Re-expansion method for circular waveguide discontinuities: Application to concentric expansion chambers

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The paper applies the re-expansion method for analyzing planar discontinuities at the junction of two axi-symmetrical circular waveguides. The normal modes in the two waveguides are expanded at the junction plane into a system of functions accounting for velocity singularities at the corner points. As the new expansion has a high convergence order, only a few terms have to be considered for obtaining the solution of most practical problems. This paper gives the equivalent impedance accounting for nonplanar waves into a plane-wave analysis and also the scattering matrix describing the coupling of arbitrary modes at each side of the discontinuity valid in the case of many propagating modes in both sides of the duct. The last section applies the re-expansion technique to some concentric expansion chambers providing an explicit formula for the transmission loss coefficient.

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I. INTRODUCTION

This work is dedicated to analysis of wave scattering by an abrupt change in cross-section (step discontinuity) or thin diaphragms (or by a combination of both of these) in axi-symmetrical parallel cylindrical waveguides having planar junctions. It is a continuation of our recently published paper,¹ which considered the discontinuity problem in the case of rectangular waveguides. The main result shows that accurate numerical results can be obtained by using very few terms in the expansion of the particle axial velocity field in the junction plane by using a Galerkin procedure and choosing the approximating functions so that they correctly model the singularities in the velocity field at the discontinuity points in the junction section of the guide. This provides an efficient means for determining the equivalent impedance of the discontinuity, useful in a plane wave-based analysis for small frequencies. It can also provide the scattering matrix of the discontinuity, which is necessary in the case where many propagating modes exist.

The broad band attenuation of expansion chamber mufflers has led to their extensive use as silencers in pulsating flows including induction and exhaust systems of internal combustion engines, ducts connected to reciprocating pumps, compressors and fans.² Basic elements of expansion chambers are the sudden area discontinuities that occur at the expansion of an inlet tube into a chamber, as well as at its contraction into an outlet tube. At last but not least, the method yields an explicit simplified formula for determining the transmission loss in the expansion chamber.

Miles³ was among the first to investigate the discontinuity problem for acoustical waves in cylindrical tubes. He wrote the fundamental equations governing the propagation of sound near the discontinuity and showed that the sudden discontinuity is analogous to a lumped impedance. Ingard⁴

used the plane piston approximation to study the radiation of sound into a circular tube by taking into account the effect of the termination. Karal⁵ investigated the acoustic inductance for the sudden discontinuities in two infinite circular ducts of different cross section joined together to form an acoustical transmission system. The given formula for the discontinuity impedance is approximate in the sense that his analysis is valid only at very low frequencies. El-Sharkawy and Nayfeh⁶ approached the problem of analytical axi-symmetrical treatment of concentric chambers and also provided comparisons with noise reduction measurements. Peat⁷ studied the effect of evanescent modes for coaxial discontinuities and clarified the valid upper frequency limit of Karal's discontinuity impedance; he determined an expression for the discontinuity impedance which is valid for all frequencies up to the cut-on frequency of nonplanar modes. He includes also in his analysis the convective effect of the mean flow.

Kergomard and Garcia⁸ used the mode matching method for obtaining Karal's correction factor of coaxial discontinuity. They have also discussed in great detail the convergence criteria and the number of modes to be considered for various values of the parameters. Finally, they provided a set of formulae for the case of the radiation of a plane piston into an infinite waveguide in the case of a change in cross-section in a circular guide. Kergomard⁹ further developed the mode matching and scattering matrix methods as applied to planar junctions. The developed "propagating modes" method, for analyzing discontinuities in waveguides, is based on a special partition of the impedance matrices.

The use of the expansion of the solution in series of approximating functions accounting for the correct modeling of singularities was considered in many physical scattering problems. Bases of weighted Jacobi polynomials satisfying the proper edge condition were used in analyzing circular aperture problems by Grigoryan^{10,11} and Lavretsky.¹² Galerkin procedures combined with the approximation of the solution in terms of special polynomials was used by Evans and Fernyhough in Refs. 13 and 14, Kanoria in Refs. 15 and 16,

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and McIver *et al.* in Ref. 17 for modeling similar problems in wave propagation. Some of the techniques developed in these papers were used in the present work for analyzing the planar junctions of axi-symmetrical cylindrical waveguides.

To illustrate the solution procedure we consider the problem of two infinitely long uniform cylindrical waveguides opening into each other through an aperture (a traverse planar junction). The two waveguides have the same symmetry axis, taken as the z -axis, and the discontinuity plane coincides with $z=0$. In order to use the mode-matching method the unknown acoustic fields in each of the two regions will be expressed as mode sums. For the case of straight cylindrical and hard-walled ducts the form of these modes is known.⁵ To solve the problem one must apply the continuity in acoustic pressure and normal velocity across the discontinuity surface $z=0$. Due to velocity singularities at some points on the discontinuity plane the convergence of the infinite mode sums is slow. Therefore, a lot of modes have to be matched before obtaining an acceptable solution. The situation is similar to the classic Gibbs phenomena when the expansion of a nonsmooth function in a finite Fourier series exhibits spurious oscillations near the region where the function is not smooth. The resolution of this phenomena will be obtained by considering a re-expansion of the velocity in the plane $z=0$ in a set of basic polynomials orthogonal with a weight function accounting for the singularity of the velocity.

The very important class of applications of cylindrical waveguides for modeling automotive dissipative silencers is subject to ongoing and extensive research. Thus, Selamet and Radavich in Ref. 18 investigated the effects of the acoustical attenuation performance of concentric expansion chambers by using an analytical approach and also a computational solution based on the boundary element method. Typical examples of recent work in this area include the work by Kirby and Lawrie¹⁹ who developed an analytic method for enforcing the continuity of the pressure and axial velocity for large silencers based on a point collocation approach and Lawrie and Guled²⁰ who used a mode-matching method for investigating the performance of a 2-D silencer which can be tuned by varying the position of an internal membrane. The nonuniform silencers with linings have been considered in Refs. 21 and 22. An application to silencers of elliptical cross section containing mean flow and a perforated pipe was given in Ref. 23. A hybrid numerical method using modal representations for the sound field in the inlet and outlet pipes and a finite element representation in a general shape geometry silencer chamber was reported by Kirby in Ref. 24. The work in Ref. 25 provides a useful overview of the analytic and numerical solution methods commonly employed for the problem of silencers. As Kirby noted in the last citation the numerically based methods (as the boundary element method and hybrid element method) are generally viewed as computationally inefficient for circular silencers and inherently slower than the equivalent analytic techniques. We note also that the presence of discontinuity points for axial velocity at the step points and especially at the tip of the diaphragms introduces supplementary difficulties in numerical methods requiring a huge number of elements for obtaining an acceptable solution.

In the last section of the paper the re-expansion technique was applied to the analysis of concentric expansion chambers involving step discontinuities and flanges (diaphragms). The explicit simplified formula for the transmission loss proposed in this section compares well with some results published in the literature.

Finally, we note that the common point of this approach as compared with that in Ref. 1 is the use of the Galerkin technique and of some special class of orthogonal polynomials for taking into account the edge singularity of velocity. This way the closed form solutions obtained have better convergence properties than classical mode matching methods. The differences between this work and that in Ref. 1 are multiple: while the mathematical apparatus in the rectangular case involves trigonometric functions and Gegenbauer polynomials, in the cylindrical case the solution is expressed as a combination of Bessel functions and a class of special built Jacobi polynomials. Rectangular waveguides containing discontinuities are involved in some acoustical applications but the rectangular “flat” structures have very important applications in microelectronics. On the other hand the cylindrical waveguides are involved in some very important applications in acoustics as the design of automotive dissipative silencers. This paper shows explicitly how the cylindrical waveguides are used for analysis of concentric expansion chambers.

II. THEORETICAL ANALYSIS

A. General relationships

Because Karal’s work is most often quoted the analysis given here follows that in Ref. 5. The analysis is restricted to two infinite coaxial ducts of circular cross section opening into each other through a traverse planar junction (aperture) Fig. 1. We denote by $P(\mathbf{x}, t)$ the pressure and consider the t -periodic solutions

$$P(\mathbf{x}, t) = p(\mathbf{x})e^{i\omega t}, \quad (1)$$

where ω is the angular frequency. Then, the complex pressure $p(\mathbf{x})$ satisfies the Helmholtz equation,

$$\nabla^2 p + k^2 p = 0, \quad (2)$$

$k = \omega/\tilde{c}$ being the wave number and $\tilde{c}$ denoting the speed of sound.

FIG. 1. (Color online) A planar junction between two circular axi-symmetric waveguides.

In addition, to make the problem more general, and inspired by the 2-D geometry considered by Schwinger and Saxon in Ref. 26, a zero thickness obstacle will be inserted in the junction plane as indicated in Fig. 2. The radii of ducts are denoted by a and c and the radius of the aperture is b . A cylindrical coordinate system is introduced in such a way that the z -axis coincides with the axes of the ducts and the origin is taken in the plane of discontinuity.

By assuming rigid duct wall boundary conditions, a separation of variables solution can be obtained for the pressure $p(r, z)$ for the two separate ducts, A and C,

$$p_A(r, z) = A_0^r e^{ikz} + A_0^a e^{-ikz} + \sum_{n=1}^{\infty} (A_n^r e^{\gamma_{a,n} z} + A_n^a e^{-\gamma_{a,n} z}) \times J_0\left(\frac{x_n r}{a}\right), \quad z < 0, \quad (3)$$

$$p_C(r, z) = C_0^r e^{-ikz} + C_0^a e^{ikz} + \sum_{n=1}^{\infty} (C_n^r e^{-\gamma_{c,n} z} + C_n^a e^{\gamma_{c,n} z}) \times J_0\left(\frac{x_n r}{c}\right), \quad z > 0, \quad (4)$$

where

$$\gamma_{a,n} = \left[\left(\frac{x_n}{a} \right)^2 - k^2 \right]^{1/2} = i \left[k^2 - \left(\frac{x_n}{a} \right)^2 \right]^{1/2} \equiv ik_{an}, \quad (5)$$

$$\gamma_{c,n} = \left[\left(\frac{x_n}{c} \right)^2 - k^2 \right]^{1/2} = i \left[k^2 - \left(\frac{x_n}{c} \right)^2 \right]^{1/2} \equiv ik_{cn}. \quad (6)$$

The numbers x_n are the roots of the equation

$$J'_0(x_n) = J_1(x_n) = 0. \quad (7)$$

The axial components of the acoustic velocity, v , are given by

$$v_A(r, z) = \frac{-1}{i\omega\rho} \frac{\partial p_A}{\partial z} = \frac{k}{\omega\rho} (A_0^a e^{-ikz} - A_0^r e^{ikz}) + \sum_{n=1}^{\infty} \frac{\gamma_{a,n}}{i\omega\rho} (A_n^a e^{-\gamma_{a,n} z} - A_n^r e^{\gamma_{a,n} z}) J_0\left(\frac{x_n r}{a}\right), \quad z \leq 0, \quad (8)$$

FIG. 2. (Color online) A longitudinal section of the waveguide in Fig. 1.

$$v_C(r, z) = \frac{-1}{i\omega\rho} \frac{\partial p_C}{\partial z} = \frac{k}{\omega\rho} (C_0^r e^{-ikz} - C_0^a e^{ikz}) + \sum_{n=1}^{\infty} \frac{\gamma_{c,n}}{i\omega\rho} (C_n^r e^{-\gamma_{c,n} z} - C_n^a e^{\gamma_{c,n} z}) J_0\left(\frac{x_n r}{c}\right), \quad z \geq 0, \quad (9)$$

Thus, at the plane of discontinuity, $z=0$, we have

$$p_A(r, 0-) = \sum_{n=0}^{\infty} (A_n^r + A_n^a) J_0\left(\frac{x_n r}{a}\right), \quad 0 \leq r \leq b, \quad (10)$$

$$p_C(r, 0+) = \sum_{n=0}^{\infty} (C_n^r + C_n^a) J_0\left(\frac{x_n r}{c}\right), \quad 0 \leq r \leq b, \quad (11)$$

$$v_A(r, 0-) = \sum_{n=0}^{\infty} Y_{a,n} (A_n^a - A_n^r) J_0\left(\frac{x_n r}{a}\right), \quad 0 \leq r < b, \quad (12)$$

$$v_C(r, 0+) = \sum_{n=0}^{\infty} Y_{c,n} (C_n^r - C_n^a) J_0\left(\frac{x_n r}{c}\right), \quad 0 \leq r < b, \quad (13)$$

where

$$Y_{a,0} \equiv \frac{\gamma_0}{i\omega\rho} = \frac{1}{\tilde{c}\rho}; \quad Y_{a,n} \equiv \frac{\gamma_{a,n}}{i\omega\rho} = \frac{\sqrt{x_n^2 - (ak)^2}}{i\tilde{c}\rho(ak)}, \quad n > 0, \quad (14)$$

and similar relationships for $Y_{c,n}$, $\gamma_{c,n}$, $Y_{d,n}$, $\gamma_{d,n}$.

The boundary conditions at the discontinuity surface are the following: the continuity of the pressure and axial particle velocity on the aperture

$$p_A(r, 0-) = p_C(r, 0+), \quad 0 \leq r < b, \quad (15)$$

$$v_A(r, 0-) = v_C(r, 0+), \quad 0 \leq r < b. \quad (16)$$

The other walls of the pipe (A) and pipe (C) included in the discontinuity plane $z=0$ are assumed as being solid impermeable surfaces. These can be solid walls, as in case of step discontinuities, or solid diaphragms of zero thickness, which model a possible flange. We can have also a combination of these two type elements. In both cases the axial particle velocity (normal to these elements) cancels out:

$$v_A(r, 0-) = 0, \quad b < r \leq a, \quad (17)$$

$$v_C(r, 0+) = 0, \quad b < r \leq c. \quad (18)$$

As will be shown in Sec. II B, in this model, the only difference between step discontinuity and diaphragm cases is the strength of the singularity in the axial velocity field. Denoting by $\tilde{v}(r)$ the axial velocity in the plane $z=0$, we can write

$$v_A(r, 0-) = \begin{cases} \tilde{v}(r), & 0 \leq r < b \\ 0, & b < r \leq a \end{cases}, \quad (19)$$

$$v_C(r, 0+) = \begin{cases} \tilde{v}(r), & 0 \leq r < b \\ 0, & b < r \leq c \end{cases}, \quad (20)$$

The formulas (15), (10), and (11) yield the basic equation of the problem

$$\sum_{n=0}^{\infty} (A_n^r + A_n^a) J_0\left(\frac{x_n r}{a}\right) = \sum_{n=0}^{\infty} (C_n^a + C_n^r) J_0\left(\frac{x_n r}{c}\right), \quad 0 \leq r < b. \quad (21)$$

B. The velocity in the aperture

The pressure and particle velocity at the plane $z=0$ have a special behavior. As has been shown in Fig. 16 and Fig. 17 in Ref. 20 (in a 2-D context) the axial velocity in the gap has some oscillations and a singularity at the point $r=b$. The authors have pointed out that by using the usual mode representations in the two pipes the situation cannot be improved by increasing the number of summed terms, a situation which reminds us of the Gibbs phenomenon. The authors claim that the used representations do not describe accurately the integrable singularities inherent to the fluid velocity at the point $z=0, r=b$.

The analysis by Evans and Fernyough,¹³ Kanoria,¹⁵ and Homencovschi and Miles¹ show that in the case of a step discontinuity the axial velocity has around the point $r=b$ a singular behavior of the type $\tilde{v}(r) = K(b-r)^{-1/3}$ (K being a constant). Also, in the case of a diaphragm of zero thickness, the axial velocity in the gap, around the point $r=b$ can be written as $\tilde{v}(r) = K(b-r)^{-1/2}$. In Ref. 1 this has been proved for the 2-D case but it can be shown that it applies also in the case of a circular edge considered in this work. More generally, in an electromagnetic context, Van Bladel in Ref. 27 has shown that when the edge is curved instead of straight, curvature effects can be neglected as long as r remains small with respect to the radius of curvature of the edge.

In order to obtain a closed form solution, accounting for the algebraic order singularity of the particle velocity at the point $r=b$, we have to consider a series expansion of this function in Jacobi-type polynomials. We write for the particle velocity in the gap a representation formula of the form

$$\tilde{v}(r) = \left[1 - \left(\frac{r}{b}\right)^2\right]^{\mu-1} \sum_{j=0}^{\infty} \tilde{v}_j^{(\mu)} \phi_j^{(\mu)}[(r/b)^2], \quad 0 \leq r < b, \quad (22)$$

where $\{\phi_n^{(\mu)}(t)\}$ are the complete system of functions defined in the interval $[0,1]$ by formula (B1) in Appendix B and $\tilde{v}_n^{(\mu)}$ being the expansion coefficients. As we can write

$$\left[1 - \left(\frac{r}{b}\right)^2\right]^{\mu-1} = \frac{(b-r)^{\mu-1}(b+r)^{\mu-1}}{b^{2\mu-2}},$$

the chosen representation formula for velocity in the junction plane accounts for the velocity behavior at the edges in both cases considered. Thus, in the case of a step discontinuity the value of parameter μ equals $2/3$ while in the case of a diaphragm of zero thickness we have $\mu = 1/2$.

Then, using the orthogonality property of Bessel functions, Eqs. (12), (19), and (B4) yield

$$Y_{a,n}(A_n^a - A_n^r) = \frac{2}{a^2 J_0^2(x_n)} \sum_{j=0}^{\infty} \tilde{v}_j \int_0^b r \left[1 - \left(\frac{r}{b}\right)^2\right]^{\mu-1} \times \phi_j^{(\mu)}[(r/b)^2] J_0\left(\frac{x_n r}{a}\right) dr. \quad (23)$$

By using formula (B4) in Eq. (23), we obtain

$$Y_{a,0}(A_0^a - A_0^r) = \frac{b^2}{a^2} \frac{v_0^{(\mu)}}{\Gamma(\mu+1)}. \quad (24)$$

Similarly,

$$Y_{a,n}(A_n^a - A_n^r) = \frac{b^2}{a^2 J_0^2(x_n)} \sum_{j=0}^{\infty} \tilde{J}_{\mu+2j} \left(x_n \frac{b}{a}\right) v_j^{(\mu)}, \quad (25)$$

where

$$v_j^{(\mu)} = \tilde{v}_j^{(\mu)} \frac{\Gamma(\mu+j)}{j!}. \quad (26)$$

Here we have used the notation

$$\tilde{J}_{\nu}(z) = \frac{J_{\nu}(z)}{(z/2)^{\nu}}. \quad (27)$$

The formula

$$\lim_{z \rightarrow 0} \tilde{J}_{\nu+j}(z) = \frac{\delta_{j,0}}{\Gamma(\nu+1)}, \quad (28)$$

where $\delta_{p,j}$ is the Kronecker delta function, will be used many times in obtaining the zero-order terms. Particularly, it was used in obtaining the formula (24).

Similarly, Eqs. (13) and (20) give

$$Y_{c,0}(C_0^r - C_0^a) = \frac{b^2}{c^2} \frac{v_0^{(\mu)}}{\Gamma(\mu+1)}, \quad (29)$$

$$Y_{c,n}(C_n^r - C_n^a) = \frac{b^2}{c^2 J_0^2(x_n)} \sum_{j=0}^{\infty} \tilde{J}_{\mu+2j} \left(x_n \frac{b}{c}\right) v_j^{(\mu)}, \quad (30)$$

To determine the coefficients $v_n^{(\mu)}$ we apply a Galerkin procedure by multiplying Eq. (21) (where $r=bt$) by $t(1-t^2)^{\mu-1} \phi_p^{(\mu)}(t^2)$ and integrating along the interval $(0, 1)$. Using again the orthogonality formula (B4) we obtain

$$\sum_{n=0}^{\infty} (A_n^r + A_n^a) \tilde{J}_{\mu+2p} \left(x_n \frac{b}{a}\right) = \sum_{n=0}^{\infty} (C_n^r + C_n^a) \tilde{J}_{\mu+2p} \left(x_n \frac{b}{c}\right), \quad p = 0, 1, 2, \dots, \quad (31)$$

which is the basic equation for determining the pressure and the velocity in the acoustical cylindrical transmission system.

III. DETERMINATION OF THE DISCONTINUITY IMPEDANCE

At the low frequencies, along a uniform duct section, only plane waves can propagate and thus plane-wave theory

is generally adequate. However, at a duct discontinuity, evanescent non-planar waves are created which cause a noticeable difference between the results from basic plane-wave analysis and the experimental results. The effect of these nonplanar wave modes can be incorporated into a plane-wave analysis by the employment of an equivalent acoustical impedance at the discontinuity introduced by Miles.³ The purpose of the present section is to determine an expression for the discontinuity impedance which is valid for all frequencies, up to the cut-on frequency of non-planar modes, with a reduced computational effort.

A. Calculation of the velocity in aperture

In the case where $ak < x_n$, $ck < x_n$ ($n=1, 2, 3, \dots$) in both waveguides (A) and (C) only the lowest plane wave propagate. Therefore,

$$A_n^a = C_n^a = 0, \quad n = 1, 2, 3, \dots$$

Also we can write

$$A_0^r + A_0^a = A_0, \quad C_0^r + C_0^a = C_0$$

assuming general termination conditions different from the anechoic termination of duct (C) supposed by Karal in Ref. 5.

Formula (31) becomes

$$\begin{aligned} \sum_{n=1}^{\infty} C_n^r \tilde{J}_{\mu+2p} \left(x_n \frac{b}{c} \right) - \sum_{n=1}^{\infty} A_n^r \tilde{J}_{\mu+2p} \left(x_n \frac{b}{a} \right) \\ = \frac{A_0 - C_0}{\Gamma(\mu+1)} \delta_{p,0} \quad p = 0, 1, 2, \dots \end{aligned} \quad (32)$$

By substituting the coefficients A_n^r and C_n^r given by formulas (25) and (30), Eq. (32) becomes

$$\sum_{j=0}^{\infty} \left[\Delta_{pj}^{(\mu)}(a) + \Delta_{pj}^{(\mu)}(c) \right] v_j^0 = \delta_{p,0}, \quad p = 0, 1, 2, \dots, \quad (33)$$

where it has been denoted

$$\Delta_{pj}^{(\mu)}(a) = \frac{b}{a} \sum_{n=0}^{\infty} \frac{\tilde{J}_{\mu+2p}(x_n b/a) \tilde{J}_{\mu+2j}(x_n b/a)}{(x_n^2 - (ak)^2)^{1/2} J_0^2(x_n)}, \quad (34)$$

and the new unknown quantities v_j^o are connected to $v_j^{(\mu)}$ by

$$v_j^{(\mu)} = \frac{(C_0 - A_0)}{i\omega\rho b\Gamma(\mu+1)} v_j^0, \quad j = 0, 1, 2, \dots \quad (35)$$

The relationships (33) give an infinite system of equations for determining the coefficients v_j^o determining the axial velocity in the aperture. As in the above analysis we took into consideration the proper behavior of the velocity at the aperture edges the resulting system has good convergence properties.

B. The discontinuity inductance

Following, Karal,⁵ we define the volume velocity V as

$$V = 2\pi \int_0^a v_A(r, z) r dr = 2\pi \int_0^c v_C(r, z) r dr. \quad (36)$$

It is easy to prove that V is a constant independent of z . Indeed we have

$$\int_0^a J_0 \left(\frac{x_n r}{a} \right) r dr = \left(\frac{a}{x_n} \right)^2 \int_0^{x_n} J_0(t) t dt = \left(\frac{a}{x_n} \right)^2 [t J_1(t)]_0^{x_n} = 0 \quad (37)$$

The relation (37) complete with the conditions (16) and (18) provide the enunciated independence. Therefore,

$$V = \pi a^2 Y_{a,0} A_0 = \pi c^2 Y_{c,0} C_0. \quad (38)$$

Equation (21) yields

$$A_0 - C_0 = \sum_{m=1}^{\infty} C_m^r J_0 \left(\frac{x_m r}{c} \right) - \sum_{n=1}^{\infty} A_n^r J_0 \left(\frac{x_n r}{a} \right), \quad (39)$$

Here $A_0 = (p_{0A})_{z=0}$ is the zeroth mode pressure in the domain (A) and $C_0 = (p_{0C})_{z=0}$ is the corresponding pressure mode in the region (C). Equation (39) indicates that the pressure in the zeroth order mode in the two regions is discontinuous across the section $z=0$. The discontinuity of pressure will be represented by a lumped impedance at the change of cross section

$$(p_0 A)_{z=0} - (p_0 C)_{z=0} = VZ, \quad (40)$$

where Z is an analogous acoustical impedance. Equations (40), (38), and (32) provide the formula for the acoustical impedance of the discontinuity as

$$Z = \frac{\mu(A_0 - C_0)}{\pi b^2 \tilde{v}_0}. \quad (41)$$

Formula (35) gives

$$Z = -i\omega\rho \frac{\Gamma^2(\mu+1)}{\pi b v_0^0}, \quad (42)$$

and hence

$$\frac{Lb}{\rho} = \frac{\Gamma^2(\mu+1)}{\pi v_0^0}, \quad (43)$$

L being the analogues acoustical inductance of the discontinuity.

C. Examples

1. Comparison with some results in analytical form for zero frequency

In the case of cylindrical geometry we miss the tool of conformal representation and, consequently, the analytical results are very rare. Fock in Ref. 30 obtained a good approximation (in the case of zero frequency) for the zero

thickness diaphragms, in an axi-symmetrical cylindrical waveguide of constant section, as a series expansion for small values of parameter $\alpha = b/a$. This solution can be also found in the book by Rschevkin³¹ as

$$L^* \frac{2b}{\rho} = 1 + \sum_{n=1}^{\infty} c_n \alpha^n, \quad (44)$$

where the first coefficients c_n are as follows:

$$\begin{aligned} c_1 &= -1.40925, & c_2 &= 0, & c_3 &= 0.33818, & c_4 &= 0 \\ c_5 &= 0.06793, & c_6 &= -0.02287, & c_7 &= 0.03015, \\ c_8 &= -0.01641 \\ c_9 &= 0.01729, & c_{10} &= -0.01248, & c_{11} &= 0.01205, \\ c_{12} &= -0.00985. \end{aligned}$$

Another asymptotic approximation, valid around $\alpha = 1$, was provided by Leppington and Levine³² in the form

$$L^* \frac{b}{\rho} = \frac{8\varepsilon^2}{16 + 3\varepsilon^2 \log \varepsilon}, \quad \varepsilon = 1 - \alpha. \quad (45)$$

The discontinuity impedance resulting by using the re-expansion method is given by the general formula (43) where the coefficient v_0^0 is obtained by solving the infinite system of Eqs. (33). As we stated before, in this case we have $\mu = 1/2$. In Fig. 3 is plotted the variation of the coefficient Lb/ρ with respect to the ratio b/a : the coefficient resulting from formula (43) when in the system (33) is considered a single equation ($N=1$) is shown by the broken line and the coefficient resulting when $N=2$ is plotted as a continuous line. In the same figure the results obtained by using Fock's formula are shown as circles (●) while the coefficients given by Leppington and Levin's formula are plotted as full squares (■). Even in the case of retaining in the infinite system of only two equations, the obtained inductance coefficient is a good approximation of that provided by the

FIG. 3. (Color online) A comparison of results given by the re-expansion formula with one term (dotted line) and two terms (continuous line) with the analytical results of Fock in Ref. 30 (full circles) and asymptotic results of Leppington and Levine in Ref. 32 (full squares).

analytical formula given by Fock. On the other hand, expression (45) provided by Leppington and Levine, was found to be a good approximation only for values of the ratio α very close to 1.

2. Some results for a diaphragm at non-zero frequency

Table I in the paper by Kergomard and Garcia⁸ contains approximate values of the inductance for a zero-thickness, circular diaphragm in a cylindrical waveguide for frequencies close to cut-off compared to the value at zero frequency. The lines in Table I are plotted as scattered data in Fig. 4. Thus the data in the third line, corresponding to the frequency $ka = 3.5$ are shown as full squares (■), the data for the frequency $ka = 3.831$ (the second line in Table I in Ref. 8 are plotted as circles (●) and, finally, the first line containing the coefficients $L(f_c = 3.831706)/L(0)$ is shown in Fig. 4 as full triangles (▲). In Fig. 4 the continuous lines show the ratio of the inductance coefficients obtained by using the re-expansion technique [formula (43) and the system (33)] and truncating the infinite system at $N=2$. As can be seen, the re-expansion method yields good approximations to the inductance coefficients in all the cases but close to the cut-off frequency and large discontinuities. However, as stated in Ref. 8 in Table I the results for small values of α and close to the cut-off frequency are questionable.

3. Comparison with some results for step discontinuity and non-zero frequency

Peat in Ref. 7 analyzed the effects of a superimposed mean flow along a cylindrical circular duct (having a step discontinuity in its cross-section) and also the effects of high frequencies up to the plane-wave cutoff limit. While the mean flow effects were found negligible the variation of impedance with frequency is important. In Fig. 5 the line plots correspond to the curves in Fig. 4 in Peat's paper Ref. 7 with the difference that instead of plotting the $H_T(x)$ curves we plotted the inductance coefficient Lb/ρ . Since in this case we

FIG. 4. (Color online) A comparison of results given by re-expansion formula for non-zero frequency with two terms retained in the linear system (continuous lines) with the results given by Kergomard and Garcia,⁸ in the case of a diaphragm, for different frequencies (full triangles, full circles, and full squares).

FIG. 5. (Color online) Comparison of the results obtained by Peat in Ref. 7, Fig. 5, for the step discontinuity in a cylindrical waveguide by solving a system of 20 equations, with the results given by the re-expansion method, with two terms, for different frequencies.

have an abrupt change of cross section of the circular waveguide we have $b = a$ and $\mu = 2/3$ and, again, the inductance coefficient can be determined by using the formula (43) and the system (33). The infinite system was truncated to two equations (with two unknowns) and the results are plotted as scattered data in Fig. 5 for different frequencies: $kc = 0.3$ ($\blacktriangleright$), $kc = 1.5$ ($\blacktriangleleft$), $kc = 2.4$ ($\blacktriangledown$), $kc = 3.0$ ($\blacktriangle$), $kc = 3.3$ ($\bullet$) and $kc = 3.6$ ($\blacksquare$). It is clear that for all the considered frequencies (up to the first cut-off limit) the results given by the re-expansion method coincide very well with those given by Peat (who solved a linear system of 20 equations) despite the fact that the infinite system was truncated to $N = 2$.

IV. THE SCATTERING MATRIX OF DISCONTINUITY

In this section a matrix solution is given which represents purely the discontinuity untied from the influence of sound source and reflecting terminations in the duct. This is the scattering matrix which describes the coupling of arbitrary modes at each side of the discontinuity. As Hudde and Letens show in Ref. 28, once having determined the scattering matrix, arbitrary arrangements with different sources and reflection coefficients of each higher mode can easily be calculated. Also, the solution remains valid at high frequencies when many propagating modes exist in both parts of the waveguide.

A. The matrix form of the basic equations

The formulas (25) and (30) are obtained by taking the projections of the velocity $\tilde{v}(r)$ on the orthogonal system for the duct (A) defined in Appendix A and, also, on the corresponding system for the waveguide (C). Denoting by ϕ_j the elements of an orthogonal basis we have

$$\sum_{j=0}^{\infty} Y_j A_j \phi_j(r) = v(r). \quad (46)$$

Taking the inner product with ϕ_k , we obtain

$$Y_k A_k \|\phi_k(r)\|^2 = \langle v(r), \phi_k(r) \rangle, \quad (47)$$

which determines the modal amplitudes A_k .

By introducing the matrices

$$[Q_a] = \text{diag} \left(\left(\frac{a}{b} \right)^2 J_0^2(x_n) Y_{a,n} \right), \quad n = 0, 1, 2, \dots, M,$$

$$[Q_c] = \text{diag} \left(\left(\frac{c}{b} \right)^2 J_0^2(x_n) Y_{c,n} \right), \quad n = 0, 1, 2, \dots, M,$$

$$[D_a] = [\tilde{J}_{\mu+2j}(x_n b/a)]_{j,n=0,1,2,\dots,M},$$

and the vectors

$$\mathbf{a}^a = [A_0^a, A_1^a, A_2^a, \dots, A_M^a]^T, \quad \mathbf{a}^r = [A_0^r, A_1^r, A_2^r, \dots, A_M^r]^T,$$

$$\mathbf{c}^a = [C_0^a, C_1^a, C_2^a, \dots, C_M^a]^T, \quad \mathbf{c}^r = [C_0^r, C_1^r, C_2^r, \dots, C_M^r]^T,$$

$$\mathbf{v} = [v_0, v_1, v_2, \dots, v_M]^T,$$

the relationships (25), (30), and (31) become

$$[Q_a](\mathbf{a}^a - \mathbf{a}^r) = [D_a]^T \mathbf{v}, \quad (48)$$

$$[Q_c](\mathbf{c}^a - \mathbf{c}^r) = [D_c]^T \mathbf{v}, \quad (49)$$

$$[D_a](\mathbf{a}^a + \mathbf{a}^r) = [D_c](\mathbf{c}^a + \mathbf{c}^r). \quad (50)$$

B. The scattering matrix

The matrix description of the discontinuity can be realized by means of reflection and transmission coefficients defined as scattering parameters relating the receding and approaching modal amplitudes. By using the definitions given in Hudde and Letens in Ref. 28 and Muehleisen and Swanson in Ref. 29 we can write

$$a_m^r = \sum_n S_{mn}^{11} a_n^a + \sum_n S_{mn}^{12} c_n^a, \quad (51)$$

$$c_m^r = \sum_n S_{mn}^{21} a_n^a + \sum_n S_{mn}^{22} c_n^a. \quad (52)$$

In these relationships, the coefficients S_{mn}^{11} , S_{mn}^{22} represent reflection coefficients from mode n to mode m in waveguide (A), (C), respectively. The coefficient S_{mn}^{12} describes the transmission from mode n in duct (C) to the mode m in the waveguide (A). Finally, S_{mn}^{21} is the transmission coefficient from mode n in waveguide (A) to the mode m in the duct (C). The relationships (51) and (52) will be written in matrix form as

$$\mathbf{a}^r = [S^{11}] \mathbf{a}^a + [S^{12}] \mathbf{c}^a, \quad (53)$$

$$\mathbf{c}^r = [S^{12}] \mathbf{a}^a + [S^{22}] \mathbf{c}^a. \quad (54)$$

The scattering coefficients S_{mn}^{11} , S_{mn}^{12} , S_{mn}^{21} , and S_{mn}^{22} are independent of the values of the modal amplitudes of approaching modes $\mathbf{a}^a$ and $\mathbf{c}^a$.

Particularly, by taking $\mathbf{c}^a = 0$ the formulas (48) and (49) yield

$$\mathbf{v} = [D_a^T]^{-1} [Q_a] [I - S^{11}] \mathbf{a}^a, \quad (55)$$

$$\mathbf{v} = [D_c^T]^{-1} [Q_c] [S^{21}] \mathbf{a}^a, \quad (56)$$

and hence,

$$[S^{21}] = [Q_c]^{-1} [D_c^T] [D_a^T]^{-1} [Q_a] [I - S^{11}]. \quad (57)$$

Also, formula (50) yields

$$[D_a] [I + S^{11}] = [D_c] [S^{21}], \quad (58)$$

and relationship (57) gives

$$[D_a] [I + S^{11}] = [H_a] [I - S^{11}], \quad (59)$$

where has been denoted

$$[H_a] = [D_c] [Q_c]^{-1} [D_c^T] [D_a^T]^{-1} [Q_a]. \quad (60)$$

Finally, Eqs. (59) and (58) give

$$[S^{11}] = [H_a + D_a]^{-1} [H_a - D_a], \quad (61)$$

$$[S^{21}] = 2[D_c]^{-1} [D_a] [H_a + D_a]^{-1} [H_a]. \quad (62)$$

By taking $\mathbf{a}^a = 0$, we obtain similarly

$$[S^{22}] = [H_c + D_c]^{-1} [H_c - D_c], \quad (63)$$

$$[S^{12}] = 2[D_a]^{-1} [D_c] [H_c + D_c]^{-1} [H_c], \quad (64)$$

where has been denoted

$$[H_c] = [D_a] [Q_a]^{-1} [D_a^T] [D_c^T]^{-1} [Q_c]. \quad (65)$$

The relationships (61)–(64) determine completely the scattering matrix of discontinuity.

C. Case study: The scattering matrix of an axi-symmetrical step discontinuity

We analyze the coupling coefficients for an axi-symmetrical junction of two circular ducts in Fig. 1 (having only step discontinuities) and calculate the reflection and transmission coefficients. This type of junction is regularly encountered in noise control at higher frequencies.

It is important to know how much of a plane wave is converted into higher order modes at a discontinuity and also how to design a series of planar discontinuities to convert plane waves to higher order modes where the energy can be more efficiently dissipated. In the case analyzed we have $b = c$ and $\mu = 2/3$.

In Fig. 6 the magnitude and phase of the plane wave reflection coefficient of the larger waveguide (A), S_{00}^{11} , for three different ratios $c/a = 0.25$, $c/a = 0.5$, and $c/a = 0.75$ are presented as functions of ka . The reflection and transmission coefficients S_{00}^{11} and S_{00}^{21} can be validated using a simple low

frequency approximation outlined in Ref. 20. In our case it is necessary to take two modes in pipe (A) and only one mode in pipe (C). By matching the pressure and velocity in the aperture whilst the average velocity is taken to be zero at $z = 0$, $b \leq r \leq a$ we obtain

$$A_0^r = \frac{(a^2/c^2 - 1)(1 + iak/R_1(a))}{a^2/c^2 + 1 + (a^2/c^2 - 1)iak/R_1(a)} \quad (66)$$

$$C_0^r = \frac{2}{a^2/c^2 + 1 + (a^2/c^2 - 1)iak/R_1(a)} \quad (67)$$

with the notation

$$R_n(a) = \sqrt{x_n^2 - (ak)^2}, x_1 = 3.83171. \quad (68)$$

At low frequencies $|S_{00}^{11}|$ is well approximated by the limit of plane wave reflection coefficient $R = (a^2 - c^2)/(a^2 + c^2)$ given by the formula (66) for $ak \ll 1$. Also the plots of the simple low frequency approximations of the amplitude of the reflection coefficients for $c/a = 0.25$; 0.5; 0.75 given by formula (66) are included in Fig. 6(a) as thin lines (blue online). As the frequency approaches the cutoff frequency of the first higher order mode in the waveguide (A) at $ka = 3.83\dots$, the evanescent higher modes begin to store part of the energy from the incident plane wave which can be seen in the increase in the magnitude and phase of S_{00}^{11} . Above the cutoff frequency of the first higher order mode, the higher order mode admittance becomes real, creating another propagating outlet for the incident energy. This leads to a sudden drop in magnitude and phase of the reflection coefficient. This scenario repeats at increasingly higher modes in the larger waveguide (A). Figure 7 show the magnitude and the phase of the plane wave transmission coefficient from the large waveguide (A) to the small waveguide (C). The magnitude starts at the low frequency value of the plane wave transmission coefficient $T = 2a^2/(a^2 + c^2)$, given by formula (67) for $ak \ll 1$, at first drops slowly and then more quickly as the frequency approaches the cutoff frequency of the first higher order mode. Above this cutoff frequency there is a sharp rise in the magnitude of the transmission coefficient. Near the cutoff frequency of the other higher order modes of the larger waveguide (A) there are additional dips in transmission coefficient. Figure 7(a) contains also as thin lines (blue online) the plots of the approximated transmission coefficients, resulted as simple frequency approximation for the values $a/c = 0.25$, 0.5, and 0.75, respectively.

Finally, we note that these results are qualitatively similar to those given, in a rectangular geometry, in Figs. 9(a), 9(b), 10(a), 10(b) in Ref. 1. (See also the paper by Muehleisen and Swanson Ref. 29.)

V. APPLICATION TO THE ANALYSIS OF CONCENTRIC EXPANSION CHAMBERS

A. General relationships

The concentric configuration of Fig. 8 has an inlet (A) of radius a , an outlet (D) of radius d and an axi-symmetrical intermediate duct (C) of radius c and length l . There are two

FIG. 6. (Color online) (a) The magnitude of the plane wave reflection coefficient S_{00}^{11} for the step discontinuity ($b=c$) junction in Fig. 1 with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$. (b) The phase of the plane wave reflection coefficient S_{00}^{11} for the axi-symmetric step discontinuity in Fig. 1 ($b=c$) with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$.

discontinuity planes: the expansion plane $z=0$ separating the pipes (A) and (C) and the contraction plane $z=l$ at the separation of the pipes (C) and (D). The gap radius in the contraction plane is denoted by b' . The point $(z=l, r=b')$ can be a step discontinuity point or the tip of a diaphragm in the case a flange is included at the joining surface of the pipes (C) and (D) (see Fig. 9). On the segments (A) and (C) the pressure and axial velocity is given by formulas (3), (4), (8), and (9) and in the outlet pipe we write

$$p_D(r, z) = D_0^r e^{-ik(z-l)} + D_0^a e^{ik(z-l)} + \sum_{n=1}^{\infty} (D_n^r e^{-\gamma_{d,n}(z-l)} + D_n^a e^{\gamma_{d,n}(z-l)}) J_0\left(\frac{x_n r}{d}\right), \quad z > l, \quad (69)$$

which gives

$$v_D(r, z) = \frac{-1}{i\omega\rho} \frac{\partial p_D}{\partial z} = \frac{k}{\omega\rho} (D_0^r e^{-ik(z-l)} - D_0^a e^{ik(z-l)}) + \sum_{n=1}^{\infty} \frac{\gamma_{d,n}}{i\omega\rho} (D_n^r e^{-\gamma_{d,n}(z-l)} - D_n^a e^{\gamma_{d,n}(z-l)}) J_0\left(\frac{x_n r}{d}\right), \quad z \geq l. \quad (70)$$

In the expansion plane the formulas (10)–(13) hold true. Then, the basic equation at the contraction plane becomes

$$\sum_{n=0}^{\infty} (C_n^r e^{-\gamma_{c,n}l} + C_n^a e^{\gamma_{c,n}l}) J_0\left(\frac{x_n r}{c}\right) = \sum_{n=0}^{\infty} (D_n^r + D_n^a) J_0\left(\frac{x_n r}{a}\right). \quad (71)$$

FIG. 7. (Color online) (a) The magnitude of the plane wave transmission coefficient S_{00}^{21} for the axis-symmetric step discontinuity ($b=c$) in Fig. 1 with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$. (b) The phase of the plane wave transmission coefficient S_{00}^{21} for the axis-symmetrical step discontinuity ($b=c$) in Fig. 1 with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$.

The axial velocity in the gap of the contraction plane is denoted by $\tilde{v}'(r)$ and, similarly as in the expansion plane $z=0$, will be represented as

$$\tilde{v}'(r) = \left[1 - \left(\frac{r}{b'}\right)^2\right]^{\mu'-1} \sum_{j=0}^{\infty} \tilde{v}_j^{(\mu')} \phi_j^{(\mu')} \left[\left(\frac{r}{b'}\right)^2\right], \quad 0 \leq r \leq b'.$$

In this formula $\mu' = 2/3$ in the case of simple step discontinuity and $\mu' = 1/2$ in the case a flange is present at $z=l$. The unknown coefficients $\tilde{v}_j^{(\mu')}$ will be determined in the same time with coefficients $\tilde{v}_j^{(\mu)}$ by using the continuity conditions.

Firstly, the continuity conditions for pressure and axial velocity at the contraction plane yield the relationships

$$Y_{c,n}(C_n^r e^{-\gamma_{c,n}l} - C_n^a e^{\gamma_{c,n}l}) = \frac{b'^2}{c^2 J_0^2(x_n)} \sum_{j=0}^{\infty} \tilde{J}_{\mu'+2j} \left(x_n \frac{b'}{c}\right) \tilde{v}_j^{(\mu')}, \quad (72)$$

$$Y_{d,n}(D_n^r - D_n^a) = \frac{b'^2}{d^2 J_0^2(x_n)} \sum_{j=0}^{\infty} \tilde{J}_{\mu'+2j} \left(x_n \frac{b'}{d}\right) \tilde{v}_j^{(\mu')}. \quad (73)$$

By using the formulas (30) and (72), we can write

$$C_n^a + C_n^r = \frac{\coth(\gamma_{c,n}l) b^2}{Y_{c,n} J_0^2(x_n) c^2} \sum_{j=0}^{\infty} \tilde{J}_{\mu'+2j} \left(x_n \frac{b}{c}\right) \tilde{v}_j^{(\mu)} - \frac{(\sinh(\gamma_{c,n}l))^{-1} b'^2}{Y_{c,n} J_0^2(x_n) c^2} \sum_{j=0}^{\infty} \tilde{J}_{\mu'+2j} \left(x_n \frac{b'}{c}\right) \tilde{v}_j^{(\mu')}, \quad (74)$$

FIG. 8. (Color online) A concentric expansion chamber.

$$C_n^a e^{\gamma_{c,n} l} + C_n^r e^{-\gamma_{c,n} l} = \frac{(\sinh(\gamma_{c,n} l))^{-1} b^2}{Y_{c,n} J_0^2(x_n)} \sum_{j=0}^{\infty} \tilde{J}_{\mu+2j} \left(x_n \frac{b}{c} \right) \tilde{v}_j^{(\mu)} - \frac{\coth(\gamma_{c,n} l) b'^2}{Y_{c,n} J_0^2(x_n)} \sum_{j=0}^{\infty} \tilde{J}_{\mu+2j} \left(x_n \frac{b'}{c} \right) \tilde{v}_j^{(\mu')}. \quad (75)$$

By applying to Eq. (71) a Galerkin procedure similar to that used for the Eq. (21) we obtain

$$\sum_{n=0}^{\infty} (C_n^r e^{-\gamma_{c,n} l} + C_n^a e^{\gamma_{c,n} l}) \tilde{J}_{\mu'+2p} \left(x_n \frac{b'}{c} \right) = \sum_{n=0}^{\infty} (D_n^r + D_n^a) \tilde{J}_{\mu'+2p} \left(x_n \frac{b'}{d} \right), \quad p = 0, 1, 2, \dots \quad (76)$$

The Eqs. (31) and (76) give the basic system for determining the pressure and the velocity in the expansion chamber.

B. Determination of the expansion coefficients

For the expansion chamber in Fig. 2, the dimensions of the inlet pipe (A) are assumed such that the incoming wave is planar and its magnitude is chosen to be unity:

$$A_0^a = 1, \quad A_n^a = 0, \quad n = 1, 2, 3, \dots, \quad (77)$$

and, also, an anechoic termination is imposed at the exit of the chamber

$$D_n^a = 0, \quad n = 0, 1, 2, 3, \dots \quad (78)$$

FIG. 9. A longitudinal section of the expansion chamber in Fig. 8.

The system of equations determining the velocity coefficients can be written as

$$\begin{cases} \sum_{j=0}^{\infty} A_{pj} w_j^{(\mu)} + \sum_{j=0}^{\infty} B_{pj} w_j^{(\mu')} = \delta_{p0} / \Gamma(\mu + 1) \\ \sum_{j=0}^{\infty} C_{pj} w_j^{(\mu)} + \sum_{j=0}^{\infty} D_{pj} w_j^{(\mu')} = 0, \end{cases} \quad (79)$$

where the coefficients $A_{pj}, \dots, D_{pj}$ have the expressions

$$A_{pj} = \sum_{n=0}^{\infty} A_{pj}^{(n)}, \quad B_{pj} = \sum_{n=0}^{\infty} B_{pj}^{(n)}, \quad (80)$$

$$C_{pj} = \sum_{n=0}^{\infty} C_{pj}^{(n)}, \quad D_{pj} = \sum_{n=0}^{\infty} D_{pj}^{(n)}, \quad (81)$$

where

$$A_{pj}^{(n)} = \frac{b \tilde{J}_{\mu+2p}(x_n b/a) \tilde{J}_{\mu+2j}(x_n b/a)}{a R_n(a) J_0^2(x_n)} + \frac{b}{c} \coth(\gamma_{c,n} l) \frac{\tilde{J}_{\mu+2p}(x_n b/c) \tilde{J}_{\mu+2j}(x_n b/c)}{R_n(c) J_0^2(x_n)}, \quad (82)$$

$$B_{pj}^{(n)} = \frac{b' \tilde{J}_{\mu+2p}(x_n b/c) \tilde{J}_{\mu+2j}(x_n b'/c)}{c \sinh(\gamma_{c,n} l) R_n(c) J_0^2(x_n)}, \quad (83)$$

$$C_{pj}^{(n)} = \frac{b \tilde{J}_{\mu'+2p}(x_n b'/c) \tilde{J}_{\mu+2j}(x_n b/c)}{c \sinh(\gamma_{c,n} l) R_n(c) J_0^2(x_n)}, \quad (84)$$

$$D_{pj}^{(n)} = \frac{b' \tilde{J}_{\mu'+2p}(x_n b'/d) \tilde{J}_{\mu'+2j}(x_n b'/d)}{d R_n(d) J_0^2(x_n)} - \frac{b'}{c} \coth(\gamma_{c,n} l) \frac{\tilde{J}_{\mu'+2p}(x_n b'/c) \tilde{J}_{\mu'+2j}(x_n b'/c)}{R_n(c) J_0^2(x_n)}. \quad (85)$$

Also the new velocity coefficients were introduced

$$w_j = i \rho \omega b \tilde{v}_j^{(\mu)}, \quad w'_j = i \rho \omega b' \tilde{v}_j^{(\mu')}. \quad (86)$$

Once Eq. (79) is solved, the formulas (86) yield the coefficients $\tilde{v}_j^{(\mu)}$ and the relationships (73) gives the pressure coefficients in the pipe (D).

C. The transmission coefficient

The pressure on the center of the pipe (D) can be written as

$$P(l', \omega) = |p_D(0, l + l') e^{i \omega t}|,$$

where $l' > 0$. The transmission loss TL in the center of the tube is defined as

$$TL = -20 \log_{10} [P(l', \omega) / P(l', \omega_0)]. \quad (87)$$

where $P(l', \omega_0)$ ($\omega_0 \simeq 10 \text{ Hz}$) is taken as the normalizing pressure instead of $P(l', 0)$ to avoid some singularities in the developed formulas associated with the zero frequency. By

expressing the coefficients D_n^r in terms of the particle velocity in the compression plane we obtain

$$P(l', \omega) = \frac{b'}{d} \left| \frac{e^{-ikl'} w'_0}{idk\Gamma(\mu' + 1)} + \sum_{j=0}^{\infty} w'_j \sum_{n=1}^{\infty} \frac{e^{-\gamma_{dn} l'} \tilde{J}_{\mu'+2j}(x_n b'/d)}{R_n(d) J_0^2(x_n)} \right|. \quad (88)$$

For the case of planar waves (the one-dimensional case) by imposing in the classical way the continuity of the pressure and axial velocity at the expansion and contraction planes there results a system of four equations for the four unknown coefficients. The solution yields the classical transmission loss of the one-dimensional expansion chamber as³³

$$TL_{lin} = 10 \log_{10} \left[1 + \frac{1}{4} \left[\left(\frac{c}{a} \right)^2 - \left(\frac{a}{c} \right)^2 \right]^2 \sin^2 kl \right], \quad (89)$$

which is a periodic function of frequency.

D. An explicit formula

To obtain an explicit analytical formula for the transmission loss we consider in the system of equations (79) only the two equations corresponding to the value $p = 0$ truncated to the value $j = 0$. In the case we take in the coefficient expressions only the values corresponding to $n = 0$ it can be shown that the result for transmission loss coincides with that corresponding to planar waves (89).

A new explicit analytical formula is obtained by considering the system of two equations corresponding to $p = 0$ with the unknowns w_0 , w'_0 and truncating the infinite series giving the coefficients at $n = 1$. Denoting the corresponding coefficients $\tilde{A} = A_{00}^{(0)} + A_{00}^{(1)}$, $\tilde{B} = B_{00}^{(0)} + B_{00}^{(1)}$, $\tilde{C} = C_{00}^{(0)} + C_{00}^{(1)}$, $\tilde{D} = D_{00}^{(0)} + D_{00}^{(1)}$, and accounting for the relationship (28) we obtain

$$\tilde{A} = \left(-\frac{b}{a^2 k} - \frac{b \cot(kl)}{c} \frac{1}{ck} \right) \frac{1}{\Gamma^2(\mu + 1)} + \frac{b}{a R_1(a) J_0^2(x_1)} + \frac{b \coth(R_1(c)l/c) \tilde{J}_\mu^2(x_1 b/c)}{R_1(c) J_0^2(x_1)}, \quad (90)$$

$$\tilde{B} = \frac{b'}{c^2} \frac{1}{\sin(kl)} \frac{1}{\Gamma(\mu + 1) \Gamma(\mu' + 1)} - \frac{b'}{c} \frac{\tilde{J}_\mu(x_1 b/c) \tilde{J}_{\mu'}(x_1 b'/c)}{\sinh(R_1(c)l/c) R_1(c) J_0^2(x_1)}, \quad (91)$$

$$\tilde{C} = -\frac{b}{c^2 k} \frac{1}{\sin(kl)} \frac{1}{\Gamma(\mu + 1) \Gamma(\mu' + 1)} + \frac{b}{c} \frac{\tilde{J}_\mu(x_1 b/c) \tilde{J}_{\mu'}(x_1 b'/c)}{\sinh(R_1(c)l/c) R_1(c) J_0^2(x_1)}, \quad (92)$$

$$\tilde{D} = \left(\frac{b'}{d^2 k} + \frac{b' \cot(kl)}{c} \frac{1}{ck} \right) \frac{1}{\Gamma^2(\mu' + 1)} + \frac{b'}{d R_1(d) J_0^2(x_1)} - \frac{b' \coth(R_1(c)l/c) \tilde{J}_{\mu'}^2(x_1 b'/c)}{R_1(c) J_0^2(x_1)}, \quad (93)$$

where we have used again the notation introduced in relationship (68).

There results

$$w'_0 = \frac{\tilde{C}}{\tilde{B}\tilde{C} - \tilde{A}\tilde{D}} \frac{1}{\Gamma(\mu + 1)}, \quad (94)$$

for the particle velocity coefficient in the contraction plane. Hence, the corresponding pressure becomes

FIG. 10. (Color online) The transmission loss of the concentric expansion chamber calculated by using the simplified formula (continued line, dotted line, broken line and dot-line-dot) and the data in Fig. 12 in Selamet and Radavich paper Ref. 18 (scattered circles).

$$P_{sim}(l', \omega) = \frac{b'}{d} \left| w'_0 \left| \frac{e^{-ikl'}}{idk\Gamma(\mu' + 1)} + \frac{e^{-R(d)l'/l} \tilde{J}_{\mu'}(x_1 b'/d)}{R_1(d)J_0^2(x_1)} \right| \right|. \quad (95)$$

The simplified transmission loss is given by the (87).

E. Numerical results

As a numerical example we consider one of the expansion chambers analyzed by Selamet and Radavich in Ref. 18. The inlet pipe has the radius $a = 2.43$ cm the expansion cylinder has the radius $c = 7.66$ cm and length $l = 15.689$ cm and the downstream tube has the same radius as the inlet pipe. The geometry analyzed in Ref. 18 has two step discontinuities. Therefore $b = b' = a$ and $\mu = \mu' = 2/3$. The transmission loss given by formulas (87), (95), and (94) was plotted versus frequency in Fig. 10 as continuous (black) line (SS1). In all the calculations we take $l' = a$ and $f_0 = 10$ Hz. The result given by the linear theory (SS0) is the periodical dotted line while some of the results taken from Fig. 12 of the paper in Ref. 18 are shown as scattered circles. We notice that the analytical results of Selamet and Radavich were obtained by solving a system of 24–64 linear equations for the expansion coefficients. The results given by the proposed explicit simplified formula are approximating quite well the analytical and experimental data.

In the case where a diaphragm of radius $b = a/2$ is introduced in the expansion plane and there is a step discontinuity at the contraction plane (denoted as *DS* case) we have $\mu = 1/2$, $\mu' = 2/3$. The results given by the simplified formula are plotted as a broken (red) line. The presence of the diaphragm in the expansion plane increases the transmission loss for all the frequencies in the studied range.

When a diaphragm of radius $b' = a/2$ is included at the contraction plane and a step discontinuity is present at the plane $z = 0$ (case denoted by *SD*), we have $b = a$, $\mu = 2/3$, $\mu' = 1/2$. The transmission loss given by the simplified formula is plotted in Fig. 10 as a dot-line-dot (green) line. Therefore, the presence of the diaphragm at the contraction plane decreases slightly the transmission loss. The presence in the same time of the both diaphragms decrease by a small amount the increase given by the upstream diaphragm.

VI. CONCLUSION

The problem of diffraction of a plane acoustic wave propagating into a cylindrical waveguide by a step discontinuity in the cross section has been approached by the method of mode matching which requires the solution of large systems (theoretically infinite) of linear equations. The presence of particle velocity singularities at the corner reentrant points makes the linear infinite system slowly convergent which gives ill conditioned finite truncations.

In this paper the problem of planar junctions of circular axi-symmetrical cylindrical waveguides, containing step discontinuities, diaphragms (thin flanges) or combinations of both of these, is approached by representing (re-expanding) the particle velocity in the aperture by means of some special

orthogonal base functions accounting for velocity singularities at the reentrant corner points. The resulting system of linear equations has very good convergence properties. Finally, the solution of most practical problems requires the solution of very few equations. This paper provides efficient formulas for determining the discontinuity impedance and the scattering matrix of discontinuity as well.

The last section of the paper considers an application of the re-expansion technique to the analysis of some concentric expansion chambers. The given simplified formulas for the transmission loss give results in good agreement with the published data determined for the case of chambers having step discontinuities. The formula is also applied to the case of step discontinuities combined with diaphragm-type discontinuities.

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APPENDIX A: DETERMINATION OF SOME EXPANSION COEFFICIENTS

Consider the system of functions

$$\mathcal{S} = \left\{ J_0 \left(x_n \frac{r}{a} \right) \right\}_{n=0,1,2,\dots}$$

x_n being the zeros of the function $J_1(x) = J_0'(x)$. Then, define the inner product

$$\langle f(r), g(r) \rangle = \int_0^a r f(r) g(r) dr.$$

We obtain

$$\begin{aligned} \left\langle J_0 \left(x_n \frac{r}{a} \right), J_0 \left(x_{n'} \frac{r}{a} \right) \right\rangle &= \left\{ \frac{ar^2}{x_n^2 - x_{n'}^2} [x_{n'} J_0 \left(x_n \frac{r}{a} \right) J_0' \left(x_{n'} \frac{r}{a} \right) \right. \\ &\quad \left. - x_n J_0 \left(x_{n'} \frac{r}{a} \right) J_0' \left(x_n \frac{r}{a} \right)] \right\}_{r=0}^{r=a} = 0, \quad \text{for } n \neq n' \end{aligned}$$

and

$$\begin{aligned} \left\| J_0 \left(x_n \frac{r}{a} \right) \right\|^2 &= \left\langle J_0 \left(x_n \frac{r}{a} \right), J_0 \left(x_n \frac{r}{a} \right) \right\rangle \\ &= \left\{ \frac{r^2}{2} \left[J_0'^2 \left(x_n \frac{r}{a} \right) + J_0^2 \left(x_n \frac{r}{a} \right) \right] \right\}_{r=0}^{r=a} = \frac{a^2}{2} J_0^2(x_n). \end{aligned}$$

Thus, the system of functions $\mathcal{S}$ is an orthogonal system (with respect to the considered inner product). It can be shown by using the results by Sneddon in Ref. 34 that the system is complete being therefore a basis.

APPENDIX B: A CLASS OF JACOBI'S HYPERGEOMETRIC POLYNOMIALS

The Jacobi's polynomials, $P_n^{(\alpha,\beta)}(z)$ are orthogonal for $z \in [-1, 1]$ with the weighting function $w(z) = (1-z)^\alpha (1+z)^\beta$. The change of variable $z = 1 - 2x$ yields the polynomials $P_n^{(\alpha,\beta)}(1-2x)$ orthogonal over the interval $[0, 1]$ with the weighting function $w_1(x) = x^\alpha (1-x)^\beta$. Finally, the change of variable $x = t^2$ yields the polynomials

$$P_n^{(\alpha,\beta)}(1-2t^2)$$

orthogonal for $t \in [0, 1]$ with the weighting function $w_2(t) = t^{2\alpha+1} (1-t^2)^\beta$. By taking $\alpha = 0$ and $\beta = \mu - 1$ we define the polynomials

$$\phi_n^{(\mu)}(t^2) = P_n^{(0,\mu-1)}(1-2t^2). \quad (\text{B1})$$

The polynomials $\phi_n^{(\mu)}(t^2)$ are a complete system of functions orthogonal over the interval $(0, 1)$ with the weight $t(1-t^2)^{\mu-1}$. Indeed the above calculation yields

$$\int_0^1 t(1-t^2)^{\mu-1} \phi_m^{(\mu)}(t^2) \phi_n^{(\mu)}(t^2) dt = \frac{1}{2(\mu+2n)} \delta_{m,n}. \quad (\text{B2})$$

By means of the formula

$$P_n^{(\alpha,\beta)}(1-2x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)} F(-n, \alpha+\beta+1+n; \alpha+1; x)$$

given in Ref. 35 (pp. 561, formula 15.4.6) the considered Jacobi's polynomials, with modified argument, can be connected to the Gauss's hypergeometric function $F(a, b; c; x)$. Also, formula §5.21(3) in Ref. 36 gives

$$J_0(tz) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n)(\mu+2n)}{n!} \phi_n^{(\mu)}(t^2) \frac{J_{\mu+2n}(z)}{(z/2)^\mu}. \quad (\text{B3})$$

By using the formulas (B2) and (B3) we can write also

$$\begin{aligned} \int_0^b r \left(1 - \frac{r^2}{b^2}\right)^{\mu-1} \phi_j^{(\mu)}\left(\frac{r^2}{b^2}\right) J_0\left(\frac{x_m r}{a}\right) dr \\ = b^2 \frac{\Gamma(\mu+j)}{2j!} \frac{J_{\mu+2j}(x_m b/a)}{[x_m b/(2a)]^\mu}. \end{aligned} \quad (\text{B4})$$

¹D. Homentcovschi and R. N. Miles, "A re-expansion method for determining the acoustical impedance and the scattering matrix for the waveguide discontinuity problem," J. Acoust. Soc. Am. **128**, 628–638 (2010).

²A. D. Sahasrabudhe, M. L. Munjal, and S. A. Ramu, "Analysis of inheritance due to the higher order mode effects in a sudden area discontinuity," J. Sound Vib. **185**, 515–529 (1995).

³J. Miles, "The analysis of plane discontinuity in cylindrical tubes. Part I and II," J. Acoust. Soc. Am. **17**, 259–284 (1946).

⁴U. Ingard, "On the theory and design of acoustic resonators," J. Acoust. Soc. Am. **25**, 1037–1061 (1953).

⁵F. C. Karal, "The analogous acoustical impedance for discontinuities of circular cross section," J. Acoust. Soc. Am. **25**, 327–334 (1953).

⁶A. I. El-Sharkawy and A. H. Nayfeh, "Effect of an expansion chamber on the propagation of sound in circular ducts," J. Acoust. Soc. Am. **63**, 667–674 (1978).

⁷K. S. Peat, "The acoustical impedance at discontinuities of ducts in the presence of a mean flow," J. Sound Vib. **127**, 123–132 (1988).

⁸J. Kergomard and A. Garcia, "Simple discontinuities in acoustic waveguides at low frequencies: Critical analysis and formulae," J. Sound Vib. **114**, 465–479 (1987).

⁹J. Kergomard, "Calculation of discontinuities in waveguides using mode-matching method: An alternative to the scattering matrix approach," J. Acoust. **4**, 111–138 (1991).

¹⁰F. E. Grigoryan, "Diffraction of sound waves by a diaphragm in a circular waveguide," Sov. Phys. Acoust. **26**, 25–28 (1980).

¹¹F. E. Grigoryan, "Theory of circular waveguide with abrupt change of cross section," Sov. Phys. Acoust. **30**, 22–26 (1984).

¹²E. I. Lavretsky, "Taking into account the edge condition in the problem of scattering from the circular aperture in circular-to-rectangular and rectangular-to-rectangular waveguide junctions," IEE Proc. Microwaves, Antennas Propag. **141**, 45–50 (1994).

¹³D. V. Evans, and M. Fernyhough, "Edge waves along periodic coastlines. Part 2," J. Fluid. Mech. **297**, 307–325 (1995).

¹⁴M. Fernyhough and D. V. Evans, "Full multimodal analysis of an open rectangular groove waveguide," IEEE Trans. Microwave Theory Tech. **46**, 97–107 (1998).

¹⁵M. Kanoria, "Water-wave scattering by thick vertical barriers," J. Eng. Math. **35**, 361–384 (1999).

¹⁶M. Kanoria, "Water wave scattering by thick rectangular slotted barriers," App. Ocean Res. **23**, 285–295 (2001).

¹⁷M. McIver, C. M. Linton, and J. Zhang, "The branch structure of embedded trapped modes in two dimensional waveguides," Q. J. Mech. Appl. Math. **55**, 313–326 (2002).

¹⁸A. Selamet and P. M. Radavich, "The effect of length on the acoustic attenuation performance of concentric expansion chambers: an analytical, computational, and experimental investigation," J. Sound Vib. **201**, 407–426 (1997).

¹⁹R. Kirby and J. B. Lawrie, "A point collocation approach to modelling large dissipative silencers," J. Sound Vib. **286**, 313–339 (2005).

²⁰J. B. Lawrie and I. M. M. Guled, "On tuning a reactive silencer by varying the position of an internal membrane," J. Acoust. Soc. Am. **120**, 780–790 (2006).

²¹W. Bi, V. Pagneux, D. Lafarge, and Y. Aurégan, "Characteristics of penalty mode scattering by rigid splices in lined ducts," J. Acoust. Soc. Am. **121**, 1303–1312 (2006).

²²W. Bi, V. Pagneux, D. Lafarge, and Y. Aurégan, "An improved multimodal method for sound propagation in nonuniform lined ducts," J. Acoust. Soc. Am. **122**, 280–290 (2007).

²³R. Kirby, "Transmission loss predictions for dissipative silencers of arbitrary cross section in the presence of mean flow," J. Acoust. Soc. Am. **114**, 200–209 (2003).

²⁴R. Kirby, "Modeling sound propagation in acoustic waveguides using a hybrid numerical method," J. Acoust. Soc. Am. **124**, 1930–1940 (2008).

²⁵R. Kirby, "A comparison between analytic and numerical methods for modelling automotive dissipative silencers with mean flow," J. Sound Vib. **325**, 565–582 (2009).

²⁶J. Schwinger and D. S. Saxon, *Discontinuities in Waveguides* (Gordon and Breach Science Publishers, New York, 1968), Chap. 5.

²⁷J. G. Van Bladel, *Electromagnetic Fields* (John Wiley, Hoboken, NJ, 2007), pp. 171.

²⁸H. Hudde and U. Letens, "Scattering matrix of a discontinuity with a non-rigid wall in a lossless circular duct," J. Acoust. Soc. Am. **78**, 1826–1837 (1985).

²⁹R. T. Muehleisen and D. C. Swanson, "Modal coupling in acoustic waveguides: Planar discontinuities," Appl. Acoust. **63**, 1375–1392 (2002).

³⁰V. A. Fock, "A theoretical investigation of the acoustical conductivity of a circular aperture in a wall put across a tube," Dokl. Akad. Nauk SSSR **31**, 875–878 (1941).

³¹S. N. Rschewkin, *A Course of Lectures on the Theory of Sound* (Pergamon Press, New York, 1963), 464 pp.

³²F. G. Leppington and H. Levine, "Some axially symmetric potential problems," Proc. Edinburgh Math. Soc. Ser. II **8**, 55–76 (1972).

³³A. P. Dowling and J. E. Ffowcs Williams, *Sound and Sources of Sound* (Ellis Horwood Limited, Chichester, UK, 1983), 64 pp.

³⁴I. Sneddon, *The Use of Integral Transforms* (McGraw Hill, New York, 1972), Chap. 5.

³⁵M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1965), 561 pp.

³⁶G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, UK, 1966), 140 pp.