Green's functions for a volume source in an elastic half-space

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(Received 23 February 2011; revised 17 November 2011; accepted 17 November 2011)

Green's functions are derived for elastic waves generated by a volume source in a homogeneous isotropic half-space. The context is sources at shallow burial depths, for which surface (Rayleigh) and bulk waves, both longitudinal and transverse, can be generated with comparable magnitudes. Two approaches are followed. First, the Green's function is expanded with respect to eigenmodes that correspond to Rayleigh waves. While bulk waves are thus ignored, this approximation is valid on the surface far from the source, where the Rayleigh wave modes dominate. The second approach employs an angular spectrum that accounts for the bulk waves and yields a solution that may be separated into two terms. One is associated with bulk waves, the other with Rayleigh waves. The latter is proved to be identical to the Green's function obtained following the first approach. The Green's function obtained via angular spectrum decomposition is analyzed numerically in the time domain for different burial depths and distances to the receiver, and for parameters relevant to seismo-acoustic detection of land mines and other buried objects.

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PACS number(s): 43.20.Bi, 43.20.Gp [ANN]

Pages: 1831–1842

I. INTRODUCTION

The present derivations of a Green's function for a volume source in an elastic half-space were motivated by seismo-acoustic techniques used for detection of land mines with compliant casings.^{1–5} The response of such casings, both linear and nonlinear, to seismo-acoustic excitation is frequently modeled as a volume oscillation.⁶ Vibrations of the ground surface produced by the pulsations may be detected by laser Doppler vibrometers (LDVs). Similar methods have been proposed for detection of compliant improvised explosive devices (IEDs). Since the resonance frequencies of such objects are often sufficiently low that the corresponding wavelengths in the scattered field are long in comparison with the size of the object, the Green's function itself may provide the entire solution for the scattered field.

Additional motivation for the present analysis was a desire for a simpler representation of the Green's function than has appeared previously in the literature. This desire stems from the use of time reversal to enhance the response of buried objects by focusing seismic energy in time and space.^{7–10} Norville and Scott¹¹ report an experimental study of time-reversal focusing of elastic surface waves scattered by a buried object. In that work, they also describe an algorithm for estimating the surface motion detected by the LDVs by iterative convolution of an appropriate Green's function with the response due to each source. The need for real time processing and detection dictates that the Green's function be in a form that is sufficiently simple for implementation with modern signal processing hardware.

As discussed by Touhei,¹² a variety of methods have been used to derive Green's functions for elastic media. Two

different methods are used in the present paper. In each, the source is at arbitrary depth in a homogeneous, isotropic elastic half-space. The first method, presented in Sec. II, is based on expansion of the Green's function in terms of eigenvectors representing only Rayleigh surface waves. Contributions from bulk compressional (longitudinal) and shear (transverse) waves are thus ignored, an approximation shown to be valid when distance to the receiver is large in comparison with source depth.

The second method, presented in Sec. III, is based on Fourier decomposition of the Green's function into its angular spectrum. This is the most common approach, used by Maradudin and Mills,¹³ Touhei,¹² and others, and it includes all modes of propagation. Like the result obtained by Touhei, our Green's function separates into two components, one corresponding to Rayleigh waves and the other to bulk waves. The field is cylindrically symmetric, and each component of the solution involves only a single integral. It is shown in Sec. IV that the Rayleigh wave component of the solution is identical to the result obtained in Sec. II. The relative importance of Rayleigh and bulk waves as a function of the ratio of source depth to receiver distance is illustrated in Sec. V. There, numerical calculations provide the impulse response by taking the inverse temporal Fourier transform of the Green's function.

Our approach based on the angular spectrum differs from others in that we separate the wave field into two regions. One region is bounded by the free surface and the parallel plane passing through the source, and the other region is the half-space on the opposite side of the plane passing through the source. Source and matching conditions in the plane passing through the source are used to obtain the angular spectrum in that plane, which is the same throughout both regions. Integration along the axis perpendicular to the free surface is thus avoided when determining the spatial form of the Green's function.

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An arbitrary force source, for example, that considered by Touhei,¹² radiates a field which does not necessarily possess cylindrical symmetry. The Green's function for an arbitrary force source obtained using the method in Sec. III has been derived and will be reported elsewhere. In Sec. III A we derive the coefficients for the angular spectrum of a Green's function for a cylindrically symmetric force source acting along an axis perpendicular to the free surface. The reason for considering this cylindrically symmetric force source is that deriving and then differentiating the result leads more easily to the Green's function for a volume source than does deriving the latter directly.

The Green's function derived in the present work has been extended to include viscous losses.¹⁴ The same methodology has also been used to derive the Green's function for a volume source in a viscous fluid bounded by viscoelastic layers. The latter was done to model the interaction of a gas bubble with a tissue interface.¹⁵

II. GREEN'S FUNCTION EXPANSION IN RAYLEIGH WAVE MODES

By expressing the Green's function as an expansion in Rayleigh wave modes the contributions from bulk waves are ignored. The solution is therefore incomplete, but as demonstrated in Sec. V it becomes increasingly accurate as distance to the receiver becomes large relative to source depth. More important for our purposes, the Green's function derived in the present section is shown in Sec. IV to be identical to, although different in form than, the Rayleigh wave contribution that separates naturally from the complete solution derived in Sec. III.

As depicted in Fig. 1, an isotropic, homogeneous elastic half-space occupies the region $z \leq 0$, bounded by a free surface at $z = 0$. A volume point source is located along the z axis at $z = z_0$. The medium is characterized by density ρ and Lamé parameters λ and μ . Derivation of the Green's function begins in Sec. II A by identifying the solution for a plane Rayleigh wave as the eigenfunction of the momentum equation satisfying the boundary conditions. In Sec. II B, the Green's function is obtained by matching to the source

FIG. 1. Geometry and material properties for a volume source buried in an elastic half space.

condition the integral of these eigenfunctions over all propagation directions in the plane $z = 0$.

A. Eigenfunctions

The momentum equation for the medium is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\mu + \lambda) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u}, \quad (1)$$

where $\mathbf{u}$ is the displacement vector and t is time. With the linear operator $\mathcal{L}$ defined to be

$$\mathcal{L} = (\mu + \lambda) \nabla \nabla \cdot + \mu \nabla^2, \quad (2)$$

and for an $e^{-i\omega t}$ time dependence, where ω is angular frequency, Eq. (1) may be rewritten as

$$\mathcal{L}\{\mathbf{u}\} + \rho \omega^2 \mathbf{u} = 0. \quad (3)$$

Boundary conditions are imposed on Eq. (3) by the free surface. The stress tensor is

$$\sigma_{ik} = 2\mu u_{ik} + \lambda u_{ll} \delta_{ik}, \quad (4)$$

where δ_{ik} is the Kronecker delta and the strain tensor is

$$u_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right). \quad (5)$$

The boundary condition is $\sigma_{iz} = 0$ (i taking the values x, y or z) at $z = 0$, which in terms of displacement is

$$\begin{aligned} \sigma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0, \\ \sigma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0, \\ \sigma_{zz} &= 2\mu \frac{\partial u_z}{\partial z} + \lambda \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) = 0. \end{aligned} \quad (6)$$

The solution of Eq. (3) for a wave propagating parallel to the x - y plane and satisfying the above boundary conditions, and the condition that displacement vanishes as $z \rightarrow -\infty$, corresponds to the Rayleigh wave. For plane wave propagation in the direction of the unit vector $\mathbf{e}_\kappa = \boldsymbol{\kappa}/\kappa$, where $\boldsymbol{\kappa} = (\kappa_x, \kappa_y)$ is the Rayleigh wave vector and κ its magnitude, this solution may be expressed in the form

$$\mathbf{u}_\kappa(\mathbf{r}, z) = \mathbf{v}(\boldsymbol{\kappa}, z) e^{i\boldsymbol{\kappa} \cdot \mathbf{r}}, \quad (7)$$

where $\mathbf{r} = (x, y)$ and

$$\mathbf{v} = v_\kappa(\kappa, z) \mathbf{e}_\kappa + v_z(\kappa, z) \mathbf{e}_z. \quad (8)$$

The components of $\mathbf{v}$ are¹⁶

$$v_\kappa(\kappa, z) = iA(\zeta_t e^{\zeta_t \kappa z} + \eta e^{\zeta_l \kappa z}), \quad (9)$$

$$v_z(\kappa, z) = A(e^{\zeta_t \kappa z} + \eta \zeta_l e^{\zeta_l \kappa z}), \quad (10)$$

where A is taken to be a real but otherwise arbitrary constant and the remaining parameters are defined as follows:

$$\xi_t = \sqrt{1 - \xi^2}, \quad (11)$$

$$\xi_l = \sqrt{1 - \xi^2 c_t^2 / c_l^2}, \quad (12)$$

$$\eta = -2 \frac{\sqrt{1 - \xi^2}}{2 - \xi^2}. \quad (13)$$

Here, $c_t = (\mu/\rho)^{1/2}$ and $c_l = [(\lambda + 2\mu)/\rho]^{1/2}$ are the transverse and longitudinal wave speeds, respectively, and $\xi = c_R/c_t$, where $c_R = \omega/\kappa$ is the Rayleigh wave speed.

The solutions $\mathbf{u}_\kappa(x, y, z)$ for different values of κ are eigenvectors that satisfy the relation

$$\mathcal{L}\{\mathbf{u}_\kappa\} = -\rho\kappa^2 c_R^2 \mathbf{u}_\kappa. \quad (14)$$

They are solutions of Eq. (3), and therefore solutions for Rayleigh waves, at frequencies for which $\kappa^2 = \omega^2/c_R^2$. They are also orthogonal, and can be made orthonormal by requiring

$$\int_{z < 0} \mathbf{u}_\kappa \cdot \mathbf{u}_{\kappa'}^* dV = \delta(\kappa - \kappa'), \quad (15)$$

which determines the constant A in Eqs. (9) and (10). The asterisk designates complex conjugate. From Eq. (7),

$$\begin{aligned} \int_{z < 0} \mathbf{u}_\kappa \cdot \mathbf{u}_{\kappa'}^* dV &= \int_{-\infty}^0 \mathbf{v}(\kappa, z) \cdot \mathbf{v}^*(\kappa', z) dz \\ &\times \int \int_{-\infty}^{\infty} e^{i(\kappa - \kappa') \cdot \mathbf{r}} dx dy. \end{aligned} \quad (16)$$

The integrals on the right-hand side are evaluated to obtain

$$\begin{aligned} &\int_{-\infty}^0 \mathbf{v}(\kappa, z) \cdot \mathbf{v}^*(\kappa', z) dz \\ &= \frac{A^2}{2\kappa} \left[\xi_t + \frac{1}{\xi_t} + \eta^2 \left(\xi_l + \frac{1}{\xi_l} \right) + 4\eta \right], \end{aligned} \quad (17)$$

$$\begin{aligned} \int \int_{-\infty}^{\infty} e^{i(\kappa - \kappa') \cdot \mathbf{r}} dx dy &= \int \int_{-\infty}^{\infty} e^{i(\kappa_x - \kappa'_x)x + i(\kappa_y - \kappa'_y)y} dx dy \\ &= (2\pi)^2 \delta(\kappa - \kappa'), \end{aligned} \quad (18)$$

and thus

$$\begin{aligned} \int_{z < 0} \mathbf{u}_\kappa \cdot \mathbf{u}_{\kappa'}^* dV &= \frac{2\pi^2 A^2}{\kappa} \left[\xi_t + \frac{1}{\xi_t} + \eta^2 \left(\xi_l + \frac{1}{\xi_l} \right) + 4\eta \right] \\ &\times \delta(\kappa - \kappa'). \end{aligned} \quad (19)$$

By equating the right-hand sides of Eqs. (15) and (19) to solve for A and recalling Eqs. (9) and (10) one obtains

$$v_\kappa(\kappa, z) = iu_0 \sqrt{\kappa} (\xi_t e^{\xi_t \kappa z} + \eta \xi_l e^{\xi_l \kappa z}), \quad (20)$$

$$v_z(\kappa, z) = u_0 \sqrt{\kappa} (e^{\xi_t \kappa z} + \eta \xi_l e^{\xi_l \kappa z}), \quad (21)$$

where the coefficient $u_0 = A/\sqrt{\kappa}$, given by

$$u_0 = \frac{1}{2^{1/2} \pi} \left[\xi_t + \frac{1}{\xi_t} + \eta^2 \left(\xi_l + \frac{1}{\xi_l} \right) + 4\eta \right]^{-1/2}, \quad (22)$$

renders the eigenvectors $\mathbf{u}_\kappa$ orthonormal.

B. Green's function

The Green's function $\mathbf{g}(\mathbf{R}|\mathbf{R}_0)$ describing the displacement field due to a volume source located at $\mathbf{R} = \mathbf{R}_0$, where $\mathbf{R} = (x, y, z)$, satisfies the following inhomogeneous momentum equation:

$$(\mathcal{L} + \rho\omega^2) \mathbf{g}(\mathbf{R}|\mathbf{R}_0) = s \nabla \delta(\mathbf{R} - \mathbf{R}_0), \quad (23)$$

where δ is the Dirac delta function. The physical significance of the source strength s may be understood by writing $\mathbf{g} = \nabla \times \boldsymbol{\Psi} + \nabla \varphi$. The resulting equation for the vector potential $\boldsymbol{\Psi}$ associated with transverse waves is homogeneous, with the source term appearing only in the equation for the scalar potential φ associated with longitudinal waves:

$$\nabla^2 \varphi + k_l^2 \varphi = \frac{s}{\rho c_l^2} \delta(\mathbf{R} - \mathbf{R}_0), \quad (24)$$

where $k_l = \omega/c_l$. Now consider the corresponding equation for the acoustic pressure p generated in a fluid by a point source with volume velocity Q .¹⁷

$$\nabla^2 p + k_l^2 p = i\omega \rho Q \delta(\mathbf{R} - \mathbf{R}_0). \quad (25)$$

Since pressure and displacement potential in a fluid are related through the momentum equation by $p = \rho\omega^2 \varphi$, comparison of Eqs. (24) and (25) indicates that the relation between source strength and volume velocity is

$$s = -\frac{\rho c_l^2}{i\omega} Q. \quad (26)$$

While the volume source in Eq. (23) generates only longitudinal waves in an infinite medium, the free surface bounding a half-space also results in the generation of transverse and Rayleigh waves through mode conversion. Of these three modes of propagation, in the present section we are interested in only that part of the Green's function that is associated with Rayleigh waves. The desired solution can thus be expressed as an expansion of the Green's function in terms of the normal modes associated with Rayleigh waves. That part of the Green's function is labeled $\mathbf{g}^{(R)}$, and its normal mode expansion, over the entire continuous spectrum of Rayleigh wave vectors, is expressed as

$$\mathbf{g}^{(R)}(\mathbf{r}, z|z_0) = \int G_\kappa^{(R)}(z|z_0) \mathbf{u}_\kappa(\mathbf{r}, z) d\kappa. \quad (27)$$

The coefficients $G_\kappa^{(R)}$ comprise the angular spectrum of the Rayleigh wave field, which following use of Eq. (15) are defined by the inverse relation

$$G_{\mathbf{k}}^{(R)}(z|z_0) = \int_{z<0} \mathbf{g}^{(R)}(\mathbf{r}, z|z_0) \cdot \mathbf{u}_{\mathbf{k}}^*(\mathbf{r}, z) dV. \quad (28)$$

The function $\mathbf{g}^{(R)}$ must satisfy Eq. (23), which for a point source located at $(r, z) = (0, z_0)$ becomes

$$(\mathcal{L} + \rho\omega^2)\mathbf{g}^{(R)}(\mathbf{r}, z|z_0) = s\nabla[\delta(\mathbf{r})\delta(z - z_0)]. \quad (29)$$

To solve Eq. (29), its dot product with $\mathbf{u}_{\mathbf{k}}^*$ is integrated over the volume of the elastic medium to obtain

$$\begin{aligned} \int_{z<0} (\mathcal{L}\mathbf{g}^{(R)}) \cdot \mathbf{u}_{\mathbf{k}}^* dV + \rho\omega^2 \int_{z<0} \mathbf{g}^{(R)} \cdot \mathbf{u}_{\mathbf{k}}^* dV \\ = s \int_{z<0} \nabla[\delta(\mathbf{r})\delta(z - z_0)] \cdot \mathbf{u}_{\mathbf{k}}^* dV. \end{aligned} \quad (30)$$

As shown in the Appendix, the operator $\mathcal{L}$ with boundary conditions given by Eq. (6) is self-adjoint, i.e.,

$$\int_{z<0} \mathbf{u}_{\mathbf{k}}^* \cdot (\mathcal{L}\mathbf{g}^{(R)}) dV = \int_{z<0} (\mathcal{L}\mathbf{u}_{\mathbf{k}})^* \cdot \mathbf{g}^{(R)} dV. \quad (31)$$

After applying Eq. (31) and recalling Eq. (14), we can rewrite Eq. (30) in the form

$$\begin{aligned} -\rho\kappa^2 c_R^2 \int_{z<0} \mathbf{g}^{(R)} \cdot \mathbf{u}_{\mathbf{k}}^* dV + \rho\omega^2 \int_{z<0} \mathbf{g}^{(R)} \cdot \mathbf{u}_{\mathbf{k}}^* dV \\ = s \int_{z<0} \mathbf{u}_{\mathbf{k}}^* \cdot \nabla[\delta(\mathbf{r})\delta(z - z_0)] dV, \end{aligned} \quad (32)$$

and, from Eq. (28), the expression for $G_{\mathbf{k}}^{(R)}$ is

$$G_{\mathbf{k}}^{(R)} = -s \frac{\nabla \cdot \mathbf{u}_{\mathbf{k}}^*|_{\mathbf{r}=0, z=z_0}}{\rho c_R^2 (k_R^2 - \kappa^2)} = \frac{s\eta\kappa^{3/2}u_0 e^{\xi_l \kappa z_0}}{\rho c_l^2 (k_R^2 - \kappa^2)}, \quad (33)$$

where $k_R = \omega/c_R$ and Eqs. (20) and (21) were used to write

$$\nabla \cdot \mathbf{u}_{\mathbf{k}}^* = -\eta\kappa^{3/2}u_0 \frac{c_R^2}{c_l^2} e^{\xi_l \kappa z} e^{-i\mathbf{k} \cdot \mathbf{r}}. \quad (34)$$

From Eq. (27) the Green's function is

$$\begin{aligned} \mathbf{g}^{(R)}(r, z|z_0) &= -\frac{s}{\rho c_l^2} \int_0^\infty \int_0^{2\pi} \frac{\eta u_0^2 \kappa^3}{\kappa^2 - k_R^2} e^{i\kappa r \cos \varphi} e^{\xi_l \kappa z_0} \\ &\quad \times [i(\xi_l e^{\xi_l \kappa z} + \eta e^{\xi_l \kappa z}) \cos \varphi \mathbf{e}_r \\ &\quad + (e^{\xi_l \kappa z} + \eta \xi_l e^{\xi_l \kappa z}) \mathbf{e}_z] d\varphi d\kappa \\ &= \frac{2\pi s}{\rho c_l^2} \int_0^\infty \frac{\kappa^3}{\kappa^2 - k_R^2} \mathbf{f}(r, z|z_0, \kappa) d\kappa, \end{aligned} \quad (35)$$

where φ is the angle between the vectors $\mathbf{k}$ and $\mathbf{r}$,

$$\begin{aligned} \mathbf{f}(r, z|z_0, \kappa) &= \eta u_0^2 e^{\xi_l \kappa z_0} [J_1(\kappa r) (\xi_l e^{\xi_l \kappa z} + \eta e^{\xi_l \kappa z}) \mathbf{e}_r \\ &\quad - J_0(\kappa r) (e^{\xi_l \kappa z} + \eta \xi_l e^{\xi_l \kappa z}) \mathbf{e}_z], \end{aligned} \quad (36)$$

$J_n(\cdot)$ are Bessel functions of the first kind, and integration with respect to φ was carried out using the identity¹⁸

$$\int_0^{2\pi} e^{i\kappa r \cos \varphi} \cos(n\varphi) d\varphi = 2\pi(i)^n J_n(\kappa r). \quad (37)$$

Taking into account that

$$\frac{\kappa}{\kappa^2 - k_R^2} = \frac{1}{2} \left(\frac{1}{\kappa - k_R} + \frac{1}{\kappa + k_R} \right), \quad (38)$$

we extend integration along the entire real line and express Eq. (35) as

$$\mathbf{g}^{(R)}(r, z|z_0) = \frac{\pi s}{\rho c_l^2} \int_{-\infty}^{\infty} \frac{\kappa_*^2 \text{sgn}(\kappa_*)}{\kappa_* - k_R} \mathbf{f}(r, z|z_0, |\kappa_*|) d\kappa_*, \quad (39)$$

where κ is related to κ_* via $\kappa = |\kappa_*|$. The integral in Eq. (39) may be evaluated using residue theory to obtain

$$\begin{aligned} \mathbf{g}^{(R)}(r, z|z_0) &= \frac{\pi^2 s}{\rho c_l^2} \{ \mathcal{H}[\kappa_*^2 \text{sgn}(\kappa_*) \mathbf{f}(r, z|z_0, |\kappa_*|)](k_R) \\ &\quad + ik_R^2 \text{sgn}(k_R) \mathbf{f}(r, z|z_0, |k_R|) \}, \end{aligned} \quad (40)$$

where $\mathcal{H}[\cdot](k_R)$ is the Hilbert transform

C. Green's function for a source on the surface

For $z_0=0$ and $z=0$ (both source and receiver are located on the free surface) the function $\mathbf{f}(r, z|z_0, \kappa)$ is

$$\mathbf{f}(r, 0|0, \kappa) = \eta u_0^2 [J_1(\kappa r) (\xi_l + \eta) \mathbf{e}_r - J_0(\kappa r) (1 + \eta \xi_l) \mathbf{e}_z], \quad (41)$$

and Eq. (40) becomes

$$\mathbf{g}^{(R)}(r, 0|0) = \frac{\pi^2 s \eta u_0^2}{\rho c_l^2} [f_{0r}(r, k_R) \mathbf{e}_r + f_{0z}(r, k_R) \mathbf{e}_z], \quad (42)$$

where

$$\begin{aligned} f_{0r}(r, k_R) &= (\xi_l + \eta) \{ \mathcal{H}[\kappa_*^2 J_1(|\kappa_*| r) \text{sgn}(\kappa_*)](k_R) \\ &\quad + ik_R^2 J_1(|k_R| r) \text{sgn}(k_R) \}, \end{aligned} \quad (43)$$

and

$$\begin{aligned} f_{0z}(r, k_R) &= -(1 + \eta \xi_l) \{ \mathcal{H}[\kappa_*^2 J_0(|\kappa_*| r) \text{sgn}(\kappa_*)](k_R) \\ &\quad + ik_R^2 J_0(|k_R| r) \text{sgn}(k_R) \} \\ &= (1 + \eta \xi_l) \nabla_r^2 \{ \mathcal{H}[J_0(|\kappa_*| r) \text{sgn}(\kappa_*)](k_R) \\ &\quad + iJ_0(|k_R| r) \text{sgn}(k_R) \} \\ &= i \text{sgn}(k_R) (1 + \eta \xi_l) \nabla_r^2 [J_0(|k_R| r)] \\ &\quad - i \text{sgn}(k_R) N_0(|k_R| r)] \\ &= -ik_R^2 (1 + \eta \xi_l) H_0^{(2)}(|k_R| r), \end{aligned} \quad (44)$$

for positive ω , where $H_0^{(2)}(\cdot)$ is the zeroth order Hankel function of the second kind and $N_0(\cdot)$ is the zeroth order Bessel function of the second kind. To obtain Eq. (44), we have used the fact that the eigenvalue of the operator ∇_r^2 applied

to the functions $J_0(|k|r)$, $N_0(|k|r)$ and $H_0^{(2)}(|k|r)$ is $-k^2$, as well as the Hilbert transform pair¹⁹

$$\mathcal{H}[J_0(|\kappa_*|r)\text{sgn}(\kappa_*)](k_R) = -N_0(|k_R|r). \quad (45)$$

III. FULL GREEN'S FUNCTION

Touhei¹² has published an expression for a Green's function separated into terms representing Rayleigh and bulk waves. In the present section, we derive a Green's function for a point volume source which includes both bulk and Rayleigh waves, comparable to Eq. (4.4) in Ref. 12. The approach is similar to that presented in Ref. 13, and was applied in Ref. 20 to derive an expression for the displacement in elastic plates with free surfaces.

It proves convenient to derive first the angular spectrum of the Green's function for a force source before proceeding to the Green's function for a volume source, the solution of Eq. (23). The Green's function $\mathbf{g}(r, z|z_0, \omega)$ describing the displacement field due to Rayleigh and bulk waves radiated by a force source located at $(r, z) = (0, z_0)$ satisfies

$$(\mu + \lambda)\nabla\nabla \cdot \mathbf{g} + \mu\nabla^2\mathbf{g} + \rho\omega^2\mathbf{g} = \mathbf{d}\delta(\mathbf{r})\delta(z - z_0), \quad (46)$$

or equivalently

$$(\mu + \lambda)\frac{\partial^2 g_k}{\partial x_i \partial x_k} + \mu\frac{\partial^2 g_i}{\partial x_k^2} + \rho\omega^2 g_i = d_i\delta(\mathbf{r})\delta(z - z_0), \quad (47)$$

where $\mathbf{d} = (d_x, d_y, d_z)$ represents the strength of the force source. Replacing $\mathbf{d}$ by $s\nabla$, or in component form d_i by

FIG. 2. Wave amplitudes for a volume source buried in an elastic half space.

$s\delta_{ik}(\partial/\partial x_k)$, recovers Eq. (23), as done below in Sec. III B. The derivation employs the angular spectrum of the Green's function, $G_i(\kappa, z|z_0)$, which is defined by the Fourier transform pair

$$g_i(r, z|z_0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_i(\kappa, z|z_0) e^{i\kappa \cdot \mathbf{r}} d\kappa_x d\kappa_y, \quad (48)$$

$$G_i(\kappa, z|z_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_i(r, z|z_0) e^{-i\kappa \cdot \mathbf{r}} dx dy. \quad (49)$$

A. Vertical force source

For a vertical force source $\mathbf{d} = (0, 0, d_z)$. In this case, in terms of the angular spectrum, Eqs. (46) and (47) become

$$\begin{aligned} i\kappa(\lambda + \mu) \left(i\kappa \cdot \mathbf{G}_\kappa + \frac{dG_z}{dz} \right) + \mu \left(-\kappa^2 \mathbf{G}_\kappa + \frac{d^2 \mathbf{G}_\kappa}{dz^2} \right) + \rho\omega^2 \mathbf{G}_\kappa &= 0, \\ (\lambda + \mu) \frac{d}{dz} \left(i\kappa \cdot \mathbf{G}_\kappa + \frac{dG_z}{dz} \right) + \mu \left(-\kappa^2 G_\kappa + \frac{d^2 G_z}{dz^2} \right) + \rho\omega^2 G_\kappa &= d_z \delta(z - z_0), \end{aligned} \quad (50)$$

where $\mathbf{G}_\kappa$ is the component of $\mathbf{G}$ along $\mathbf{e}_\kappa$, G_z is the component along $\mathbf{e}_z$, and d_z is the force acting along $\mathbf{e}_z$.

The medium supports both transverse and longitudinal wave motion, respectively, represented by the complex amplitudes T and L in Fig. 2. As Fig. 2 illustrates, the medium may be divided into two regions: one between the source and the surface ($0 \leq z \leq z_0$) and another below the source ($z \leq z_0$). The amplitude subscript signs, when present, were chosen to match those of the arguments of the exponentials in Eqs. (51), (52), (55), and (56).

The general form of the solution satisfying the homogeneous form of Eq. (50) for $0 \leq z \leq z_0$ is¹⁶

$$G_\kappa = \kappa_t (T_+ e^{\kappa_t z} - T_- e^{-\kappa_t z}) + \kappa_l (L_+ e^{\kappa_l z} + L_- e^{-\kappa_l z}), \quad (51)$$

$$G_z = -i[\kappa(T_+ e^{\kappa_t z} + T_- e^{-\kappa_t z}) + \kappa_l(L_+ e^{\kappa_l z} - L_- e^{-\kappa_l z})], \quad (52)$$

where the eigenvalues are

$$\kappa_t = \sqrt{\kappa^2 - k_t^2}, \quad (53)$$

$$\kappa_l = \sqrt{\kappa^2 - k_l^2}, \quad (54)$$

and the wavenumbers are $k_t = \omega/c_t$ and $k_l = \omega/c_l$ for transverse and longitudinal waves, respectively. When κ is less than k_t or k_l the appropriate branch of the square root in Eqs. (53) and (54) must be chosen to ensure that G_κ and G_z decay in the direction of propagation. The notation is such that real values of κ_t and κ_l in Eqs. (51) and (52) correspond to exponentially damped motion (evanescent waves), and imaginary values to propagating waves. The general form of

the solution satisfying the homogeneous form of Eq. (50) for $z \leq z_0$ is

$$G_\kappa = \kappa_t T e^{\kappa_t z} + \kappa_l L e^{\kappa_l z}, \quad (55)$$

$$G_z = -i(\kappa_t T e^{\kappa_t z} + \kappa_l L e^{\kappa_l z}). \quad (56)$$

The six amplitudes T , L , $T_\pm$, and $L_\pm$ in Fig. 2 and Eqs. (51), (52), (55) and (56) are determined by the conditions at $z=0$ and $z=z_0$. The boundary conditions at the surface $z=0$ are represented by Eq. (6), i.e.,

$$(\kappa^2 + \kappa_t^2)(T_+ + T_-) + 2\kappa\kappa_l(L_+ - L_-) = 0, \quad (57)$$

$$2\kappa\kappa_l(T_+ - T_-) + (\kappa^2 + \kappa_t^2)(L_+ + L_-) = 0. \quad (58)$$

At the plane $z=z_0$ containing the source, continuity of displacement dictates that Eq. (51) equal Eq. (55) and Eq. (52) equal Eq. (56), or

$$\kappa_t(T'_+ - T'_- - T') + \kappa(L'_+ + L'_- - L') = 0, \quad (59)$$

$$\kappa(T'_+ + T'_- - T') + \kappa_l(L'_+ - L'_- - L') = 0, \quad (60)$$

where the quantities $T'_\pm = T_\pm e^{\pm\kappa_l z_0}$, $T' = T e^{\kappa_l z_0}$, $L'_\pm = L_\pm e^{\pm\kappa_l z_0}$, and $L' = L e^{\kappa_l z_0}$ are introduced for notational convenience. Another set of conditions is given by integration of Eqs. (50) along the z axis from $z_0 - \varepsilon$ to $z_0 + \varepsilon$ and taking the limit as ε approaches zero. Since $\mathbf{G}$ is continuous at $z=z_0$ the integral of terms in Eq. (50) proportional to G_κ and G_z or their first derivatives with respect to z are zero. The remaining terms in Eq. (50) involving second derivatives of G_κ and G_z with respect to z satisfy

$$\kappa_t^2(T'_+ + T'_- - T') + \kappa\kappa_l(L'_+ - L'_- - L') = 0, \quad (61)$$

$$\kappa\kappa_l(T'_+ - T'_- - T') + \kappa_l^2(L'_+ + L'_- - L') = \frac{id_z k_l^2}{\rho\omega^2}. \quad (62)$$

Equations (59)–(62) may be solved for amplitudes

$$T'_- = -\frac{ikd_z}{2\rho\omega^2\kappa_t}, \quad (63)$$

$$L'_- = -\frac{id_z}{2\rho\omega^2}, \quad (64)$$

representing waves traveling from the source to the surface. Thus T_- and L_- are

$$T_- = -\frac{ikd_z}{2\rho\omega^2\kappa_t} e^{\kappa_t z_0}, \quad (65)$$

$$L_- = -\frac{id_z}{2\rho\omega^2} e^{\kappa_l z_0}, \quad (66)$$

and Eqs. (57) and (58) may be used to express T_+ and L_+ in terms of T_- and L_- as

$$T_+ = \frac{1}{D} \left\{ -\left[(\kappa^2 + \kappa_t^2)^2 + 4\kappa^2\kappa_l\kappa_t \right] T_- + 4\kappa\kappa_l(\kappa^2 + \kappa_t^2) L_- \right\}, \quad (67)$$

$$L_+ = \frac{1}{D} \left\{ 4\kappa\kappa_t(\kappa^2 + \kappa_t^2) T_- - \left[4\kappa^2\kappa_l\kappa_t + (\kappa^2 + \kappa_t^2)^2 \right] L_- \right\}, \quad (68)$$

where

$$D = (\kappa^2 + \kappa_t^2)^2 - 4\kappa^2\kappa_l\kappa_t \quad (69)$$

is the determinant of the system composed of Eqs. (57) and (58). The Green's function in the region $0 \geq z \geq z_0$ may now be found by substituting Eqs. (65)–(68) into Eqs. (51) and (52). However, since a force source is not the focus of the present work we now return to the case of a volume source.

B. Volume source

As mentioned, the solution for a volume source is recovered with the substitution of $s\delta_{ik}(\partial/\partial x_k)$ for d_i in Eq. (47). Due to symmetry about the z axis it is convenient to consider two cases separately, (a) $s_{11}=s_{22}=s$, $s_{33}=0$ and (b) $s_{11}=s_{22}=0$, $s_{33}=s$. Since the problem is linear we may add the results obtained from each case to obtain the full solution for a volume source.

For case (a), Eq. (47) becomes

$$\begin{aligned} i\kappa(\lambda + \mu) \left(i\boldsymbol{\kappa} \cdot \mathbf{G}_\kappa + \frac{dG_z}{dz} \right) + \mu \left(-\kappa^2 \mathbf{G}_\kappa + \frac{d^2 \mathbf{G}_\kappa}{dz^2} \right) + \rho\omega^2 \mathbf{G}_\kappa &= -i\kappa s \delta(z - z_0), \\ (\lambda + \mu) \frac{d}{dz} \left(i\boldsymbol{\kappa} \cdot \mathbf{G}_\kappa + \frac{dG_z}{dz} \right) + \mu \left(-\kappa^2 G_z + \frac{d^2 G_\kappa}{dz^2} \right) + \rho\omega^2 G_z &= 0, \end{aligned} \quad (70)$$

which, after integration along the z axis from $z_0 - \varepsilon$ to $z_0 + \varepsilon$ gives, in the limit $\varepsilon \rightarrow 0$,

$$\kappa_t^2(T'_+ + T'_- - T') + \kappa\kappa_l(L'_+ - L'_- - L') = \frac{i\kappa s}{\mu}, \quad (71)$$

$$\kappa\kappa_l(T'_+ - T'_- - T') + \kappa_l^2(L'_+ + L'_- - L') = 0. \quad (72)$$

Equations (59), (60), (71) and (72) may be used to obtain

$$T_- = -\frac{i\kappa s}{2\rho\omega^2} e^{\kappa_t z_0}, \quad (73)$$

$$L_- = -\frac{i\kappa^2 s}{2\rho\omega^2 \kappa_l} e^{\kappa_l z_0}. \quad (74)$$

For case (b), with $s_{11}=s_{22}=0$ and $s_{33}=s$, Eq. (47) becomes

$$\begin{aligned} i\kappa(\lambda + \mu) \left(i\boldsymbol{\kappa} \cdot \mathbf{G}_\kappa + \frac{dG_z}{dz} \right) + \mu \left(-\kappa^2 \mathbf{G}_\kappa + \frac{d^2 \mathbf{G}_\kappa}{dz^2} \right) + \rho\omega^2 \mathbf{G}_\kappa &= \mathbf{0}, \\ (\lambda + \mu) \frac{d}{dz} \left(i\boldsymbol{\kappa} \cdot \mathbf{G}_\kappa + \frac{dG_z}{dz} \right) + \mu \left(-\kappa^2 G_z + \frac{d^2 G_z}{dz^2} \right) + \rho\omega^2 G_z &= s \frac{d}{dz} \delta(z - z_0). \end{aligned} \quad (75)$$

Comparing the right-hand side of Eq. (75) to that of Eq. (50), it is apparent that the solution for case (b) may be derived from Eqs. (65) and (66) by differentiating with respect to z_0 and replacing d_z with $-s$, i.e.,

$$T_- = \frac{i\kappa s}{2\rho\omega^2} e^{\kappa_l z_0}, \quad (76)$$

$$L_- = \frac{i\kappa_l s}{2\rho\omega^2} e^{\kappa_l z_0}. \quad (77)$$

The right-hand sides of Eqs. (73) and (74) may now be added to those of Eqs. (76) and (77) to determine the coefficients for a volume source:

$$T_- = 0, \quad (78)$$

$$L_- = -\frac{isk_l^2 e^{\kappa_l z_0}}{2\rho\omega^2 \kappa_l}. \quad (79)$$

Substitution of Eqs. (67), (68), (78), and (79), into Eqs. (51) and (52) yields the angular spectrum components for a volume source in the region $z_0 \leq z \leq 0$:

$$\begin{aligned} G_\kappa &= -\frac{is\zeta^2}{2\mu D} \left\{ 4\kappa\kappa_l (\kappa^2 + \kappa_l^2) e^{\kappa_l z} \right. \\ &\quad \left. - \frac{\kappa}{\kappa_l} \left[4\kappa^2 \kappa_l \kappa_l + (\kappa^2 + \kappa_l^2)^2 \right] e^{\kappa_l z} + D \frac{\kappa}{\kappa_l} e^{-\kappa_l z} \right\} e^{\kappa_l z_0}, \end{aligned} \quad (80)$$

$$\begin{aligned} G_z &= -\frac{s\zeta^2}{2\mu D} \left\{ 4\kappa^2 (\kappa^2 + \kappa_l^2) e^{\kappa_l z} \right. \\ &\quad \left. - \left[4\kappa^2 \kappa_l \kappa_l + (\kappa^2 + \kappa_l^2)^2 \right] e^{\kappa_l z} - D e^{-\kappa_l z} \right\} e^{\kappa_l z_0}, \end{aligned} \quad (81)$$

where D is given by Eq. (69). For an observer on the free surface ($z=0$) Eqs. (80) and (81) may be combined and the angular spectrum expressed more compactly as

$$\begin{aligned} \mathbf{G}(\kappa, 0|z_0) &= G_\kappa \mathbf{e}_\kappa + G_z \mathbf{e}_z \\ &= \frac{s\rho\omega^2 \zeta^2 \left[(\kappa^2 + \kappa_l^2)^2 + 4\kappa^2 \kappa_l \kappa_l \right]}{\mu^2 \left[(\kappa^2 + \kappa_l^2)^4 - 16\kappa^4 \kappa_l^2 \kappa_l^2 \right]} \\ &\quad \times \left[2i\kappa\kappa_l \mathbf{e}_\kappa - (\kappa^2 + \kappa_l^2) \mathbf{e}_z \right] e^{\kappa_l z_0} \end{aligned} \quad (82)$$

by evaluating Eqs. (80) and (81) at $z=0$ and multiplying the numerators and denominators by $[(\kappa^2 + \kappa_l^2)^2 + 4\kappa^2 \kappa_l \kappa_l]$.

The polynomial $(\kappa^2 + \kappa_l^2)^4 - 16\kappa^4 \kappa_l^2 \kappa_l^2$ in the denominator of Eq. (82) may be expressed as

$$\begin{aligned} &(\kappa^2 + \kappa_l^2)^4 - 16\kappa^4 \kappa_l^2 \kappa_l^2 \\ &= 8\kappa_l^8 \left[\tilde{\kappa}^4 (3 - 2\zeta^2) - \tilde{\kappa}^2 + \frac{1}{8} - 2\tilde{\kappa}^6 (1 - \zeta^2) \right] \\ &= -16\kappa_l^8 (1 - \zeta^2) (\tilde{\kappa}^2 - \tilde{\kappa}_R^2) (\tilde{\kappa}^4 + C_1 \tilde{\kappa}^2 + C_0), \end{aligned} \quad (83)$$

where

$$\zeta = c_t/c_l, \quad (84)$$

$$\tilde{\kappa} = \kappa/k_t, \quad (85)$$

$$\tilde{\kappa}_l = \kappa_l/k_t = \sqrt{\tilde{\kappa}^2 - \zeta^2}, \quad (86)$$

$$\tilde{\kappa}_t = \kappa_t/k_t = \sqrt{\tilde{\kappa}^2 - 1}, \quad (87)$$

and the fact that the denominator is equal to zero when $\tilde{\kappa}$ is equal to $\tilde{\kappa}_R = k_R/k_t$ was used.¹⁶ Coefficients C_0 and C_1 are given by

$$C_0 = \frac{1}{2(1 - \zeta^2)} \left(1 + \frac{C_1 \tilde{\kappa}_R^2}{8} \right), \quad (88)$$

$$C_1 = \frac{2\zeta^2 - 3}{2(1 - \zeta^2)} + \tilde{\kappa}_R^2. \quad (89)$$

As verified numerically, Eq. (82) possesses no singularities because there are no real roots of the equation $\tilde{\kappa}^4 + C_1 \tilde{\kappa}^2 + C_0 = 0$.

Substitution of Eq. (82) into Eq. (48) and scaling wave numbers and distances by k_t [see Eqs. (85)–(87)] yields the frequency domain Green's function

$$\begin{aligned} \mathbf{g}(r, 0|z_0) &= \frac{s\rho\omega^2 \zeta^2}{64\pi\mu^2 (1 - \zeta^2)} \\ &\quad \times \left[\mathbf{e}_r \int_{-\infty}^{\infty} f_r(|\tilde{\kappa}_*|) \frac{J_1(|\tilde{\kappa}_*| \tilde{r})}{\tilde{\kappa}_* - \tilde{\kappa}_R} \text{sgn}(\tilde{\kappa}_*) d\tilde{\kappa}_* \right. \\ &\quad \left. + \mathbf{e}_z \int_{-\infty}^{\infty} f_z(|\tilde{\kappa}_*|) \frac{J_0(|\tilde{\kappa}_*| \tilde{r})}{\tilde{\kappa}_* - \tilde{\kappa}_R} \text{sgn}(\tilde{\kappa}_*) d\tilde{\kappa}_* \right], \end{aligned} \quad (90)$$

where $\tilde{r} = rk_t$,

$$f_r(\tilde{\kappa}) = 2\tilde{\kappa}_t \left[\frac{(2\tilde{\kappa}^2 - 1)^2 + 4\tilde{\kappa}^2 \tilde{\kappa}_t \tilde{\kappa}_l}{\tilde{\kappa}^4 + C_1 \tilde{\kappa}^2 + C_0} \right] e^{-\tilde{\kappa}_l \tilde{r}/\gamma}, \quad (91)$$

$$f_z(\tilde{\kappa}) = (2\tilde{\kappa}^2 - 1) \left[\frac{(2\tilde{\kappa}^2 - 1)^2 + 4\tilde{\kappa}^2 \tilde{\kappa}_t \tilde{\kappa}_l}{\tilde{\kappa}^4 + C_1 \tilde{\kappa}^2 + C_0} \right] e^{-\tilde{\kappa}_l \tilde{r}/\gamma}, \quad (92)$$

and $\gamma = r/|z_0|$ is the ratio of the radial distance to the observer to source depth.

The Green's function may be separated into terms for bulk and Rayleigh waves and written as the sum

$$\mathbf{g}(r, 0|z_0) = \mathbf{g}^{(b)}(r, 0|z_0) + \mathbf{g}^{(R)}(r, 0|z_0), \quad (93)$$

where

$$\mathbf{g}^{(b)}(r, 0|z_0) = \frac{\rho\omega^2\Lambda}{\pi\mu} \tilde{\mathbf{g}}^{(b)}, \quad (94)$$

$$\mathbf{g}^{(R)}(r, 0|z_0) = \frac{\rho\omega^2\Lambda}{\pi\mu} \tilde{\mathbf{g}}^{(R)}, \quad (95)$$

represent bulk and Rayleigh wave contributions, respectively,

$$\Lambda = \frac{s\zeta^2}{64\mu(1 - \zeta^2)}, \quad (96)$$

and the dimensionless quantities $\tilde{\mathbf{g}}^{(b)}$ and $\tilde{\mathbf{g}}^{(R)}$ are given by

$$\begin{aligned} \tilde{\mathbf{g}}^{(b)} = & \mathbf{e}_r \int_{-\infty}^{\infty} [f_r(|\tilde{\kappa}_*|) - f_r(\tilde{\kappa}_R)] \frac{J_1(|\tilde{\kappa}_*|\tilde{r})}{\tilde{\kappa}_* - \tilde{\kappa}_R} \text{sgn}(\tilde{\kappa}_*) d\tilde{\kappa}_* \\ & + \mathbf{e}_z \int_{-\infty}^{\infty} [f_z(|\tilde{\kappa}_*|) - f_z(\tilde{\kappa}_R)] \frac{J_0(|\tilde{\kappa}_*|\tilde{r})}{\tilde{\kappa}_* - \tilde{\kappa}_R} \text{sgn}(\tilde{\kappa}_*) d\tilde{\kappa}_*, \end{aligned} \quad (97)$$

$$\begin{aligned} \tilde{\mathbf{g}}^{(R)} = & \mathbf{e}_r \int_{-\infty}^{\infty} f_r(\tilde{\kappa}_R) \frac{J_1(|\tilde{\kappa}_*|\tilde{r})}{\tilde{\kappa}_* - \tilde{\kappa}_R} \text{sgn}(\tilde{\kappa}_*) d\tilde{\kappa}_* \\ & + \mathbf{e}_z \int_{-\infty}^{\infty} f_z(\tilde{\kappa}_R) \frac{J_0(|\tilde{\kappa}_*|\tilde{r})}{\tilde{\kappa}_* - \tilde{\kappa}_R} \text{sgn}(\tilde{\kappa}_*) d\tilde{\kappa}_* \\ = & \pi \left[\mathbf{e}_r \left\{ \mathcal{H}[f_r(\tilde{\kappa}_R) J_1(|\tilde{\kappa}_*|\tilde{r}) \text{sgn}(\tilde{\kappa}_*)] (\tilde{\kappa}_R) \right. \right. \\ & + i f_r(\tilde{\kappa}_R) J_1(|\tilde{\kappa}_R|\tilde{r}) \} \\ & + \mathbf{e}_z \left\{ \mathcal{H}[f_z(\tilde{\kappa}_R) J_0(|\tilde{\kappa}_*|\tilde{r}) \text{sgn}(\tilde{\kappa}_*)] (\tilde{\kappa}_R) \right. \\ & \left. \left. + i f_z(\tilde{\kappa}_R) J_0(|\tilde{\kappa}_R|\tilde{r}) \right\} \right] \end{aligned} \quad (98)$$

for positive ω . If the source is located on the surface ($\gamma \rightarrow \infty$) then the component of Eq. (95) along the z axis may be expressed in terms of a Hankel function as

$$g_z^{(R)}(r, 0|0) = i \frac{\Lambda\rho\omega^2}{\pi\mu} f_z(\tilde{\kappa}_R) \Big|_{\gamma \rightarrow \infty} H_0^{(2)}(|\tilde{\kappa}_R|\tilde{r}). \quad (99)$$

IV. EQUIVALENCE OF THE TWO RAYLEIGH WAVE GREEN'S FUNCTIONS

The expressions derived for receivers on the free surface for the Green's functions representing Rayleigh waves derived in Secs. II and III are now proved to be equivalent. The Green's function derived in Sec. II, denoted here as $\mathbf{g}^{(R_1)}$, is given by Eq. (40):

$$\begin{aligned} \mathbf{g}^{(R_1)}(r, 0|z_0) = & \frac{\pi^2 s}{\rho c_l^2} \left\{ \mathcal{H}[\kappa_*^2 \text{sgn}(\kappa_*) \mathbf{f}(r, 0|z_0, |\kappa_*|)] (k_R) \right. \\ & \left. + i k_R^2 \text{sgn}(k_R) \mathbf{f}(r, 0|z_0, |k_R|) \right\}. \end{aligned} \quad (100)$$

The parameters ζ_t , ζ_l , and η given by Eqs. (11)–(13) and present in the expressions for u_0 and $\mathbf{f}(r, 0|z_0, |k_R|)$ may be expressed in terms of the dimensionless quantities $\tilde{\kappa}$, $\tilde{\kappa}_l$, and $\tilde{\kappa}_t$ as

$$\zeta_t = \frac{\tilde{\kappa}_t}{\tilde{\kappa}}, \quad \zeta_l = \frac{\tilde{\kappa}_l}{\tilde{\kappa}}, \quad \eta = -\frac{2\tilde{\kappa}\tilde{\kappa}_t}{2\tilde{\kappa}^2 - 1}. \quad (101)$$

After substitution of Eq. (22) for u_0 and Eq. (36) for the function $\mathbf{f}(r, 0|z_0, |k_R|)$ Eq. (100) may be written as

$$\begin{aligned} \mathbf{g}^{(R_1)}(r, 0|z_0) = & \frac{s\rho\omega^2\zeta^2\tilde{\kappa}^2}{2\mu^2(2\tilde{\kappa}^2 - 1)\Delta} \\ & \times \left\{ \sqrt{\frac{\tilde{\kappa}_t}{\tilde{\kappa}_l}} (1 - i\mathcal{H}) [iJ_1(|\tilde{\kappa}_R|\tilde{r}) e^{\tilde{\kappa}_l \tilde{z}_0} \text{sgn}(\tilde{\kappa}_R)] \mathbf{e}_r \right. \\ & \left. + (1 - i\mathcal{H}) [iJ_0(|\tilde{\kappa}_R|\tilde{r}) e^{\tilde{\kappa}_l \tilde{z}_0} \text{sgn}(\tilde{\kappa}_R)] \mathbf{e}_z \right\}, \end{aligned} \quad (102)$$

where

$$\Delta = 2 + \tilde{\kappa}^2 \left(\frac{1}{\tilde{\kappa}_t^2} + \frac{1}{\tilde{\kappa}_l^2} - \frac{8}{2\tilde{\kappa}^2 - 1} \right) \quad (103)$$

and the identities

$$\zeta_t + \eta = \frac{\sqrt{\tilde{\kappa}_t/\tilde{\kappa}_l}}{2\tilde{\kappa}^2}, \quad (104)$$

$$1 + \zeta_l \eta = \frac{1}{2\tilde{\kappa}^2}, \quad (105)$$

have been used. We have also used the fact that for Rayleigh waves the polynomial given by Eq. (69) equals zero and therefore

$$(2\tilde{\kappa}^2 - 1) = 2\tilde{\kappa}\sqrt{\tilde{\kappa}_t\tilde{\kappa}_l}. \quad (106)$$

The Green's function associated with Rayleigh waves derived in Sec. III, denoted here as $\mathbf{g}^{(R_2)}$, is given by Eq. (95). Substitution of Eqs. (91) and (92) into Eq. (95) yields

$$\begin{aligned} \mathbf{g}^{(R_2)}(r, 0|z_0) = & \frac{s\rho\omega^2\zeta^2}{32\mu^2(1-\zeta^2)} \frac{(2\tilde{\kappa}^2-1)^2}{(\tilde{\kappa}^4+C_1\tilde{\kappa}^2+C_0)} \\ & \times \{2\tilde{\kappa}\tilde{\kappa}_l(1-i\mathcal{H})[iJ_1(|\tilde{\kappa}_R|\tilde{r})e^{\tilde{\kappa}_l\tilde{z}_0}\text{sgn}(\tilde{\kappa}_R)]\mathbf{e}_r \\ & + (2\tilde{\kappa}^2-1)(1-i\mathcal{H})[iJ_0(|\tilde{\kappa}_R|\tilde{r}) \\ & \times e^{\tilde{\kappa}_l\tilde{z}_0}\text{sgn}(\tilde{\kappa}_R)]\mathbf{e}_z\}. \end{aligned} \quad (107)$$

The ratio of the components g_r along $\mathbf{e}_r$ in Eqs. (102) and (107) is

$$\begin{aligned} \frac{g_r^{(R_1)}(r, 0|z_0)}{g_r^{(R_2)}(r, 0|z_0)} = & 16 \frac{\tilde{\kappa}^2(1-\zeta^2)}{(2\tilde{\kappa}^2-1)^3\Delta} (\tilde{\kappa}^4+C_1\tilde{\kappa}^2+C_0) \frac{1}{2\tilde{\kappa}\sqrt{\tilde{\kappa}_l\tilde{\kappa}_l}} \\ = & \frac{(1-\zeta^2)}{\tilde{\kappa}^2\tilde{\kappa}_l^2\tilde{\kappa}_l^2\Delta} (\tilde{\kappa}^4+C_1\tilde{\kappa}^2+C_0), \end{aligned} \quad (108)$$

and the ratio of the components g_z along $\mathbf{e}_z$ is

$$\begin{aligned} \frac{g_z^{(R_1)}(r, 0|z_0)}{g_z^{(R_2)}(r, 0|z_0)} = & 16 \frac{\tilde{\kappa}^2(1-\zeta^2)}{(2\tilde{\kappa}^2-1)^4\Delta} (\tilde{\kappa}^4+C_1\tilde{\kappa}^2+C_0) \\ = & \frac{(1-\zeta^2)}{\tilde{\kappa}^2\tilde{\kappa}_l^2\tilde{\kappa}_l^2\Delta} (\tilde{\kappa}^4+C_1\tilde{\kappa}^2+C_0), \end{aligned} \quad (109)$$

where we have made use of Eq. (106). Note that Eqs. (108) and (109) are equal, i.e., the ratio is the same for both components. Proving that this ratio is equal to unity verifies that Eqs. (102) and (107) are identical.

Consider the polynomial on the right-hand side of Eq. (83):

$$P_d^{(1)} = -16k_t^8(1-\zeta^2)(\tilde{\kappa}^2-\tilde{\kappa}_R^2)(\tilde{\kappa}^4+C_1\tilde{\kappa}^2+C_0). \quad (110)$$

Calculating the derivative of Eq. (110) with respect to $\tilde{\kappa}^2$ at the point $\tilde{\kappa} = \tilde{\kappa}_R$ we find

$$\frac{dP_d^{(1)}}{d\tilde{\kappa}^2} = -16k_t^8(1-\zeta^2)(\tilde{\kappa}^4+C_1\tilde{\kappa}^2+C_0), \quad (111)$$

which is $-16k_t^8$ times the numerator of Eqs. (108) or (109). However, according to the left-hand side of Eq. (83) we may also express the same polynomial as

$$\begin{aligned} P_d^{(2)} = & (\kappa^2 + \kappa_l^2)^4 - 16\kappa^4\kappa_l^2\kappa_l^2 \\ = & k_t^8 \left[(2\tilde{\kappa}^2-1)^2 - 4\tilde{\kappa}^2\tilde{\kappa}_l\tilde{\kappa}_l \right] \left[(2\tilde{\kappa}^2-1)^2 + 4\tilde{\kappa}^2\tilde{\kappa}_l\tilde{\kappa}_l \right]. \end{aligned} \quad (112)$$

The derivative of Eq. (112) with respect to $\tilde{\kappa}^2$ at the point $\tilde{\kappa} = \tilde{\kappa}_R$ may be written, after some manipulation, as

$$\begin{aligned} \frac{dP_d^{(2)}}{d\tilde{\kappa}^2} = & -4k_t^8\tilde{\kappa}_l\tilde{\kappa}_l(2\tilde{\kappa}^2-1)^2 \left(2 + \frac{\tilde{\kappa}^2}{\tilde{\kappa}_l^2} + \frac{\tilde{\kappa}^2}{\tilde{\kappa}_l^2} - \frac{8\tilde{\kappa}^2}{2\tilde{\kappa}^2-1} \right) \\ = & -16k_t^8\tilde{\kappa}^2\tilde{\kappa}_l^2\tilde{\kappa}_l^2\Delta, \end{aligned} \quad (113)$$

which is $-16k_t^8$ times the denominator of Eqs. (108) or (109). Since Eqs. (111) and (113) are equal and are propor-

tional to the numerator and denominator of Eqs. (108) and (109) by the same factor it follows that

$$\frac{g_r^{(R_1)}(r, 0|z_0)}{g_r^{(R_2)}(r, 0|z_0)} = \frac{g_z^{(R_1)}(r, 0|z_0)}{g_z^{(R_2)}(r, 0|z_0)} = 1, \quad (114)$$

proving that the Green's functions for Rayleigh waves obtained in Secs. II and III are identical

V. SIMULATIONS

In this section, time domain forms of the Green's functions derived in Sec. III will be presented to illustrate the relative influence of bulk and Rayleigh waves on motion at the free surface due to a volume source. In the time domain, Eqs. (94) and (95), representing bulk and Rayleigh waves, respectively, are

$$\mathbf{g}^{(b)}(r, \gamma, \tau) = -\frac{\Lambda}{\pi r^2} \mathcal{F}_\tau^{-1} \left\{ \Omega^2 \tilde{\mathbf{g}}^{(b)}(\Omega, \gamma) \right\}, \quad (115)$$

$$\mathbf{g}^{(R)}(r, \gamma, \tau) = -\frac{\Lambda}{\pi r^2} \mathcal{F}_\tau^{-1} \left\{ \Omega^2 \tilde{\mathbf{g}}^{(R)}(\Omega, \gamma) \right\}, \quad (116)$$

where the operator $\mathcal{F}_\tau^{-1}\{\cdot\}$ is the inverse temporal Fourier transform in terms of dimensionless time $\tau = c_t t/r$ and frequency $\Omega = \omega r/c_t = \tilde{r}$.

Examination of Eqs. (97) and (98) shows that the only geometric parameter in $\tilde{\mathbf{g}}^{(b)}(\Omega, \gamma)$ and $\tilde{\mathbf{g}}^{(R)}(\Omega, \gamma)$ is the ratio of the burial depth to observation distance $\gamma = r/|z_0|$, and that the elastic medium is characterized by the ratio

$$\zeta = \frac{c_t}{c_l} = \sqrt{\frac{\mu}{\lambda + 2\mu}}, \quad (117)$$

which is a convenient way to define the relationship between the bulk and shear moduli of an isotropic elastic medium. In the present section we assume that $\zeta = 1/2$. Figures 3 and 4 illustrate the radial (g_r) and vertical (g_z) components, respectively, of Eqs. (115) and (116) as a function of τ for bulk (solid lines) and Rayleigh (dashed lines) waves. In the figures the Green's functions are normalized by

$$\Gamma = \frac{\Lambda}{\pi r^2}, \quad (118)$$

and the plotted quantities are therefore dimensionless. Note that both axes must be scaled appropriately if results are desired for a given medium at a particular location r . The inset plots provide a detailed view of the Green's function for bulk waves in the neighborhood of its peak value.

Three cases are considered: (a) $\gamma = 100$, (b) $\gamma = 10$ and (c) $\gamma = 1$. For a fixed source depth $|z_0|$ decreasing γ corresponds to moving the observation point closer to the source. Note that for each value of γ the amplitudes of the Green's functions for the radial component of displacement (Fig. 3) are moderately larger than the amplitudes for the vertical component (Fig. 4). For large γ (observer far from the source) Figs. 3(a) and 4(a) show that contributions from

FIG. 3. Time-domain Green's functions for displacement parallel to the free surface $z=0$ for bulk (solid) and Rayleigh (dashed) wave modes as a function of dimensionless time τ . The elastic medium is characterized by $\zeta = c_t/c_l = 1/2$ and the Green's functions are scaled by $\Gamma = \Lambda/\pi r^2$. Results are shown for three values of the ratio of observer distance to source depth, $\gamma = r/|z_0| = 100$ (a), 10 (b), and 1 (c). Inset plots show detail of the bulk wave component around its peak.

Rayleigh waves dominate those due to bulk waves. For smaller γ (observer close to the source) parts (b) and (c) of Figs. 3 and 4 show that the bulk wave is dominant. The Green's function for the Rayleigh wave is not visible in parts (b) or (c) of either figure. For example, in Fig. 3(b) the Rayleigh wave arrives at approximately $\tau = 1.0$ and has a maximum amplitude of approximately 0.1, on the order of 10^2 times less than the maximum amplitude of the Rayleigh wave for $\gamma = 100$ shown in Fig. 3(a). Examination of the

FIG. 4. Time-domain Green's functions for displacement perpendicular to the free surface $z=0$ for bulk (solid) and Rayleigh (dashed) wave modes as a function of dimensionless time τ . The elastic medium is characterized by $\zeta = c_t/c_l = 1/2$ and the Green's functions are scaled by $\Gamma = \Lambda/\pi r^2$. Results are shown for three values of the ratio of observer distance to source depth, $\gamma = r/|z_0| = 100$ (a), 10 (b), and 1 (c). Inset plots show detail of the bulk wave component around its peak.

simulation results for the radial component of the Green's function g_r at $\gamma = 10$ [Fig. 3(b)] reveals that the bulk wave amplitude is on the order of 10^3 times larger than the amplitude of the Rayleigh wave. At $\gamma = 1$ [Fig. 3(c)] the bulk wave amplitude is on the order of 10^5 times larger than that of the Rayleigh wave. These relationships are similar for the vertical component of the Green's function g_z shown in Fig. 4. Since locations on the surface approximately above a buried object correspond to $\gamma \simeq 1$ it is clear that accounting for the bulk wave is of critical importance in the context of buried object detection.

VI. SUMMARY

Two different approaches, eigenfunction expansion with respect to surface waves, and angular spectrum decomposition, were used to derive Green's functions for particle displacement in a semi-infinite elastic medium containing a volume source. The Green's function obtained via angular spectrum decomposition was separated into a term representing surface waves and another representing bulk waves. The Green's function obtained via eigenfunction expansion was shown to be equivalent to the term representing surface waves obtained via the angular spectrum approach. Simulations illustrating the impulse response for various values of the ratio between observation distance and source depth showed that displacement at the surface is dominated by surface waves at observation points far from the source and bulk waves for observers close to the source.

ACKNOWLEDGMENTS

This work is supported by the Office of Naval Research and NIH Grant Nos. DK070618 and EB011603.

APPENDIX: SELF-ADJOINTNESS OF $\mathcal{L}$

According to Eq. (2), the operator $\mathcal{L}$ may be written

$$\mathcal{L} = (\mu + \lambda) \frac{\partial^2}{\partial x_i \partial x_k} + \mu \frac{\partial^2}{\partial x_k^2}. \quad (\text{A1})$$

To prove that $\mathcal{L}$ is self adjoint we must show that

$$\begin{aligned} \int_{z < 0} u_i^* \left[(\mu + \lambda) \frac{\partial^2 g_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 g_i}{\partial x_k^2} \right] dV \\ = \int_{z < 0} g_i \left[(\mu + \lambda) \frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 u_i}{\partial x_k^2} \right]^* dV. \end{aligned} \quad (\text{A2})$$

The derivatives on the left-hand side of Eq. (A2) may be expressed as

$$\begin{aligned} u_i^* \frac{\partial^2 g_k}{\partial x_i \partial x_k} &= \frac{\partial}{\partial x_i} \left(u_i^* \frac{\partial g_k}{\partial x_k} \right) - \frac{\partial u_i^*}{\partial x_i} \frac{\partial g_k}{\partial x_k}, \\ u_i^* \frac{\partial^2 g_i}{\partial x_k^2} &= \frac{\partial}{\partial x_k} \left(u_i^* \frac{\partial g_i}{\partial x_k} \right) - \frac{\partial u_i^*}{\partial x_k} \frac{\partial g_i}{\partial x_k}. \end{aligned} \quad (\text{A3})$$

After repeating this procedure once more one may write the left-hand side of Eq. (A2) as

$$\begin{aligned} \int_{z < 0} u_i^* \left[(\mu + \lambda) \frac{\partial^2 g_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 g_i}{\partial x_k^2} \right] dV \\ = (\mu + \lambda) \int_{z < 0} g_k \frac{\partial^2 u_i^*}{\partial x_i \partial x_k} dV + \mu \int_{z < 0} g_i \frac{\partial^2 u_i^*}{\partial x_k^2} dV \\ + \int_{z=0} \left[(\mu + \lambda) \left(u_i^* \frac{\partial g_k}{\partial x_k} - \frac{\partial u_i^*}{\partial x_i} g_k \right) \right. \\ \left. + \mu \left(u_i^* \frac{\partial g_i}{\partial z} - \frac{\partial u_i^*}{\partial z} g_i \right) \right] dS. \end{aligned} \quad (\text{A4})$$

The boundary conditions at $z=0$, Eq. (6) written in terms of the Green's function, are

$$\begin{aligned} \frac{\partial g_x}{\partial z} + \frac{\partial g_z}{\partial x} &= 0, \\ \frac{\partial g_y}{\partial z} + \frac{\partial g_z}{\partial y} &= 0, \\ 2\mu \frac{\partial g_z}{\partial z} + \lambda \left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} \right) &= 0. \end{aligned} \quad (\text{A5})$$

Applying Eqs. (A5) to the surface integral in Eq. (A4) gives

$$\begin{aligned} \int_{z=0} \left[(\mu + \lambda) \left(u_i^* \frac{\partial g_k}{\partial x_k} - \frac{\partial u_i^*}{\partial x_i} g_k \right) + \mu \left(u_i^* \frac{\partial g_i}{\partial z} - \frac{\partial u_i^*}{\partial z} g_i \right) \right] dS \\ = \mu \int_{z=0} \left[\frac{\partial}{\partial x} (u_z^* g_x) + \frac{\partial}{\partial y} (u_z^* g_y) \right. \\ \left. - \frac{\partial}{\partial x} (u_x^* g_z) - \frac{\partial}{\partial y} (u_y^* g_z) \right] dS = 0, \end{aligned} \quad (\text{A6})$$

and therefore we have shown that $\mathcal{L}$ is self-adjoint, i.e.,

$$\begin{aligned} \int_{z < 0} u_i^* \left[(\mu + \lambda) \frac{\partial^2 g_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 g_i}{\partial x_k^2} \right] dV \\ = (\mu + \lambda) \int_{z < 0} g_k \frac{\partial^2 u_i^*}{\partial x_i \partial x_k} dV + \mu \int_{z < 0} g_i \frac{\partial^2 u_i^*}{\partial x_k^2} dV. \end{aligned} \quad (\text{A7})$$

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