TOPOLOGICAL GROUP STRUCTURES OF INFINITE SYMMETRIC GROUPS*

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Communicated by S. M. Ulam, July 19, 1967

Let M be an infinite set and let $S(M)$ denote the group of all permutations on M . For each $x \in M$, define E_x to be the subgroup of $S(M)$ consisting of all permutations which leave x fixed. Assume $\mathfrak{J}$ is a topology on $S(M)$ such that $(S(M), \cdot, \mathfrak{J})$ is a topological group. In this paper, it is proved that the following conditions are equivalent: (i) E_x is closed for all $x \in M$; (ii) E_x is open for all $x \in M$; and (iii) $\mathfrak{J}$ is a Hausdorff topology. As a consequence of this equivalence, it is proved that if $\mathfrak{J}$ is a Hausdorff nondiscrete topology on $S(M)$, then $(S(M), \cdot, \mathfrak{J})$ cannot be a locally bounded topological group. This yields the result that there is no nontrivial Weil-Borel measure on $S(M)$.

1. *Notation.*—Throughout this paper, M will denote a fixed infinite set and $S(M)$ will denote the group of all permutations on M . If $N \subseteq M$, define $E_N = \{\sigma \in S(M) : \sigma(n) = n \text{ for all } n \in N\}$; we will take the liberty to write E_x in place of $E_{\{x\}}$ and $E_{x,y}$ in place of $E_{\{x,y\}}$. For each $\sigma \in S(M)$, define $\text{spt } \sigma = \{x \in M : \sigma(x) \neq x\}$. If $A \subset S(M)$ and $x \in M$, define $A(x) = \{\sigma(x) : \sigma \in A\}$. Denote the cardinal number of a set K by $|K|$.

2. *Properties of E_x .*—Throughout this section, we will assume that $\mathfrak{J}$ is a topology on $S(M)$ such that $(S(M), \cdot, \mathfrak{J})$ is a topological group.

LEMMA 1. *Suppose that $A \subseteq S(M)$, $e \in A$, and $|M \setminus A(x)| = |M|$. Then there is $\sigma \in S(M)$ such that $[\sigma A \sigma^{-1} \cap A](x) = \{x\}$.*

Proof: Note that for all $x \in M$, $\sigma \in S(M)$, and $A \subset S(M)$, $[\sigma A \sigma^{-1}](x) = \sigma[A(\sigma^{-1}(x))]$ and $[\sigma A \sigma^{-1} \cap A](x) \subseteq [\sigma A \sigma^{-1}](x) \cap A(x)$.

Since $|M \setminus A(x)| = |M|$, there is $\sigma \in S(M)$ such that $\sigma(x) = x$ and $\sigma(A(x) \setminus \{x\}) \subset M \setminus A(x)$. Thus $x \in [\sigma A \sigma^{-1} \cap A](x) \subseteq [\sigma A \sigma^{-1}](x) \cap A(x) = \{x\}$. Hence $[\sigma A \sigma^{-1} \cap A](x) = \{x\}$.

It is well known that E_x is a maximal subgroup of $S(M)$ for each $x \in M$. Thus the closure of E_x is either E_x or $S(M)$, i.e., E_x is either closed or dense in $S(M)$. Since E_x and E_y are conjugate subgroups of $S(M)$ for x and y in M , it follows that either E_x is closed in $S(M)$ for all $x \in M$, or E_x is dense in $S(M)$ for all $x \in M$. In the next theorem we show that the word "closed" in the preceding statement may be replaced by the word "open."

THEOREM 1. *For each $x \in M$, E_x is closed iff E_x is open.*

Proof: It is well known that an open subgroup of a topological group is also closed, hence if E_x is open in $S(M)$, then E_x is also closed.

Suppose now that E_x is closed. Then $\{\sigma \in S(M) : \sigma(x) \neq x\}$ is open, hence for $y \in M \setminus \{x\}$, $V = \{\sigma \in S(M) : \sigma(x) \neq y\}$ is a neighborhood of e . There is a neighborhood W of e such that $W^{-1}W \subseteq V$. If ρ and σ belong to W , then $\rho^{-1}\sigma(x) \neq y$, i.e., $\sigma(x) \neq \rho(y)$. This implies that $W(x)$ and $W(y)$ are disjoint; thus either $M \setminus W(x)$ or $M \setminus W(y)$ must have the cardinality of M . Since $e \in W$, $x \in W(x)$, and $y \in W(y)$. Hence by Lemma 1, there is $\sigma \in S(M)$ and $u \in M$ (u is either x or y as $M \setminus W(x)$ or $M \setminus W(y)$ has the cardinality of M) such that

$$[\sigma W \sigma^{-1} \cap W](u) = \{u\}.$$

The set $\sigma W \sigma^{-1} \cap W$ is a neighborhood of e and $\sigma W \sigma^{-1} \cap W \subset E_u$, hence E_u is open. As remarked above, E_u and E_x are conjugate subgroups, hence E_x must also be open.

LEMMA 2. *Let n be a positive integer and let F be a finite subset of M with n elements. Then E_F is dense in $S(M)$ iff each neighborhood of e is n -transitive.*

Proof: Recall that $\bar{B} = \bigcap \{AB: A \text{ is a neighborhood of } e\}$ for each $B \subseteq S(M)$. Therefore E_F is dense in $S(M)$ iff $AE_F = S(M)$ for all neighborhoods A of e . If F_1 and F_2 are finite subsets of M with the same cardinality, then E_{F_1} and E_{F_2} are conjugate subgroups of $S(M)$, hence E_{F_1} is dense in $S(M)$ iff E_{F_2} is dense in $S(M)$. With these remarks in mind, the proof of this lemma is quite obvious.

LEMMA 3. *If E_x is dense in $S(M)$, then $E_{x,y}$ is dense in $S(M)$.*

Proof: Let $G = \bar{E}_{x,y}$ and assume $G \neq S(M)$. The subgroup $G' = G \cap E_x$ is the closure of $E_{x,y}$ in the relative topology on E_x . Since $E_{x,y}$ is a maximal subgroup of E_x , we must have $G' = E_x$ or $G' = E_{x,y}$. If $G' = E_x$, then $E_x \subset G$, hence $G = S(M)$ since G is closed and E_x is dense in $S(M)$. Similarly, $G'' = G \cap E_y$ must be $E_{x,y}$ if we assume $G \neq S(M)$.

Suppose now that $G' = G'' = E_{x,y}$. This means that $E_{x,y}$ is closed in E_x , hence by Theorem 1, $E_{x,y}$ is open in E_x . Therefore, by the fact that E_x is dense in $S(M)$, $G = \bar{E}_{x,y}$ is open in $S(M)$. It is clear that $G \neq E_{x,y}$ since $E_{x,y}$ open implies E_x is open contrary to E_x being dense in $S(M)$. Now if $\sigma \in G \setminus E_{x,y}$, then $\sigma \notin E_x$ and $\sigma \notin E_y$, hence $\{x, y\} \subseteq \text{spt } \sigma$.

Consider $\sigma \in G \setminus E_{x,y}$ with $\sigma(x) = u$ and $\sigma(y) = v$. There are three cases to consider.

Case 1: $u = y$ and $v = x$. Then $\sigma = (xy) \rho$ where $\rho \in E_{x,y}$, hence the 2-cycle (xy) belongs to G .

Case 2: $u \neq y$, $v \neq x$. Then $(uw) \in E_{x,y}$ and $\sigma^{-1}(uw)\sigma = (xy) \in G$.

Case 3: $v \neq x$, $u = y$. Let $w = \sigma^{-1}(x)$. If $v \neq w$, then $(vw) \in E_{x,y}$, and $\rho = \sigma \cdot (vw) \in G$ and $\rho = \alpha \cdot (xyv)$ where $\alpha \in E_{x,y}$, hence $(xyv) \in G$.

If $v = w$, then $\sigma = \alpha \cdot (xyv)$ where $\alpha \in E_{x,y}$, hence $(xyv) \in G$. Choose $z \in M \setminus \{x, y, v\}$ and let $\beta = (vz)$. Then $\beta \in E_{x,y}$ and $(xv) \cdot (yz) = (xyv) \cdot (vz) \cdot (xyv) \cdot (vz) \in G$, hence by Case 2, $(xy) \in G$.

Thus, in any case, the 2-cycle (xy) belongs to G . Ball¹ has shown that for all $x, y \in M$ with $x \neq y$, the subgroup generated by $E_{x,y} \cup \{(xy)\}$ is maximal in $S(M)$, hence must be equal to G as we have assumed $G \neq S(M)$. Choose $u \in M \setminus \{x, y\}$ and let $\sigma = (uy)$. Then $\sigma G \sigma^{-1}$ is open. By the choice of σ and the fact that G is the subgroup generated by $E_{x,y} \cup \{(xy)\}$, each $\rho \in G \cap \sigma G \sigma^{-1}$ must have the property that $\rho(x) = x$. Hence $G \cap \sigma G \sigma^{-1} \subset E_x$ and E_x is both open and closed contrary to E_x being dense in $S(M)$. Therefore, we must have $G = S(M)$, i.e., $E_{x,y}$ is dense in $S(M)$.

THEOREM 2. *If $\mathfrak{J}$ is a topology on $S(M)$ such that $(S(M), \cdot, \mathfrak{J})$ is a topological group, then the following are equivalent.*

- (1) $(S(M), \mathfrak{J})$ is a Hausdorff space.
- (2) For each $x \in M$, E_x is closed.
- (3) For each $x \in M$, E_x is open.

Proof: Statements (2) and (3) are equivalent by Theorem 1.

Suppose $x \in M$ and E_x is dense in $S(M)$. Then by Lemmas 2 and 3, each neighborhood of e is 2-transitive. Let a, b , and c be distinct elements of M and A an arbitrary neighborhood of e . There is a neighborhood B of e such that $B \cdot B^{-1} \subset A$. Let $\sigma = (ab)$ and define $D = \sigma B \sigma^{-1} \cap B$; D is also a neighborhood of e and $\sigma D \sigma^{-1} = D$ since $\sigma^2 = e$. Since D is 2-transitive, there is $\rho \in D$ such that $\rho(a) = a$ and $\rho(b) = c$. Then

$$(\sigma cb) = \sigma \rho \sigma^{-1} \rho^{-1} \in (\sigma D \sigma^{-1}) \cdot D^{-1} \subseteq B \cdot B^{-1} \subset A.$$

Therefore A must contain every 3-cycle. Since A was an arbitrary neighborhood of e , it follows that $\mathfrak{J}$ is not Hausdorff since

$$\overline{\{e\}} = \bigcap \{A: A \text{ is a neighborhood of } e\} \neq \{e\}.$$

Thus (1) implies (2).

If $\mathfrak{J}$ is not Hausdorff, then $\overline{\{e\}}$ is a normal subgroup of $S(M)$ which is not $\{e\}$, hence is n -transitive for each positive integer n . Therefore each neighborhood of e must be n -transitive for each positive integer n . Thus by Lemma 2, E_x is dense in $S(M)$ for each $x \in M$. This shows that (2) implies (1).

3. *Locally Bounded Topologies.*—Let $(G, \cdot, \mathfrak{J})$ be a topological group. A subset A of G is *left (right) bounded* iff for each neighborhood B of e , there is a finite set $F \subset G$ such that $A \subseteq F \cdot B$, ($A \subseteq B \cdot F$). It is clear that A is left bounded iff A^{-1} is right bounded. A subset $A \subset G$ is *bounded* iff A is both left and right bounded. A topological group G is *locally bounded* iff there is a bounded neighborhood of e . It is clear that G is locally bounded iff there is a neighborhood of e which is either left or right bounded.

THEOREM 3. *There does not exist a nondiscrete Hausdorff topology $\mathfrak{J}$ on $S(M)$ such that $(S(M), \cdot, \mathfrak{J})$ is a locally bounded topological group.*

Proof: Suppose $\mathfrak{J}$ is a Hausdorff topology on $S(M)$ such that $(S(M), \cdot, \mathfrak{J})$ is a locally bounded topological group. This means that there exists a neighborhood A of e which is bounded. By Theorem 2, E_x is open for each $x \in M$, hence there are $\sigma_1, \dots, \sigma_n \in S(M)$ such that $A \subset \bigcup \{\sigma_i E_x: i = 1, 2, \dots, n\}$. Hence for each $\alpha \in A$, $\alpha(x) \in \{\sigma_1(x), \dots, \sigma_n(x)\}$. Since x was an arbitrary member of M , it follows that $A(x)$ is finite for each $x \in M$.

There is a neighborhood B of e such that $B^{-1} \cdot B \subset A$. If $z \in B(x) \cap B(y)$, then there are σ and ρ in B such that $\sigma(x) = z = \rho(y)$. Then

$$\rho^{-1} \sigma(x) = y, \rho^{-1} \sigma \in B^{-1} \cdot B \subset A,$$

and hence $y \in A(x)$.

Define $\mathfrak{O} = \{K \subset M: \text{for all distinct } x, y \in K, B(x) \cap B(y) = \Phi\}$. The collection $\mathfrak{O}$ is certainly nonvoid and if $\mathfrak{O}$ is ordered by set inclusion, it is clear that every chain in $\mathfrak{O}$ has an upper bound in $\mathfrak{O}$. Thus by Zorn's lemma, $\mathfrak{O}$ has a maximal element, call it K_0 . Let $N = \bigcup \{A(x): x \in K_0\}$. If $y \in M \setminus N$, then $y \notin K_0$, hence by the maximality of K_0 , there is $u \in K_0$ such that $B(u) \cap B(y) \neq \Phi$. This implies that $y \in A(u) \subset N$ contrary to the choice of y . Hence $M = N$, and since $A(x)$ is finite for each $x \in M$, $|M| = |K_0|$.

Choose $K_1 \subset K_0$ and $K_2 \subset K_0$ such that $K_1 \cap K_2 = \Phi$, $K_0 = K_1 \cup K_2$, and $|K_1| = |K_2| = |K_0|$.

Define $P = \bigcup \{B(x): x \in K_1\}$, $Q = P \setminus K_1$, and $R = M \setminus P$. Now $K_2 \subset R$,

hence $|R| = |K_2| = |M|$. There is $\sigma \in S(M)$ such that $\sigma(x) = x$ for all $x \in K_1$ and $\sigma(Q) \subseteq R$. Then $C = B \cap \sigma B \sigma^{-1}$ is a neighborhood of e . For each $x \in K_1$,

$$C(x) = [B \cap \sigma B \sigma^{-1}](x) \subseteq B(x) \cap [\sigma B \sigma^{-1}](x) = \{x\}.$$

Thus $C \subseteq E_{K_1}$ and E_{K_1} is open with $|K_1| = |M \setminus K_1| = |M|$. There is $\alpha \in S(M)$ such that $\alpha(K_1) = M \setminus K_1$. Then $\{e\} = E_{M \setminus K_1} \cap E_{K_1} = E_{K_1} \alpha^{-1} \cap E_{K_1}$ is also open, hence $\mathfrak{I}$ is the discrete topology on $S(M)$.

4. *Consequences of Theorem 3.*—Since any locally compact topological group is locally bounded, Theorem 3 asserts that there does not exist a nondiscrete Hausdorff topology $\mathfrak{I}$ on $S(M)$ such that $(S(M), \cdot, \mathfrak{I})$ is a locally compact topological group. This gives a negative answer to a question posed by Ulam (ref. 3, p. 57) concerning the existence of a metric ρ on $S(M)$, M countable, such that $S(M)$ with the metric topology is a locally compact topological group.

A *measurable group* is a σ -finite measure space (G, R, μ) such that

- (1) μ is not identically zero.
- (2) G is a group.
- (3) The σ -ring R and the measure μ are invariant under left translations.
- (4) The function $S: G \times G \rightarrow G \times G$ defined by $S(x, y) = (x, xy)$ is measurability preserving.

A measurable group is *separated* iff for each $x \in G$, with $x \neq e$, there is a measurable set $E \subset G$ such that $\mu(E)$ is finite and nonzero and $\mu(E \Delta xE) > 0$.

Let (G, R, μ) be a separated measurable group. The *Weil topology associated with μ* is the topology on G obtained by taking the sets of the form MM^{-1} , M measurable of finite positive measure, for a base for the neighborhood system of e . It is a well-known result of Weil (ref. 4, pp. 140–146) that, equipped with this topology, G becomes a locally bounded, Hausdorff, topological group. If this is the discrete topology on G , the μ is the σ -finite contraction of the Haar measure on G determined by the discrete topology on G , i.e., R = collection of all countable subsets of G and $\mu(\{x\}) = \mu(\{e\}) \neq 0$ for all $x \in G$. Thus Theorem 3 asserts that this is the only measure on $S(M)$ such that $(S(M), R, \mu)$ is a separated measurable group. This answers a question posed by Oxtoby (ref. 2, p. 236) and Ulam (ref. 3, p. 58) as to the existence of a Weil-Borel measure on $S(M)$, M countable.

* This paper was written while the author held an ARMU Faculty Orientation grant at Los Alamos Scientific Laboratory.

¹ Ball, R. W., "Maximal subgroups of symmetric groups," *Trans. Am. Math. Soc.*, **121**, 393–407 (1966).

² Oxtoby, J. C., "Invariant measures in groups which are not locally compact," *Trans. Am. Math. Soc.*, **60**, 215–237 (1946).

³ Ulam, S. M., *A Collection of Mathematical Problems* (New York: Interscience Publishers, Inc., 1960).

⁴ Weil, A., "L'intégration dans les groupes topologiques et ses applications," *Actualités Scientifiques et Industrielles*, no. 869, Paris, 1940.