	
	Recon Image	Null-space Image	Recon Image	Null-space Image
(j) Sparse sampling (15 views over 180° , $\Delta=0.5$) NUA, Yes
(k) No truncation, Sparse (60 views over 180° , $\Delta=1$) NA, No

Table 4'

SVD method results. (NA – no attenuation, UA – Uniform attenuation, NUA – Non uniform attenuation, No – Do not enforce a zero sub-region, Yes - Enforce a zero sub-region).

	Cut-off index r & Cond. Num.	ROI mean $\pm$ STD	ROI min & max	Comments
(a) Dense sampling (120 views over 180°, $\Delta=0.5$) NA, No	3863 1.4058×10^8	0.0095 ± 0.0014	0.0080 & 0.0187	ROI $\approx$ 0
(b)) Dense sampling (120 views over 180°, $\Delta=0.5$) NA, Yes	3863 2.025×10^8	0.0075 ± 0.00154	0.0052 & 0.0157	ROI $\approx$ 0
(c) Dense sampling (120 views over 180°, $\Delta=0.5$) UA, No	3370 1.2175×10^{10}	0.0220 ± 0.00268	0.0165 & 0.0375	ROI $\approx$ 0
(d) Dense sampling (120 views over 180°, $\Delta=0.5$) UA, Yes	3373 5.0293×10^{10}	0.00657 ± 0.00205	-0.00129 & 0.0158	ROI $\approx$ 0
(e) Dense sampling (120 views over 180°, $\Delta=0.5$) NUA, No	3500 4.0194×10^9	0.0318 ± 0.00485	0.0318 & 0.0495	ROI $\approx$ 0
(f) Dense sampling (120 views over 180°, $\Delta=0.5$) NUA, Yes	3506 2.8852×10^9	0.00374 ± 0.00160	-0.00099 & 0.00730	ROI $\approx$ 0
(g) Sparse sampling (120 views over 180°, $\Delta=1$) NUA, Yes	2803 1.6321×10^{10}	0.0364 ± 0.0293	-0.0939 & 0.1512	Medium errors
(h) Sparse sampling (60 views over 180°, $\Delta=0.5$) NUA, Yes	2727 5.5841×10^{10}	0.0276 ± 0.0190	-0.0363 & 0.1056	Medium errors
(i) Sparse sampling (30 views over 180°, $\Delta=0.5$) NUA, Yes	1429 2.3891×10^{11}	0.0977 ± 0.0520	-0.00876 & 0.3485	Large errors
(j) Sparse sampling (15 views over 180°, $\Delta=0.5$) NUA, Yes	733 1.8277×10^{12}	0.1678 ± 0.1649	-0.3084 & 0.5469	Large errors
(k) No truncation, Sparse (60 views over 180°, $\Delta=1$) NA, No	3760 1.7082×10^9	-0.000032 ± 0.0064	-0.03278 & 0.03303	ROI $\approx$ 0