Published in final edited form as:

IEEE Trans Ultrason Ferroelectr Freq Control. 2012 March ; 59(3): 590–604. doi:10.1109/TUFFC.2012.22233.

A Singular-Value Method for Reconstruction of Nonradial and Lossy Objects

Wei Jiang[Member, IEEE],

Department of Electrical and Computer Engineering, University of Rochester, Rochester, NY

Jeffrey P. Astheimer, and

Department of Electrical and Computer Engineering, University of Rochester, Rochester, NY

Robert C. Waag[Life Fellow, IEEE]

Department of Electrical and Computer Engineering, University of Rochester, Rochester, NY.

Department of Imaging Sciences, University of Rochester, Rochester, NY

Robert C. Waag: waag@ece.rochester.edu

Abstract

Efficient inverse scattering algorithms for nonradial lossy objects are presented using singular-value decomposition to form reduced-rank representations of the scattering operator. These algorithms extend eigenfunction methods that are not applicable to nonradial lossy scattering objects because the scattering operators for these objects do not have orthonormal eigenfunction decompositions. A method of local reconstruction by segregation of scattering contributions from different local regions is also presented. Scattering from each region is isolated by forming a reduced-rank representation of the scattering operator that has domain and range spaces comprised of far-field patterns with retransmitted fields that focus on the local region. Methods for the estimation of the boundary, average sound speed, and average attenuation slope of the scattering object are also given. These methods yielded approximations of scattering objects that were sufficiently accurate to allow residual variations to be reconstructed in a single iteration. Calculated scattering from a lossy elliptical object with a random background, internal features, and white noise is used to evaluate the proposed methods. Local reconstruction yielded images with spatial resolution that is finer than a half wavelength of the center frequency and reproduces sound speed and attenuation slope with relative root-mean-square errors of 1.09% and 11.45%, respectively.

I. Introduction

Ultrasonic imaging is widely used in detection and diagnosis of disease because ultrasound is non-ionizing, noninvasive, and inexpensive [1], [2]. The specific clinical application that motivates this study is breast imaging. Ultrasound images of the breast are usually b-scans that are generated from pulse-echo measurements. These scans currently play an important role in the detection and diagnosis of breast cancer as well as other diseases of the breast. However, the usefulness of b-scans is limited by speckle artifacts, degradation resulting from inhomogeneous transmission paths, and resolution that is generally coarser than the wavelength of the center frequency of the transmitted pulse [3]. Aberration correction methods can be used to compensate for inhomogeneous transmission paths [4]–[8], but these corrections are only effective in small isoplanatic regions. Furthermore, the envelope detection process used to form scan line intensities does not result in image values that provide quantitative estimates of tissue parameters that may be used to improve diagnosis.

Although a measurement system that acquires scattering measurements over a large range of angles is required (e.g., a ring transducer), inverse scattering is a different imaging modality that overcomes many of the disadvantages of b-scan imaging.

Time-of-flight tomography [9]–[11] is one approach to inverse scattering. Images formed using this method do not contain speckle. However, refraction and diffraction effects introduce error in the time-of-flight propagation model, resulting in image resolution that is no better than the resolution of b-scan images [10], [11]. Higher resolution can be realized from diffraction tomography [12], but these methods are only effective when a weak-scattering assumption is applicable. If the scattering is not weak, then iterative reconstruction methods, called distorted Born iteration methods (DBIMs) [13]–[15], are required to form high-quality images. The computational cost of each distorted Born iteration is very high and, furthermore, the iterations are not guaranteed to converge to the true medium variations. Computation may be reduced by simplifying the linear sub-problems that are solved in each iteration [16], [17], and the convergence issue can be addressed by using methods such as frequency hopping [17]–[19] that initialize the iterations with low frequency reconstructions. The inverse scattering method presented here employs simplified linear sub-problems that can be adapted to image different local regions. The convergence issue is addressed by deriving an *a priori* initial estimate of the scattering object from the measurements that is accurate enough so that only a single distorted Born iteration is needed to realize a well-resolved image.

The inverse scattering method described in [20] was developed to minimize computation by solving linear sub-problems of reduced rank that are constructed from eigen-decompositions of the scattering operators. This monochromatic approach is extended in [21] to use the full range of temporal frequencies in the bandwidth of the incident pulse. A very efficient form of the eigenfunction method that applies to large-scale and high-contrast objects that are approximately radial (e.g., breast cross-sections) is developed in [22]. In this study, experimental measurements from a unique ring transducer system [23] were used to generate speckle-free, quantitative images of sound speed and slope of attenuation that have sub-wavelength resolution. Efficient methods for forming *a priori* estimates of the global characteristics of scattering objects are described in [24]. Approximation of scattering objects from global characteristics allows subsequent reconstruction of residual variations that include image details. It also provides a starting point to improve the rate of convergence if the reconstruction is implemented iteratively (i.e., by distorted Born iteration methods) as well as to reduce the risk that the iterations fail to converge to the true medium variations.

The eigenfunction methods in [20] and [21] do not apply to lossy scattering objects because the scattering operators for lossy objects are not normal¹ and, therefore, do not have unitary eigenfunction decompositions. This limitation is significant because biological tissues always have appreciable absorption coefficients in the frequency bands of interest. Expansions in terms of circular harmonics [22] may be used to represent scattering operators of lossy radial objects in two dimensions and expansions in terms of spherical harmonics may be used to represent scattering operators of lossy radial objects in three dimensions, but these expansions are not suitable for objects that lack radial symmetry. Thus, a new method is required to efficiently reconstruct lossy scattering objects that are not approximately radial. In this study, a reduced-rank form for the linear sub-problem is derived from the singular-value decomposition of the residual scattering operator rather than the eigen-

¹An operator A is called normal if $A^\dagger A = AA^\dagger$, where $A^\dagger$ is the Hermitian transpose of A . Normality is a necessary and sufficient condition for the operator to have an orthonormal eigenvector decomposition.

decomposition. Numerical trials indicate that this natural extension of the eigenfunction method also produces quantitative images with sub-wavelength resolution.

Inverse scattering computations usually entail solving a nonlinear system of equations where the number of unknowns is proportional to the volume of the scattering object. If the scattering object is large, then the cost of these computations becomes prohibitive. Furthermore, there is often only a small region of interest must be imaged, such as a cyst or a region with a concentration of calcifications. To isolate the residual scattering from a local region, a reduced-rank representation of a simulated residual scattering operator is formed in which the domain and range of the operator are comprised of incident and scattered far-field patterns with retransmitted fields that focus on the local region. These far-field patterns are obtained from the singular-value decomposition of the simulated residual scattering operator for the characteristic function of the local region. The details of these methods are described in Section II and the performance of these methods is investigated in the numerical studies that follow.

The purpose of this paper is to develop a new framework for inverse scattering reconstructions of nonradial lossy scattering objects. The theory is presented in Section II. Section III describes the numerical methods used to simulate scattering measurements, and also the methods used to form the reconstructions. Section IV describes the results of the numerical experiments, and the results are discussed in Section V. Finally, conclusions are drawn in Section VI.

II. Theory

A. Estimation of Global Characteristics for Scattering Objects

1) Estimation of Object Boundary—The boundary of a scattering object is derived from the first-reflection times of backscatter measurements. These measurements are obtained from plane-wave transmissions, and the first-reflection times are found by cross correlation of the received signals with the transmitted pulse. The boundary is estimated as the envelope of a family of parabolas in which each parabola is the set of spatial locations that can produce echoes from the incident plane wave that arrive at the receiving element in the backscatter direction at the estimated first reflection time. Referring to the geometric configuration in [24, Fig. 1], the envelope is given by

$$\mathbf{g}(\varphi) = -s(\varphi)\mathbf{e}_{\perp}(\varphi) + \left[\frac{s^2(\varphi)}{4[R - r(\varphi)]} + r(\varphi) \right] \mathbf{e}(\varphi), \quad (1)$$

where $s(\varphi)$ is the intercept parameter for the parabola associated with the plane-wave transmission in the φ direction, $\mathbf{e}_{\perp}(\varphi)$ is a unit vector orthogonal to $\mathbf{e}(\varphi)$ in the counter-clockwise direction, R is the radius of the ring transducer, and $R - r(\varphi)$ is the earliest reflection distance. The intercept parameter $s(\varphi)$ is related to $r(\varphi)$ by

$$s(\varphi) = -r'(\varphi) / \left[1 + \frac{r(\varphi)}{2[R - r(\varphi)]} \right]. \quad (2)$$

Together, (1) and (2) determine the boundary of the scattering object based on the earliest reflection times. Detailed description of the boundary determination is found in [24].

2) Estimation of Average Sound Speed—If the scattering object is convex and has dimensions much larger than the wavelength of the center frequency of the transmitted

pulse, then the average sound speed in the scattering object can be derived from the arrival times of through transmissions. Like the first-reflection times of backscatter measurements, these arrival times are also found by cross correlation of the received signals on the far side of the scattering object with the transmitted pulse. The time difference $\Delta t(\varphi)$ of the arrival times of through transmissions in the φ direction that are measured with and without the presence of the scattering object can be expressed by

$$\Delta t(\varphi) = \frac{d(\varphi)}{c(\varphi)} - \frac{d(\varphi)}{c_0}, \quad (3)$$

where $d(\varphi)$ is the object dimension in the direction φ determined from the estimated boundary, c_0 is the sound speed in the background medium, and $c(\varphi)$ is the average sound speed in the scattering object along the ray path in the direction φ . Therefore, the average sound speed for each incident direction is determined by

$$c(\varphi) = \frac{d(\varphi)}{d(\varphi) + c_0 \Delta t(\varphi)} c_0. \quad (4)$$

The average sound speed of the scattering object is estimated by averaging $c(\varphi)$ over all incident angles.

Refraction effects, especially at the boundary of the scattering object, may cause the shortest ray path through the scattering object to deviate from the straight path assumed in (3). The error that results from these effects is reduced by averaging sound speed estimates over multiple ray paths and, for more eccentric scattering objects, by assigning more weight to sound speed estimates for diameters that are nearly normal to the object boundaries.

3) Estimation of Average Attenuation Slope—The rate at which energy is lost from a monochromatic plane wave incident on a scattering object can be decomposed into a component that is due to absorption and a component that is due to scattering. This relationship is expressed by

$$E_{\text{tot}} = E_{\text{scat}} + E_{\text{abs}}, \quad (5)$$

in which E_{tot} is the total rate of energy loss, E_{scat} is the rate of energy loss from scattering, and E_{abs} is the rate of energy loss from absorption. Using the extinction theorem [25], these energy losses are defined by the expressions

$$E_{\text{tot}} = -\frac{1}{2c_0\rho_0} \text{Im}[A(0)], \quad (6)$$

$$E_{\text{scat}} = \frac{1}{8c_0\rho_0} \left[\frac{1}{2\pi} \int_0^{2\pi} |A(\theta)|^2 d\theta \right], \quad (7)$$

$$E_{\text{abs}} = -\frac{k_0}{2c_0\rho_0} \text{Im} \left[\int_{x^2+y^2 < \text{Bdry}} q(x,y) |p(x,y)|^2 dx dy \right], \quad (8)$$

where k_0 , c_0 , and ρ_0 are the wavenumber, sound speed, and density in the background medium, respectively, q is the variation in the scattering object, p is the total pressure, $A(\theta)$ is the normalized far-field scattering pattern, and $\text{Im}[A(0)]$ denotes the imaginary part of $A(0)$.

The measured value of energy lost to absorption, E_{abs} , is computed as the difference between the terms E_{tot} and E_{scat} that are calculated by applying (6) and (7) to the measured scattering pattern $A(\theta)$. In [24], measured estimates of energy loss due to absorption are fitted to theoretical values of the energy absorbed by a homogeneous lossy cylinder by iteratively adjusting the attenuation-slope parameter β of the cylinder. However, for more general scattering objects, theoretical expressions for the absorbed energy are not available. Numerical methods must then be used to compute the theoretical values of absorbed energy for different values of attenuation slope.

Let $q(x, y)$ denote the scattering potential of a large two-dimensional scattering object. The object is assumed to be homogeneous so that

$$q(x, y) = \begin{cases} \lambda & \text{for } x, y \text{ inside the object} \\ 0 & \text{for } x, y \text{ outside the object.} \end{cases} \quad (9)$$

Furthermore, the illumination of the object is assumed to be a monochromatic plane wave with the form $p(x, y) = e^{jk_0 x}$, in which k_0 is the wavenumber of transmission through the background. The complex wavenumber k_1 inside the object is defined as [26]

$$k_1 = 2\pi f / c_1 + j\alpha = 2\pi f / c_1 + j\beta f, \quad (10)$$

where α is the attenuation coefficient, β is the slope of attenuation defined by $\alpha = \beta f$ [27], and c_1 is the estimated average sound speed.

The theoretical value of the energy loss due to absorption can be computed using (8) by approximating the total field $p(x, y)$ in the interior of the scattering object as a multiplicative perturbation of the unobstructed incident field that converts the background wavenumber $k_0 = 2\pi f / c_0$ to the complex wavenumber $k_1 = 2\pi f / c_1 + j\beta f$ in interior of the object. Thus, referring to the geometry illustrated in Fig. 1, $p(x, y)$ is defined as

$$p(x, y) = e^{jk_0 x} \times \begin{cases} 1 & \text{for } x, y \text{ in the:} \\ e^{j(k_1 - k_0)[x - x_L(y)]} & \text{left exterior region,} \\ e^{j(k_1 - k_0)[x_R(y) - x_L(y)]} & \text{interior of the object,} \\ & \text{right exterior region,} \end{cases} \quad (11)$$

where $x_L(y)$ traces the x coordinate of the boundary of the object along the left side and $x_R(y)$ traces the x coordinate of the boundary of the object along the right side. Both $x_L(y)$ and $x_R(y)$ are determined by the estimated object boundary. The parameter λ in (9) that determines the value of the scattering potential inside the object is related to k_0 and k_1 by

$$\lambda = 1 - k_1^2 / k_0^2. \quad (12)$$

Substitution of (9) and (11) into (8) gives the formula

$$\begin{aligned}
E_{\text{abs}} &= -\frac{k_0}{2c_0\rho_0} \text{Im}[\lambda] \int_{y_{\min}}^{y_{\max}} \int_{x_L(y)}^{x_R(y)} e^{-2\text{Im}[k_1][x-x_L(y)]} dx dy \\
&= -\frac{k_0}{4c_0\rho_0} \frac{\text{Im}[\lambda]}{\text{Im}[k_1]} \int_{y_{\min}}^{y_{\max}} \{1 - e^{-2\text{Im}[k_1][x_R(y)-x_L(y)]}\} dy
\end{aligned} \tag{13}$$

for the absorbed energy E_{abs} . The average attenuation slope β of the scattering object is found by fitting theoretical estimates for E_{abs} given by (13) to measured values of $E_{\text{abs}} = E_{\text{tot}} - E_{\text{scat}}$, obtained from far-field scattering patterns $A(\theta)$ of the scattering object, over a range of frequencies.

B. Image Reconstruction of a Scattering Potential

1) Configuration of the Inverse Scattering Problem—The objective of the inverse scattering problem is to reconstruct the scattering potential $q(\mathbf{x})$ using the measured scattering operator $A(\theta, \alpha)$. The scattering object is characterized by the scattering potential

$$q(\mathbf{x}) = 1 - k_1(\mathbf{x})^2 / k_0^2, \tag{14}$$

where $k_0 = 2\pi f / c_0$ is the background wavenumber, c_0 is the sound speed in the background medium, and $k_1(\mathbf{x}) = 2\pi f / c_1(\mathbf{x}) + j\beta(\mathbf{x})$ is the wavenumber distribution that can be complex if the scattering object is lossy [i.e., the attenuation slope $\beta(\mathbf{x}) \neq 0$]. The scattering operator $A(\theta, \alpha)$ that maps incident wave angles α to scattered wave angles θ is defined as

$$A(\theta, \alpha) = \lim_{r \rightarrow \infty} [j \sqrt{8\pi j k_0 r} e^{-j k_0 r} p_s(r, \theta, \alpha)], \tag{15}$$

where r is the measurement radius, p_s is the measured scattered pressure, and $j \sqrt{8\pi j k_0 r} e^{-j k_0 r}$ is a normalization factor that removes the cylindrical wave factor contained in p_s .

If $q_0(\mathbf{x})$ is the approximate scattering potential, the inverse problem reduces to reconstruction of the residual scattering potential,

$$\Delta q(\mathbf{x}) = q(\mathbf{x}) - q_0(\mathbf{x}), \tag{16}$$

by using the residual scattering operator,

$$A_{\Delta q}^0 = A_q - A_{q_0}, \tag{17}$$

where A_{q_0} is the scattering operator for the approximate potential.

2) Image Reconstruction Based on Singular-Value Decomposition—The inverse scattering methods described in [20]–[22] employ an eigenfunction decomposition of the scattering operators to regularize the inverse problem and improve efficiency. However, these eigenfunction methods are not applicable to the scattering operators of nonradial lossy objects because the scattering operators are not normal matrices ($A^\dagger A \neq A A^\dagger$, where $A^\dagger$ is the Hermitian transpose of A) and complex matrices that are not normal do not have orthonormal eigenfunction decompositions. A more general approach that accomplishes the same objectives will be developed here using a singular-value decomposition of the scattering operator instead of an eigen-decomposition. In this method, a lower-dimensional

representation of the residual scattering operator $A_{\Delta q}^0$ is obtained by the truncation of the singular-value decomposition:

$$A_{\Delta q}^0 = (\psi_1, \psi_2, \dots, \psi_N) \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_N \end{pmatrix} \begin{pmatrix} \phi_1^* \\ \phi_2^* \\ \vdots \\ \phi_N^* \end{pmatrix}. \quad (18)$$

The elements of the vectors ϕ_i and ψ_n in this decomposition are samples of functions $\phi_i(\alpha)$ and $\psi_n(\theta)$ that are coefficients of plane-wave expansions for the incident and scattered waves in the far field. These coefficients also determine incident fields $\Phi_i(\mathbf{x})$ and back propagated fields $\bar{\Psi}_n^*(\mathbf{x})$ inside the scattering volume that are given by the expansions

$$\Phi_i(\mathbf{x}) = \int_0^{2\pi} \phi_i(\alpha) p(\mathbf{x}, \alpha) d\alpha \quad (19a)$$

$$\bar{\Psi}_n^*(\mathbf{x}) = \int_0^{2\pi} \psi_n^*(\theta) \bar{p}(\mathbf{x}, \theta) d\theta, \quad (19b)$$

where $p(\mathbf{x}, \alpha)$ denotes the total pressure produced by an incident plane wave that propagates through the scattering potential $q(\mathbf{x})$ in the α direction, and $\bar{p}(\mathbf{x}, \theta) = p(\mathbf{x}, \theta + \pi)$ denotes the backpropagated field of a plane wave in the θ direction that also propagates through $q(\mathbf{x})$.

The far-field scattering pattern $A_{\Delta q}^0(\theta, \alpha)$ can also be expanded using Lippman-Schwinger equation as

$$A_{\Delta q}^0(\theta, \alpha) = \int p(\mathbf{x}, \alpha) \Delta q(\mathbf{x}) \bar{p}_0(\mathbf{x}, \theta) d^2\mathbf{x}, \quad (20)$$

where $\bar{p}_0(\mathbf{x}, \theta)$ denotes the back propagated field of a plane wave in the θ direction that propagates through the approximate potential $q_0(\mathbf{x})$ rather than $q(\mathbf{x})$. Substituting (19a) and (19b) into (20) gives

$$\begin{aligned} \langle A_{\Delta q}^0 \phi_i, \psi_n \rangle_{[0, 2\pi]} &= \int_0^{2\pi} \int_0^{2\pi} A_{\Delta q}^0(\theta, \alpha) \phi_i(\alpha) \psi_n^*(\theta) d\alpha d\theta \\ &= \int \Delta q(\mathbf{x}) \left[\int_0^{2\pi} \phi_i(\alpha) p(\mathbf{x}, \alpha) d\alpha \right] \left[\int_0^{2\pi} \psi_n^*(\theta) \bar{p}_0(\mathbf{x}, \theta) d\theta \right] d^2\mathbf{x} \\ &= \int \Delta q(\mathbf{x}) \Phi_i(\mathbf{x}) \bar{\Psi}_n^{0*}(\mathbf{x}) d^2\mathbf{x}, \end{aligned} \quad (21)$$

where $\langle \cdot, \cdot \rangle_{[0, 2\pi]}$ denotes the Hermitian inner product for the interval $[0, 2\pi]$ and $\bar{\Psi}_n^{0*}(\mathbf{x})$ is the back propagated field for the far-field pattern $\psi_n^*(\theta)$ that propagates through $q_0(\mathbf{x})$ rather than $q(\mathbf{x})$. Eq. (21) gives the explicit relationship between the residual far-field scattering pattern $A_{\Delta q}^0(\theta, \alpha)$ and the residual medium variations.

Given an approximate scattering potential $q_0(\mathbf{x})$, the residual potential $\Delta q(\mathbf{x})$ may be estimated by minimizing the weighted L^2 norm

$$\|\Delta q(\mathbf{x})\|_{W_R}^2 = \int |\Delta q(\mathbf{x})|^2 W_R(\mathbf{x}) d^2 \mathbf{x} \quad (22)$$

subject to the constraints

$$s_i \delta_{in} = \int \Delta q(\mathbf{x}) \Phi_i(\mathbf{x}) \bar{\Psi}_n^{0*}(\mathbf{x}) d^2 \mathbf{x} \quad (23)$$

that result from using (18) to evaluate the left side of (21). The weighting factor $W_R(\mathbf{x})$ primarily serves to limit the domain of integration. The right side of (23) has a nonlinear dependency on $\Delta q(\mathbf{x})$ because $\Phi_i(\mathbf{x})$ depends on $\Delta q(\mathbf{x})$. For this reason, the method of Lagrange multipliers is needed to solve the minimization problem. The variation of (22) with respect to the perturbation $\Delta q \rightarrow \Delta q + \delta[\Delta q]$ is given by

$$\delta \left[\int |\Delta q(\mathbf{x})|^2 W_R(\mathbf{x}) d^2 \mathbf{x} \right] = 2 \int \text{Re}[\Delta q(\mathbf{x}) \delta[\Delta q^*(\mathbf{x})]] W_R(\mathbf{x}) d^2 \mathbf{x}, \quad (24)$$

and the variation of (23) with respect to perturbation $\Delta q \rightarrow \Delta q + \delta[\Delta q]$ (using the two potential formula [28]) gives

$$\delta[s_i \delta_{in}] = \int \delta[\Delta q(\mathbf{x})] \Phi_i(\mathbf{x}) \bar{\Psi}_n^*(\mathbf{x}) d^2 \mathbf{x}. \quad (25)$$

The term $\bar{\Psi}_n^*(\mathbf{x})$ appears on the right side of (25) rather than $\bar{\Psi}_n^{0*}(\mathbf{x})$ that appears on the right side of (23).

To apply the method of Lagrange multipliers, the right side of (24) must be equated to a linear combination of the constraint variations given by (25). However, because the constraint variations are complex valued, they must be split into pairs of variations given by

$$\delta[s_i \delta_{in}] \rightarrow \begin{cases} \text{Re} \left[\int \delta[\Delta q(\mathbf{x})] \Phi_i(\mathbf{x}) \bar{\Psi}_n^*(\mathbf{x}) d^2 \mathbf{x} \right] \\ \text{Im} \left[\int \delta[\Delta q(\mathbf{x})] \Phi_i(\mathbf{x}) \bar{\Psi}_n^*(\mathbf{x}) d^2 \mathbf{x} \right] \end{cases} \quad (26)$$

that are assigned separate real-valued Lagrange multipliers. Using a single complex multiplier to weight the complex variation is not sufficient because of the specialized form of linear combination that results from complex multiplication. Thus, the Lagrange multiplier condition for the minimum is

$$2 \int \text{Re}[\Delta q(\mathbf{x}) \delta[\Delta q^*(\mathbf{x})]] W_R(\mathbf{x}) d^2 \mathbf{x} = \sum_{i,n} \lambda_{in}^{\text{Re}} \text{Re} \left[\int \delta[\Delta q(\mathbf{x})] \Phi_i(\mathbf{x}) \bar{\Psi}_n^*(\mathbf{x}) d^2 \mathbf{x} \right] + \lambda_{in}^{\text{Im}} \text{Im} \left[\int \delta[\Delta q(\mathbf{x})] \Phi_i(\mathbf{x}) \bar{\Psi}_n^*(\mathbf{x}) d^2 \mathbf{x} \right],$$

where λ_{in}^{Re} and λ_{in}^{Im} are the real-valued Lagrange multipliers. If $\delta[\Delta q(\mathbf{x})]$ is a real perturbation, then (27) simplifies to

$$2 \int \text{Re}[\Delta q(\mathbf{x})] \delta[\Delta q(\mathbf{x})] W_R(\mathbf{x}) d^2 \mathbf{x} = \sum_{i,n} \lambda_{in}^{\text{Re}} \int \delta[\Delta q(\mathbf{x})] \text{Re}[\Phi_i(\mathbf{x}) \bar{\Psi}_n^*(\mathbf{x})] d^2 \mathbf{x} + \lambda_{in}^{\text{Im}} \int \delta[\Delta q(\mathbf{x})] \text{Im}[\Phi_i(\mathbf{x}) \bar{\Psi}_n^*(\mathbf{x})] d^2 \mathbf{x}.$$

This equation implies that

$$\text{Re}[\Delta q(\mathbf{x})W_R(\mathbf{x})] = \frac{1}{2} \sum_{i,n} \lambda_{in}^{\text{Re}} \text{Re}[\Phi_i(\mathbf{x})\bar{\Psi}_n^*(\mathbf{x})] + \lambda_{in}^{\text{Im}} \text{Im}[\Phi_i(\mathbf{x})\bar{\Psi}_n^*(\mathbf{x})]. \quad (29)$$

Similarly, if $\delta[\Delta q(\mathbf{x})]$ is an imaginary perturbation, then (27) simplifies to

$$-2 \int \text{Im}[\Delta q(\mathbf{x})] \delta[\Delta q(\mathbf{x})] W_R(\mathbf{x}) d^2 \mathbf{x} = \sum_{i,n} \lambda_{in}^{\text{Re}} \int \delta[\Delta q(\mathbf{x})] \text{Im}[\Phi_i(\mathbf{x})\bar{\Psi}_n^*(\mathbf{x})] d^2 \mathbf{x} - \lambda_{in}^{\text{Im}} \int \delta[\Delta q(\mathbf{x})] \text{Re}[\Phi_i(\mathbf{x})\bar{\Psi}_n^*(\mathbf{x})] d^2 \mathbf{x}.$$

This equation implies that

$$\text{Im}[\Delta q(\mathbf{x})W_R(\mathbf{x})] = \frac{1}{2} \sum_{i,n} -\lambda_{in}^{\text{Re}} \text{Im}[\Phi_i(\mathbf{x})\bar{\Psi}_n^*(\mathbf{x})] + \lambda_{in}^{\text{Im}} \text{Re}[\Phi_i(\mathbf{x})\bar{\Psi}_n^*(\mathbf{x})]. \quad (31)$$

Eqs. (30) and (31) can be rewritten as the single expression

$$\Delta q(\mathbf{x}) = \frac{1}{W_R(\mathbf{x})} \sum_{l,m} Q_{lm} [\Phi_l(\mathbf{x})\bar{\Psi}_m^*(\mathbf{x})]^* \quad (32)$$

by choosing $Q_{lm} = (\lambda_{lm}^{\text{Re}} + j\lambda_{lm}^{\text{Im}})/2$. Substitution of (32) into the constraints equation, i.e., (23), gives

$$s_i \delta_{in} = \sum_{l,m} \left[\int \frac{1}{W_R(\mathbf{x})} \Phi_i(\mathbf{x})\bar{\Psi}_n^{0*}(\mathbf{x}) [\Phi_l(\mathbf{x})\bar{\Psi}_m^*(\mathbf{x})]^* d^2 \mathbf{x} \right] Q_{lm}, \quad (33)$$

which is a system of linear equations with the Q_{lm} coefficients as the unknowns. However, the fields $\Phi(\mathbf{x})$ and $\bar{\Psi}_m^*(\mathbf{x})$ that appear on the right side of (32) and that also appear in the integrand of (33) are not available because these fields propagate through the unknown potential $q(\mathbf{x}) = q_0(\mathbf{x}) + \Delta q(\mathbf{x})$. Eqs. (32) and (33) are, therefore, replaced with the approximate expressions

$$\Delta q(\mathbf{x}) = \frac{1}{W_R(\mathbf{x})} \sum_{l,m} Q_{lm} [\Phi_l^0(\mathbf{x})\bar{\Psi}_m^{0*}(\mathbf{x})]^*, \quad (34)$$

and

$$s_i \delta_{in} = \sum_{l,m} \left[\int \frac{1}{W_R(\mathbf{x})} \Phi_i^0(\mathbf{x})\bar{\Psi}_n^{0*}(\mathbf{x}) [\Phi_l^0(\mathbf{x})\bar{\Psi}_m^{0*}(\mathbf{x})]^* d^2 \mathbf{x} \right] Q_{lm} \quad (35)$$

obtained by substituting fields that propagate through $q_0(\mathbf{x})$ for fields that propagate through $q(\mathbf{x})$. This results in a distorted-wave Born approximation that estimates the residual scattering potential as a weak (i.e., linear) perturbation of the scattering from $q_0(\mathbf{x})$. Eq. (35) may be solved for the Q_{lm} coefficients. Then, the residual potential $\Delta q(\mathbf{x})$ can be computed from the expansion in (34). The residual potential $\Delta q(\mathbf{x})$ may next be combined with the approximate potential $q_0(\mathbf{x})$ to obtain a refined estimate of $q(\mathbf{x})$. The image reconstruction may also be implemented iteratively. For each potential estimate, background components of the scattering measurements that are attributed to the current estimate of $q(\mathbf{x})$ are

calculated and removed from the overall scattering measurements. Corresponding fields $\Phi_l^0(\mathbf{x})$ and $\bar{\Psi}_m^{0*}(\mathbf{x})$ that propagate through the current estimate of $q(\mathbf{x})$, together with the residual scattering measurements, are then used to obtain a correction to the potential and the correction is appended to the current estimate. Repeated estimation allows accurate reconstruction of the residual variations that include image details.

3) Local Image Reconstruction—Reconstruction of a small region of interest within a large scattering volume is investigated here as a way to reduce the cost of the inverse scattering computations. In general, the number of singular values needed to image a scattering object from the far-field pattern for a fixed temporal frequency is proportional to the size of the object. Thus, local reconstruction requires much less computation and may be used to resolve isolated features when imaging the entire object is prohibitive.

The method developed below is motivated by observed characteristics of the singular-value fields $\Phi_i^0(\mathbf{x})$ and $\bar{\Psi}_n^{0*}(\mathbf{x})$ that are obtained by propagating the $\phi(\alpha)$ incident planewave distributions through approximate scattering potential $q_0(\mathbf{x})$ and also backpropagating the $\psi_n^*(\theta)$ incident plane-wave distributions through $q_0(\mathbf{x})$. If $\phi(\alpha)$, $\psi_n^*(\theta)$ correspond to large singular values s_i , s_n of the residual scattering operator $A_{\Delta q}^0(\theta, \alpha)$, then the fields $\Phi_i^0(\mathbf{x})$ and $\bar{\Psi}_n^{0*}(\mathbf{x})$ will be concentrated in areas where the residual potential $\Delta q(\mathbf{x})$ is appreciable. Local reconstruction requires a different set of incident plane-wave distributions $\tilde{\phi}(\alpha)$, $\tilde{\psi}_n(\theta)$ that propagate through $q_0(\mathbf{x})$ to produce localized singular-value fields $\tilde{\Phi}_i^0(\mathbf{x})$, $\tilde{\bar{\Psi}}_n^{0*}(\mathbf{x})$ that are concentrated in the local region of interest.

The local incident plane-wave distributions can be derived from an artificially produced residual scattering operator $\tilde{A}_\chi^0(\theta, \alpha)$ for the residual potential $\Delta q(\mathbf{x}) = \chi(\mathbf{x})$, where $\chi(\mathbf{x})$ is the characteristic function of the local region of interest, as shown in Fig. 2. The artificial residual scattering operator $\tilde{A}_\chi^0(\theta, \alpha)$ is computed from the Lippman-Schwinger equation

$$\tilde{A}_\chi^0(\theta, \alpha) = \int p_0(\mathbf{x}, \alpha) \chi(\mathbf{x}) \bar{p}_0(\mathbf{x}, \theta) d^2 \mathbf{x}, \quad (36)$$

and has the truncated singular-value decomposition

$$\tilde{A}_\chi^0 = (\tilde{\psi}_1, \tilde{\psi}_2, \dots, \tilde{\psi}_N) \begin{pmatrix} \tilde{s}_1 & & & \\ & \tilde{s}_2 & & \\ & & \ddots & \\ & & & \tilde{s}_N \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1^* \\ \tilde{\phi}_2^* \\ \vdots \\ \tilde{\phi}_N^* \end{pmatrix}. \quad (37)$$

The localized singular-value fields $\tilde{\Phi}_i^0(\mathbf{x})$, $\tilde{\bar{\Psi}}_n^{0*}(\mathbf{x})$ are calculated using (19a) and (19b) with the local incident planewave distributions $\tilde{\psi}_i$ and $\tilde{\phi}_n$.

Local reconstruction of the medium variations is obtained by minimizing (22) subject to the constraints

$$\langle A_{\Delta q}^0 \tilde{\phi}_i, \tilde{\psi}_n \rangle_{[0, 2\pi]} = \int \Delta q(\mathbf{x}) \tilde{\Phi}_i^0(\mathbf{x}) \tilde{\bar{\Psi}}_n^{0*}(\mathbf{x}) d^2 \mathbf{x}. \quad (38)$$

This problem is also solved by the Lagrange multiplier method described in the previous section, which yields the expansion

$$\Delta\tilde{q}(\mathbf{x}) = \frac{1}{\tilde{W}_R(\mathbf{x})} \sum_{l,m} \tilde{Q}_{lm} \left[\tilde{\Phi}_l^0(\mathbf{x}) \tilde{\Psi}_m^{0*}(\mathbf{x}) \right]^* \quad (39)$$

for the residual potential with coefficients $\tilde{Q}_{lm}$ that satisfy the linear system

$$\langle A_{\Delta q}^0 \tilde{\phi}_i, \tilde{\psi}_n \rangle_{[0,2\pi]} = \sum_{l,m} \left[\int \frac{1}{\tilde{W}_R(\mathbf{x})} \tilde{\Phi}_i^0(\mathbf{x}) \tilde{\Psi}_n^{0*}(\mathbf{x}) \left[\tilde{\Phi}_l^0(\mathbf{x}) \tilde{\Psi}_m^{0*}(\mathbf{x}) \right]^* d^2\mathbf{x} \right] \tilde{Q}_{lm}. \quad (40)$$

III. Methods

An inhomogeneous elliptical object, diagrammed in Fig. 3, was used for numerical studies of the proposed methods that estimate global characteristics of a scattering object and also perform global image reconstruction. The elliptical boundary of the object has a semi-major axis of 6 mm and a semi-minor axis of 4 mm, and is offset by (0.50 mm, 0.25 mm) from the center of an assumed ring transducer measurement system. In the interior of the ellipse are two small cylinders and three circular filaments clustered together in a local volume that is approximately 4×4 mm in size. The elliptical scattering object is embedded in a water background with a sound speed c_0 of 1.509 mm/ μ s, a density ρ_0 of 0.997 g/cm³, and no attenuation.

A different elliptical object, diagrammed in Fig. 4, was used to demonstrate the effectiveness of the proposed method for both global and local image reconstructions. This object has the same boundary as the original object, but contains two clusters of internal features instead of one cluster. The cluster on the right side of the ellipse is a translation of the internal features in the original object with three extra filaments that form a row of filaments with graduated separations that are used to demonstrate the spatial resolution of the reconstructions. The additional cluster on the left side of the ellipse is comprised of a single larger cylinder and three circular filaments. The cluster on the right is contained in a disk with a radius of 2 mm and the cluster on the left is contained in a disk with a radius of 1.75 mm. These disks, shown in the figure as dashed circles, are the regions used in the local reconstructions. The centers of the two disks are separated by a distance of 5 mm. The background object contains randomly distributed subwavelength scatterers that are simulated by convolving Gaussian white noise with a Gaussian structure function that has a 0.2 mm standard deviation. The amplitudes of the resulting random components of the sound speed and attenuation slope were chosen to realize variations with standard deviations that are 2% of the assigned values of the background object. These random fluctuations are consistent with the tissue models used in [29] to simulate different breast tissues. Random fluctuations are not included in the vicinity of the right cluster to allow clear characterization of spatial resolution. The calculated scattering in this simulation was corrupted with Gaussian white noise with an rms value that is 40 dB lower than the rms value of the measurements. This results in a signal-to-noise ratio that is representative of the signal-to-noise ratio in actual systems.

Scattering measurements were simulated by using a k -space method [30], [31] to propagate plane-wave transmissions through the scattering object. This method is a time-domain forward solver that computes the time-varying pressures at the vertices of a rectangular grid of spatial locations. A model for absorption and dispersion based on relaxation parameters is incorporated in the k -space algorithm, and perfectly matched layer (PML) boundary

conditions are used to suppress reflections from the edges of the computational grid. Previous studies [32]–[34] have demonstrated that the k -space method is an efficient and accurate way to simulate ultrasonic propagation over a large scale (i.e., distances of many wavelengths) in soft tissues.

Pressure fields were calculated for both scattering objects for a set of 256 incident plane waves traveling in directions that are uniformly spaced around the unit circle. A modulated Gaussian pulse with a center frequency of 2.5 MHz and a –6-dB bandwidth of 1.7 MHz was used as the waveform for each plane wave transmission. The timevarying pressure field for each transmission was sampled at 256 points around a circle that was centered at the origin and had a 14 mm diameter. These signals emulate scattering measurements from a ring transducer with a 14 mm diameter.

The number of incident wave angles and the number of receiver locations were both chosen to satisfy $N \geq 4\pi f_{\max} a_{\max}/c_0$, where $a_{\max}$ is the maximum half dimension of the scattering object. (If the object position with respect to the aperture is known, then $a_{\max}$ is replaced by the maximum distance from the boundary of the object to the center of the aperture [35].) This formula is obtained by applying the Nyquist criteria to a uniform sampling of the Ewald circle. However, because the spatial frequencies of the actual measurements are not spaced uniformly, this condition is only approximate. The minimum value for N given by this estimate is 176, but a value of 256 was used for both the number of transmissions and the number of receiver locations to ensure accurate reconstruction.

The following processing steps were applied to scattering measurements:

1. Computation of Scattering Operator A_q :

The pressure fields of the incident plane waves are subtracted from the total pressure fields at each receiver location to obtain time-varying measurements of the scattered pressure. The Fourier transforms of these waveforms give a separate set of monochromatic scattering amplitudes for each temporal frequency and each incident plane wave. Each set of monochromatic scattering amplitudes is interpreted as the values of a monochromatic scattered field at the receiver locations, and a far-field pattern for this field is extrapolated from the values on the ring. This results in 256 far-field patterns that correspond to the 256 incident plane waves for each temporal frequency. These patterns are assembled into a 256×256 scattering matrix A_q (i.e., the scattering operator) in which the row index determines the transmit direction and the column index determines the receive direction.

2. Estimation of Global Characteristics:

The boundary of the scattering object is derived from the first-reflection times of backscatter measurements, as described in the first part of Section II. Each backscatter measurement is the pressure field at the receiver in the ring location that is antipodal to the direction of the incident plane wave. Once the boundary is identified, the average sound speed of the scattering object is derived from the earliest arrival times of through-transmission measurements, as also described in Section II. Each through-transmission measurement is the pressure field at the receiver in the ring location that is oriented in the same direction as the incident plane wave. Finally, to estimate the average attenuation slope of the scattering object, (13) is used to compute the energy lost to absorption at 65 different frequencies that range from 1.5 to 3.5 MHz. These losses are fitted to the measured losses given by (6) and (7) by adjusting the attenuation slope to minimize the mean-square error.

The approximate scattering potential $q_0(\mathbf{x})$ used in the subsequent processing steps is the characteristic function of the region enclosed by the estimated boundary times a complex constant determined from the estimates for the average sound speed and average attenuation slope.

3. Computation of Residual Scattering Operator $A_{\Delta q}^0$:

The k -space method is used to calculate the pressure fields $p_0(\mathbf{x}, \alpha)$ that result from propagating each of the 256 incident plane waves through the approximate potential $q_0(\mathbf{x})$. These computations are also used to compute the scattering operator A_{q_0} for $q_0(\mathbf{x})$. Subtraction of the scattering operator A_{q_0} from the overall scattering operator A_q gives the residual scattering operator $A_{\Delta q}^0$.

4. global Reconstruction:

Global reconstruction starts with the singular-value decomposition of the residual scattering operator $A_{\Delta q}^0$ given by (18). The singular-value fields $\Phi_i^0(\mathbf{x})$, $\bar{\Psi}_n^{0*}(\mathbf{x})$ are then calculated using (19a) and (19b). These fields are used to form numerical estimates of the integral in (35) so that (35) can be solved for the unknown Q_{lm} coefficients. These coefficients are used in (34) to compute the residual potential $\Delta q(\mathbf{x})$. This residual potential is added to the approximate potential $q_0(\mathbf{x})$ to form a revised estimate $q(\mathbf{x}) = q_0(\mathbf{x}) + \Delta q(\mathbf{x})$ of the scattering potential.

5. Local Reconstruction:

Local reconstruction begins with computation of the artificial scattering operator $\tilde{A}_{\chi}^0$ given by (36), which is explicitly constructed for the selected region. Different scattering operators can be used to reconstruct the medium variations in different local regions. The functions $\tilde{\phi}_i, \tilde{\psi}_n^*$ from the singular-value decomposition of $\tilde{A}_{\chi}^0$, (37), are used to compute the fields $\tilde{\Phi}_i^0(\mathbf{x})$, $\tilde{\bar{\Psi}}_n^{0*}(\mathbf{x})$. These fields are used to form numerical estimates of the integral in (40) so that (40) can be solved for the unknown $\tilde{Q}_{lm}$ coefficients. These coefficients are used in (39) to compute the local residual potential $\Delta \tilde{q}(\mathbf{x})$ that is added to the approximate potential to form a revised estimate, $q(\mathbf{x}) = q_0(\mathbf{x}) + \Delta \tilde{q}(\mathbf{x})$, of the scattering potential.

6. Compounding and Smoothing:

Image reconstruction is performed at five temporal frequencies: 1.5, 2.0, 2.5, 3.0, and 3.5 MHz. A final image is formed as the sum of the reconstructed images for the five temporal frequencies. Because electronic noise at different temporal frequencies is expected to be independent, this summation will improve the signal-to-noise ratio of the reconstruction while also suppressing frequency-dependent reconstruction artifacts. A two-dimensional cosine-power filter is applied to smooth the final image. The spatial- frequency response of the filter is given by

$$f(k) = \cos^\gamma(k\pi/2k_c), \quad (41)$$

where k is wavenumber, k_c is the cutoff wavenumber determined by the highest temporal frequency employed, and γ is the power coefficient that determines the response shape.

IV. Numerical Results

A. Results for the Scattering Object with One Set of Internal Features

The boundary of the scattering object shown in Fig. 3 was estimated as the envelope of a family of parabolas, as described in processing Step 1 in Section III. For comparison, the boundary was also estimated by a relatively simpler radial-ray method that assumes the first reflection in each backscattered signal is a reflection from a point along the ray from the center of the transducer ring to the receiving element. In that case, the first-reflection times for the set of all 256 incident plane wave directions translate into a polar equation for the object boundary. The boundaries obtained from both of these methods are plotted in Fig. 5 along with the true boundary of the scattering object. The estimate obtained from the parabola-envelope method matches the true boundary almost perfectly, but the estimate obtained from the radial-ray method deviates significantly from the true boundary at radial angles that are between the axes of the ellipse because the first reflection in the backscattered signal does not occur at a point along the ray from the center of the transducer ring to the receiving element.

The average sound speed and average slope of attenuation were also estimated using the methods described in processing Step 1 in Section III. These estimates are reported in Table I along with the true values of the parameters used in the k -space computations of the simulated measurements. The dimensions and center coordinates of an ellipse that is fitted to the parabola-envelope estimate are also reported in Table I along with the dimensions and center coordinates for the actual ellipse. The close agreement of the estimated and true values indicates that the proposed methods for estimating global characteristics of a scattering object are highly effective.

Representative singular-value fields and reconstructed images of the first scattering object using global reconstruction are shown in Fig. 6. The top two panels of Fig. 6 show the magnitude of a first-order singular-value field and a 16th-order singular-value field for the 2.5-MHz residual scattering operator. These images demonstrate that the retransmitted fields corresponding to the singular-value far-field patterns [i.e., the expansions given by (19a) and (19b)] produce fields that are concentrated in areas where the residual medium variations are largest. The bottom two panels of Fig. 6 show the reconstructed sound speed and attenuation slope over the entire scattering area. The internal features of the scattering object are clearly resolved in the reconstruction. The relative rms errors are 0.43% and 5.22% for the reconstructed sound speed and attenuation slope, respectively. The reconstructed images are blurred as a result of spatial-frequency limitations inherent in the reconstruction. The image of attenuation slope is blurred more severely than the image of sound speed. This blur is due to the small magnitude of the imaginary variations compared with the real variations of the scattering potentials. The singular-value orders for the 1.5-, 2.0-, 2.5-, 3.0-, and 3.5-MHz reconstructions are 18, 24, 28, 32, and 37, respectively. All of these orders are much smaller than the 256 transmit and receive directions that comprise each full set of monochromatic measurements; hence, the singular-value representations allow the residual scattering potential to be estimated much more efficiently.

Enlarged images of the reconstructed sound speed and attenuation slope in the region of interest indicated in Fig. 3 are shown in the top two panels of Fig. 7. The cross sections of the sound speed and attenuation slope along the profile line in Fig. 3 are shown in the bottom two panels of Fig. 7. For comparison, the middle two panels in Fig. 7 show the reconstructed sound speed and attenuation slope that result from use of the filtered backpropagation algorithm. This efficient algorithm is often used in diffraction tomography, but is based on a weak scattering assumption. These weak scattering reconstructions are much worse than the results in the top two panels, which highlights the importance of the

initial estimates. The large errors exhibited in these images support the expectation that filtered back-propagation is not an effective method for reconstructing large-scale or high-contrast objects.

B. Results for the Scattering Object with Two Sets of Internal Features

The global characteristics of the more complicated object that includes randomly distributed scatterers were also estimated using the procedure described in processing Step 1 in Section III. The results are in Table II. The reported estimates for the dimensions and center coordinates of the ellipse were obtained by fitting an ellipse to the parabola-envelope estimate for the boundary. The values reported in Table II are almost as accurate as the values reported in Table I. This indicates that the increased complexity of the second object and the inclusion of random scatterers as well as Gaussian white noise do not corrupt the estimation procedures.

The top two panels in Fig. 8 show the magnitude of a first-order singular-value field and a 16th-order singular-value field for the 2.5 MHz residual scattering operator when global reconstruction is used. The magnitudes of these representative singular-value fields are, again, spatially concentrated in the vicinity of the residual medium variations. Thus, each field has areas of concentration in both sets of internal features.

Reconstructed images of the second scattering object that are obtained from global reconstruction of the entire scattering area are shown in the middle two panels of Fig. 8. The internal features are well resolved, but distortion from the random scatterers is evident in the reconstructed images, particularly in the image of the reconstructed attenuation slope. The relative rms errors are 1.12% and 12.49% for the reconstructed sound speed and attenuation slope in the right cluster that does not contain random scatterers. The attenuation slope image exhibits some corruption in each local region that appears to be due to the influence of scattering from the other local region. The bottom two panels in Fig. 8 show the profiles of sound speed and attenuation slope in the cross section through the row of neighboring filaments. The four filaments are clearly discerned in these profiles as four isolated peaks. Because the smallest separation between the filaments is 0.3 mm, this demonstrates that the spatial resolution of the reconstruction is finer than one-half the wavelength of the center frequency of the transmitted pulse.

Fig. 9 shows representative singular-value fields when local reconstruction is used. Unlike the singular-value fields of the global reconstruction in Fig. 8, the magnitudes of the representative localized singular-value fields are spatially concentrated in the vicinity of only one of the local regions.

The middle two panels in Fig. 9 show the reconstructed images in the two local regions of interest, as indicated by the dashed circles in Fig. 4. The bottom two panels in Fig. 9 show the profiles of sound speed and attenuation slope through the row of four neighboring filaments. The values of the reconstructed potential that are outside the local regions of interest are determined by the estimates of the global characteristics. The images obtained from local reconstructions have better spatial resolution than the images obtained from global reconstructions shown in Fig. 8. The relative rms errors for the reconstructed sound speed and attenuation slope in the right cluster are 1.09% and 11.45%, respectively. The improvement is attributed to the use of localized singular-value fields that are spatially concentrated in only one of the two local regions. The local reconstructions are also more efficient than the global reconstructions. At 2.5 MHz, for example, the scattering operator needed for global reconstruction has rank $N = 50$, whereas the scattering operator needed for local reconstruction of the left region has rank $N = 38$ and the scattering operator needed for local reconstruction of the right region has rank $N = 42$. Because the most expensive part of

the computation involves evaluation of the inner product matrices in (35) and (40), each of which has a computational cost proportional to N^4 , reducing N from 50 to either 38 or 42 results in a reconstruction that is more than twice as fast.

V. Discussion

The proposed methods for estimating global characteristics of scattering objects yielded accurate results for both of the nonradial, lossy objects under the representative conditions used in this study. The parabola-envelope method, used in [24] to determine the boundary of nearly circular objects, was used here to determine the boundary of more eccentric elliptical objects. The methods for estimating average sound speed and average slope of attenuation require boundary estimates that are accurate. Furthermore, the utility of the approximate potential is strongly influenced by the accuracy of the estimated boundary. Because the parabola-envelope algorithm yielded much better boundary estimates than the simpler radial-ray algorithm, the parabola-envelope method is preferable.

The proposed method for estimation of the average sound speed in the scattering object is based on the earliest arrival times of through-transmission measurements. The results reported here, together with other numerical experiments not reported here for brevity, indicate that this technique is accurate for large objects but becomes inaccurate when the object size approaches the wavelength of the center frequency of the transmitted pulse. The method does not work well for such small scattering objects because the ray theory of propagation is not applicable to objects of that size. The proposed method for estimating the average attenuation slope is based on fitting measured values of the energy lost to absorption that are calculated from the scattering operator to theoretical values of the energy lost to absorption that are computed from ray-path transmissions. This method is also not accurate for small scattering objects because the ray theory of propagation is not applicable to those objects.

The internal variations of both scattering objects are small relative to the size of the transitions in sound speed and attenuation at the boundaries of the scattering objects. As a result, the estimates for average sound speed and average attenuation slope are very close to the values assigned to the elliptical region. Differences between the estimated parameters and the parameters for the background ellipse would be larger if the internal variations were more extensive or more pronounced. However, the proposed methods should still accurately estimate averages that include internal variations.

The weighting factors $W_R(\mathbf{x})$ and $\tilde{W}_R(\mathbf{x})$ merit comment. The appearance of $W_R(\mathbf{x})$ in the denominator of the expansion in (34) causes the residual potential to be small in regions where $W_R(\mathbf{x})$ is large. Thus, $W_R(\mathbf{x})$ should be large in regions where the residual potential is expected to be negligible and small in regions where the residual potential is expected to be appreciable. The weighting factors $W_R(\mathbf{x})$ and $\tilde{W}_R(\mathbf{x})$ used for the reconstructions of both scattering objects were defined to be one inside the region enclosed by the estimated boundary of the scattering object, and to be infinite outside this region.

The described singular-value method that solves inverse scattering problems for nonradial and lossy objects is a generalization of the eigenfunction methods described in [20] and [21]. This is important in clinical applications because the absorption coefficients of biological tissues are appreciable and deviations from radial symmetry in anatomical cross sections can be appreciable. The eigenfunction method is not applicable to nonradial lossy objects because the scattering operators for these objects are not normal and, therefore, do not have unitary eigenfunction decompositions. However, the results of this study demonstrate that the functions associated with singular values of the scattering operator can be just as

effective as eigenfunctions in reducing the number of computations in inverse scattering calculations. Furthermore, images of the retransmitted fields of these singular-value functions exhibit the same focusing properties as the retransmitted fields of eigenfunctions.

The images of the reconstructed attenuation slope are not as good as the images of the reconstructed sound speed. This is due to the relative magnitude of the real and imaginary components of the scattering potential. The real component is much larger (i.e., approximately ten times larger) than the imaginary component, and because the real and imaginary errors are intermingled by the reconstruction process, the signal-to-noise ratio of the imaginary component is significantly smaller than the signal-to-noise ratio of the real component. Nevertheless, meaningful attenuation-slope estimates are imaged by both the global and local reconstruction procedures even when the background object includes randomly distributed scatterers and the scattering measurements are corrupted by Gaussian white noise.

Dispersion relations result in variations in sound speed and attenuation with frequency that are different for different tissues. These relations do not affect monochromatic reconstructions, but they can introduce error when reconstructions from different frequencies are summed. However, the sound speeds and slopes of attenuation of soft tissues are often very nearly constant in the frequency ranges of common ultrasonic systems [27].

The Cartesian plots in the bottom panels of Fig. 7 show cross sections of sound speed and attenuation slope that are derived from the real and imaginary parts of the reconstructed potential. Both cross sections pass through the centers of two filaments, but the reconstructed tissue parameters at these centers differ from the parameters that are assigned to the filaments. This discrepancy also occurs in the filament reconstructions reported in [20]–[22] and is attributed to the finite size of the local point responses of the reconstruction process. Because the filaments are smaller than the point responses, the reconstructed values at the filament centers are averages of the parameters of the filament and the parameters of the surrounding ellipse. Because the point responses are much smaller than the cylinders, the parameters of the reconstructed cylinders are only affected near the cylinder boundaries.

Spatial resolution of the proposed reconstruction procedures can be estimated from the profiles of the reconstructed sound speed and attenuation slope through the row of four neighboring filaments shown in Figs. 8 and 9. The graduated separations between these filaments are 0.3, 0.4, and 0.5 mm, and all four filaments are all clearly resolved in both the global and local reconstructions. The spatial resolution of these reconstructions must therefore be finer than 0.3 mm, which is equal to one-half the wavelength of the center frequency of the transmitted pulse. This resolution is appreciably better than the resolutions of current b-scan imaging instruments and is also better than the resolution achieved by current time-of-flight ultrasound tomography systems that use ring transducer arrays [10], [11].

Comparisons of global and local reconstruction indicate that local reconstruction yields more accurate images and better spatial resolution. This improvement is attributed to the use of localized singular-value fields that are spatially concentrated in the local region. These fields reduce distortion caused by scattering that originates from sources that are outside the local region. The comparisons also indicate that local reconstruction is more efficient than global reconstruction. Furthermore, increasing the complexity of the internal features of the scattering object will result in a global scattering operator of increased rank, whereas the scattering operator for a local region, which is determined by an artificial medium variation, will not be affected by changes in the complexity of the scattering object. This suggests that, in some instances, the improved efficiency of local reconstruction will be more substantial.

The reported simulation results support the conclusion that the proposed global and local methods are applicable in the clinical imaging environment. Experience indicates that if the random variation strength is further increased (e.g., the standard deviations are 3% of the assigned values), then the accuracy of object boundary estimation is degraded because the random scattering makes peaks in temporal cross correlation harder to detect. Less accurate boundary estimates would introduce additional error in reconstruction of the residual scattering potential. However, experience also indicates that further increased electronic noise (e.g., a 20 dB signal-to-noise ratio) does not significantly degrade the accuracy of the boundary estimates.

VI. Conclusion

Efficient inverse scattering algorithms for nonradial lossy objects have been described. These methods are extensions of the eigenfunction methods developed in [20]– [22]. However, reduced-rank representations are obtained from a singular-value decomposition rather than from an eigenfunction decomposition because an eigenfunction decomposition is only applicable to normal scattering operators. A method of local reconstruction by segregation of scattering contributions from different local regions has also been described. This segregation is accomplished by formation of a reduced-rank representation of the scattering operator for domain and range spaces of far-field patterns that have retransmitted fields that focus on the local region. Methods for estimation of the boundary, average sound speed, and average attenuation slope of the scattering object have also been identified. These methods were used to obtain accurate reconstructions of the internal features of scattering objects in a single iteration under conditions chosen to emulate those encountered in practice.

Calculated scattering from two lossy elliptical objects was used to evaluate the performance of the proposed methods. The first elliptical object has a homogeneous background and the second object contains randomly distributed sub-wavelength scatterers. Global reconstruction of the first object yielded images of sound speed and attenuation slope with relative rms errors of 0.43% and 5.22%, respectively. Local reconstruction of the second object resulted in more accurate images of tissue parameters than global reconstruction, and both methods produced images with finer spatial resolution than one-half the wavelength of the center frequency of the transmitted pulse.

Acknowledgments

The research was supported by National Institutes of Health grants EB009692 and EB010069, and by the National Semiconductor Corporation.

D. P. Duncan is thanked for his assistance in the use of his k -space code. J. Jin is thanked for her initial study in image reconstruction of nonradial objects. Helpful discussions with J. C. Tillet and A. J. Hesford are acknowledged with appreciation.

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Biographies

Wei Jiang received a B.S. degree in electronic information engineering and a M.S. degree in underwater acoustic engineering, both from the Northwestern Polytechnical University, Xi'an, China, in 2004 and 2007, respectively. He received a Ph.D. degree in electrical engineering in 2011 from the University of Rochester. Since September 2007, he has been with the Diagnostic Ultrasound Research Laboratory of the University of Rochester, where he currently is a post-doctoral research associate in the Department of Electrical and Computer Engineering. His research interests include aberration correction, ultrasonic propagation, global estimation of scattering object characteristics, and image reconstruction based on inverse scattering.

Jeffrey P. Astheimer received the B.A., M.A., and Ph.D. degrees, all in mathematics, from the University of Rochester in 1975, 1977, and 1982, respectively. After completing his Ph.D., he joined the faculty of Colgate University in 1984, but left in 1985 to co-found the Adaptable Laboratory Software company, which developed the Asyst software system. Dr. Astheimer spent 19 years in commercial scientific software development, but has also retained a long-term interest in the mathematics of wave propagation, especially as it relates to applications in medical ultrasound.

Robert C. Waag received his B.E.E., M.S., and Ph.D. degrees from Cornell University in 1961, 1963, and 1965, respectively. After completing his Ph.D. studies, he became a member of the technical staff at Sandia Laboratories, Albuquerque, NM, and then served as an officer in the United States Air Force from 1966 to 1969 at the Rome Air Development Center, Griffiss Air Force Base, NY. In 1969, he joined the faculty of the University of Rochester, where he is now Arthur Gold Yates Professor in the Department of Electrical Engineering, School of Engineering and Applied Science, and also holds an appointment in the Department of Radiology, School of Medicine and Dentistry. Prof. Waag's recent research has treated ultrasonic scattering, propagation, and imaging in medical applications. In 1992, he received the Joseph H. Holmes Pioneer Award from the American Institute of Ultrasound in Medicine. He is a life fellow of the Institute of Electrical and Electronics Engineers and a fellow of the Acoustical Society of America and the American Institute of Ultrasound in Medicine.

Fig. 1.
A two-dimensional scattering object and its boundary.

Fig. 2.
An example of the domain of the characteristic function (dark gray circle) enclosing a local region of interest inside the scattering region.

Fig. 3.

An elliptical scattering object. The dashed square of $5 \times 5 \text{ mm}$ is the region of interest. The dash-dotted line is a profile line through the center of the right cylinder and the two right-most filaments.

Fig. 4.

An inhomogeneous elliptical scattering object containing two clusters of scatterers. Random scatterers are distributed through the whole object background except for the right cluster area. The dashed circles specify the domains of the characteristic functions used to isolate the local regions of reconstruction. The left circle is centered at $(-3.5, 0.0) \text{ mm}$, with a radius of 1.75 mm , and the right circle is positioned at $(1.5, 0.0) \text{ mm}$, with a radius of 2.0 mm . The dash-dotted lines identify cross sections of the four neighboring filaments.

Fig. 5. Boundary comparison. The estimated boundary is obtained by using the parabola-envelope method described in the Section II. The approximate boundary is obtained from the arrival time of the first echoes using the radial-ray method.

Fig. 6.

Representative singular-value fields at 2.5 MHz and reconstructed images using global reconstruction. The top two panels show the magnitude distribution of first-order and 16th-order singular-value fields, respectively. Red lines indicate the positions of the scatterers as well as the object boundary. The bottom two panels show the reconstructed sound speed and attenuation slope, respectively. The orders used for the 1.5-, 2.0-, 2.5-, 3.0-, and 3.5-MHz reconstructions are 18, 24, 28, 32, and 37, respectively.

Fig. 7.

Reconstructed sound speed and attenuation slope. The top two panels show the sound speed and attenuation slope in the region of interest that results from global reconstruction. The middle two panels show the reconstruction obtained from the low-pass filtered back-propagation formula. The bottom two panels show the profiles of sound speed and attenuation slope in the cross section shown in Fig. 3.

Fig. 8.

Representative singular-value fields at 2.5 MHz and reconstructed images using global image reconstructions. The top two panels show the magnitude distribution of representative singular-value fields and red lines indicate the positions of the scatterers as well as the object boundary. The middle two panels show the reconstructed sound speed and attenuation slope, respectively. The bottom two panels show the profiles of sound speed and attenuation slope in the cross section through the four neighboring filaments, as shown in Fig. 4. The orders used for the 1.5-, 2.0-, 2.5-, 3.0-, and 3.5-MHz reconstructions are 40, 45, 50, 55, and 60, respectively.

Fig. 9.

Representative singular-value fields at 2.5 MHz and reconstructed images using local image reconstructions. The panel formats are the same as in Fig. 8. In the bottom two panels, blue solid lines denote the results of local reconstructions and red dashed lines denote the results of global reconstructions. The used orders (N_{left} , N_{right}) for left and right local regions are (26, 30), (32, 36), (38, 42), (44, 48), and (50, 54) for 1.5, 2.0, 2.5, 3.0, and 3.5 MHz, respectively.

True Background and Estimated Global Characteristics of an Inhomogeneous Elliptical Object with One Set of Internal Features.

TABLE I

	$2a$ (mm)	$2b$ (mm)	(x_0, y_0) (mm)	c (m/s)	β (dB/cm·MHz)
True values	12.000	8.000	(−0.50, −0.25)	1570.00	0.500
Estimated values	11.992	8.001	(−0.50, −0.25)	1570.36	0.501

TABLE II

True Background and Estimated Global Characteristics of an Inhomogeneous Elliptical Object with Two Sets of Internal Features.

	$2a$ (mm)	$2b$ (mm)	(x_0, y_0) (mm)	c (m/s)	β (dB/cm·MHz)
True values	12.000	8.000	(−0.50, −0.25)	1570.00	0.500
Estimated values	11.977	8.042	(−0.50, −0.25)	1570.63	0.520