ON OBSTRUCTIONS TO DEFORMATIONS OF COMPLEX ANALYTIC SURFACES*

BY ARNOLD KAS

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY

Communicated by D. C. Spencer, June 19, 1967

Let $\{\Phi: \mathfrak{M} \rightarrow N\}$ be a complex analytic family of compact complex analytic manifolds in the sense of Kodaira and Spencer.¹ To each point $t \in N$, there is a map of infinitesimal deformation $\rho_t: T_t(N) \rightarrow H^1(M_t, \Theta_t)$ where $T_t(N)$ = the tangent space of N at t , $M_t = \Phi^{-1}(t)$, and Θ_t = the sheaf of germs of holomorphic vector fields on M_t . If M is a given compact complex analytic manifold, an element $\eta \in H^1(M, \Theta)$ is said to be unobstructed if there exists a complex analytic family $\{\Phi: \mathfrak{M} \rightarrow N\}$ with $M = M_0$, $0 \in N$, such that $\eta \in \rho_0(T_0(N))$; if no such family exists, η is said to be obstructed.

A complex analytic family $\{\Phi: \mathfrak{M} \rightarrow N\}$ is said to be effectively parametrized at $t \in N$ if the map $\rho_t: T_t(N) \rightarrow H^1(M_t, \Theta_t)$ is injective. For a given complex manifold M , one says that the number of moduli $\mu(M)$ is defined, if there exists a complex analytic family $\{\Phi: \mathfrak{M} \rightarrow N\}$, $M = M_0$, which is effectively parametrized at each point $t \in N$ and which is complete at each point $t \in N$ in the sense that every family of deformations of M_t is "locally induced" from the family $\{\Phi: \mathfrak{M} \rightarrow N\}$ (cf. ref. 1, p. 337). In this case we define $\mu(M) = \dim N$.

It follows from results of Kuranishi^{4, 5} that if $H^1(M, \Theta)$ contains no obstructed elements, and if $\mu(M)$ is defined, then $\mu(M) = \dim H^1(M, \Theta)$. In reference 1, Kodaira and Spencer give examples of complex manifolds M such that $H^1(M, \Theta)$ contains obstructed elements. Later, Mumford⁷ gave an example of a three-dimensional complex manifold M such that $\mu(M)$ is defined and $\mu(M) < \dim H^1(M, \Theta)$. An important problem is to determine geometric conditions on M , such that $H^1(M, \Theta)$ contains obstructed elements. In particular, no example was known of a complex analytic surface S such that $H^1(S, \Theta)$ contains obstructed elements, and a fortiori no example was known of a complex analytic surface S such that $\mu(S)$ is defined with $\mu(S) < \dim H^1(S, \Theta)$. The purpose of this note is to give examples of both of these phenomena. Full proofs will appear elsewhere.

Definition: By an elliptic surface, we mean a compact complex analytic surface S such that there exists a holomorphic map $\Psi: S \rightarrow \Delta$ onto a nonsingular algebraic curve Δ , such that for a general point $u \in \Delta$, $C_u = \Psi^{-1}(u)$ is a nonsingular elliptic curve on S . In this paper, when we speak of an elliptic surface S , we will assume that S is not a fiber bundle of elliptic curves over Δ .

If S is an elliptic surface with base curve Δ , there exists a meromorphic function j defined on Δ , such that if $u \in \Delta$, and C_u is a nonsingular elliptic curve on S , then $j(u) = J(C_u)$, the elliptic modular invariant of the elliptic curve C_u (cf. ref. 2, p. 574). j is called the functional invariant of the elliptic surface S .

For any surface S , let p_g = the geometric genus of S , and q = the irregularity of S . It is well known³ that p_g and q depend only on the topological structure of S .

THEOREM 1. *Let S be an elliptic surface with base curve Δ , and let d = genus Δ . Assume that:* (i) S contains no multiple fiber (ref. 2, p. 564); (ii) no fiber of S con-

tains an exceptional curve; (iii) S is neither a K3 surface nor a rational surface, i.e., we exclude the cases where $q = 0$, $p_g \leq 1$.

Then

$$\dim H^1(S, \Theta) = \begin{cases} 11(p_g - q + 1) + 3d - 3 & \text{if } j \neq \text{constant} \\ 11(p_g - q + 1) + 3d - 3 + \dim H^0(\Delta, D), & \text{if } j = \text{constant} \end{cases}$$

where D is a divisor on Δ , and degree $D = 2d - 2 + 2(p_g - q + 1) - \#(\text{singular fibers of } S)$. Moreover, if $j = \text{constant}$ and $j \neq 0, 1$, then D is a canonical divisor on Δ , and $\dim H^1(S, \Theta) = 11(p_g - q + 1) + 4d - 3$.

It is known (ref. 3, p. 797) that every deformation of such an elliptic surface is an elliptic surface. We consider elliptic surfaces S for which the functional invariant $j = \text{constant}$ and such that $\dim H^0(\Delta, D) > 0$. By Theorem, 1, $\dim H^1(S, \Theta)$ is greater for these special surfaces than for the corresponding "general" elliptic surface.

THEOREM 2. *Let S be an elliptic surface with base curve Δ satisfying the assumptions of Theorem 1. If $j = \text{constant}$ and $\dim H^0(\Delta, D) > 0$, then $H^1(S, \Theta)$ contains obstructed elements.*

We will give an explicit construction of an elliptic surface S with constant functional invariant. Let Δ be a nonsingular algebraic curve and C a nonsingular elliptic curve. Let $\tilde{\Delta}$ be a two-sheeted covering of Δ with $2r$ branch points for some integer $r > 0$. We let S be the nonsingular model of the quotient of $\tilde{\Delta} \times C$ by the involution: $(z, \xi) \rightarrow (z^*, -\xi)$ of $\tilde{\Delta} \times C$, where $z \rightarrow z^*$ is the involution of "sheet interchange" on $\tilde{\Delta}$, and $\xi \rightarrow -\xi$ is reflection about the origin on C . It is obvious that S is an elliptic surface with base curve Δ . It can be shown that for this surface S , $p_g - q + 1 = r$, the Euler-Poincaré characteristic $\chi(S) = 12r$, and $\dim H^1(S, \Theta) = 11r + 4d - 3$, where $d = \text{genus } \Delta$.

It is known⁶ that on a general algebraic curve of genus d , every nonconstant meromorphic function has degree $\geq (d + 2)/2$. It follows from results of Kodaira (ref. 2, p. 575) that if S is any elliptic surface with functional invariant j , then $\chi(S) \geq \text{degree } j$. Let r, d be any positive integers such that $12r < (d + 2)/2$. Let Δ be a sufficiently general algebraic curve of genus d , and let S be the elliptic surface with constant functional invariant constructed as above with respect to Δ and r . Then every small deformation of S must have a constant functional invariant, for if S_i were a small deformation of S with nonconstant functional invariant j_i , then $\chi(S_i) \geq \text{degree } j_i \geq (d + 2)/2 > 12r = \chi(S)$, which is impossible. It follows that the number of moduli $\mu(S) = 3r + 4d - 3$.

THEOREM 3. *To each pair of positive integers r, d such that $12r < (d + 2)/2$, there exists a compact complex analytic surface S such that:*

$$\mu(S) = 3r + 4d - 3$$

$$\dim H^1(S, \Theta) = 11r + 4d - 3.$$

* This research is part of the author's doctoral dissertation submitted to Stanford University in 1966. Research supported by U.S. Army research contract DA-31-124-ARO (D)-151.

¹ Kodaira, K., and D. C. Spencer, "On deformations of complex analytic structures, I-II," *Ann. Math.*, **67**, 328 (1958).

² Kodaira, K., "On compact analytic surfaces, II," *Ann. Math.*, **77**, 563 (1963).

³ Kodaira, K., "On the structure of compact complex analytic surfaces," these PROCEEDINGS, *Am. J. Math.*, **86**, 751(1964).

⁴ Kuranishi, M., "On the locally complete families of complex analytic structures," *Ann. Math.*, **75**, 536 (1962).

⁵ Kuranishi, M., "New proof for the existence of locally complete families of complex structures," in *Proceedings of the Conference on Complex Analysis*, Minneapolis (1964), pp. 142-154.

⁶ Meis, Th., "Die minimale Blätterzahl der Konkretisierung einer kompakten Riemannschen Fläche," *Schr. Math. Inst. Univ. Munster* (1960) pp. 1-61.

⁷ Mumford, D., "Further pathologies in algebraic geometry," *Am. J. Math.*, **84**, 642 (1962).