Published in final edited form as:

Inverse Probl. 2012 January 1; 28(1): .

High-order total variation minimization for interior SPECT

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Abstract

Recently, we developed an approach for solving the computed tomography (CT) interior problem based on the high-order TV (HOT) minimization, assuming that a region-of-interest (ROI) is piecewise polynomial. In this paper, we generalize this finding from the CT field to the single-photon emission computed tomography (SPECT) field, and prove that if an ROI is piecewise polynomial, then the ROI can be uniquely reconstructed from the SPECT projection data associated with the ROI through the HOT minimization. Also, we propose a new formulation of HOT, which has an explicit formula for any n -order piecewise polynomial function, while the original formulation has no explicit formula for $n \geq 2$. Finally, we verify our theoretical results in numerical simulation, and discuss relevant issues.

1. Introduction

Inspired by the development of reconstruction algorithms for computed tomography (CT), the single-photon emission computed tomography (SPECT) techniques are being rapidly improved [1–8]. As a unique biomedical tomographic imaging technique, SPECT is to reconstruct a radioactive source distribution from externally measured data. In a typical SPECT application, a gamma camera is employed to acquire multiple 1D/2D projections from various angles. Then, a tomographic reconstruction algorithm is applied to these projections, yielding a 2D/3D image. Different from the line integral model for x-ray imaging, SPECT projections can be mathematically modeled as an exponentially attenuated Radon transform [5, 9]. In this context, the CT reconstruction may be regarded as a special case of SPECT (all the attenuation coefficients are zeros), but CT reconstruction techniques cannot be directly used for SPECT reconstruction in general.

Given the increasing concern over radiation risk from CT examinations, a number of image reconstruction algorithms were developed to reduce the amount of raw data. In this context, in 2007, it was proved that the interior problem, the reconstruction of an interior region-of-

interest (ROI) or volume-of-interest (VOI) only from data associated with lines through the ROI/VOI, can be accurately solved if a sub-region in the ROI/VOI is known [10–13]. Similar results were also independently reported [14, 15]. If there is no prior information available, it has been proved that the interior problem does not have a unique solution in general [16]. Based on the aforementioned CT interior reconstruction results [10–15], in 2008 we proved that theoretically exact SPECT of an ROI is feasible from uniformly attenuated local projection data, aided by prior knowledge of a sub-region in the ROI [6].

Although the CT numbers of certain sub-regions such as air in a trachea and blood in an aorta can indeed be assumed, how to obtain precise knowledge of a sub-region can be difficult in important cases such as in contrast-enhanced functional studies. Therefore, it would be very valuable to develop more powerful interior tomography techniques [17]. Fortunately, the compressive sampling (CS) theory has recently emerged which shows that high-quality signals and images can be reconstructed from far fewer data than is usually considered necessary according to the Nyquist sampling theory [18–20]. The main idea of CS is that most signals are sparse in an appropriate system, that is, a majority of their coefficients are close or equal to zero when represented in an appropriate domain. In light of the CS theory and using the specific gradient transform, we proved that it is possible to accurately reconstruct an ROI only from truncated projections by minimizing the total variation (TV) if the ROI is piecewise constant, without knowledge of any known sub-region in the ROI, which is required by our previous interior tomography theory [21–23]. Very recently, we extended this piece-wise constant result to allow a piecewise polynomial model and proposed interior tomography by the high-order TV (HOT) minimization [24].

Then, we were motivated to test the hypothesis that HOT-minimization-based interior tomography can be generalized for interior SPECT. That is, we seek to accurately reconstruct an ROI only from the uniformly attenuated local SPECT projections through the HOT minimization under the assumption that the underlying distribution function is piecewise polynomial. In this paper, we will report our initial results in this aspect. Particularly, we will develop a new formulation of HOT, which allows an explicit expression for any n -order piecewise polynomial function, while in our previous work [24] no explicit formula was derived for $n \geq 2$. Because of its explicit representation, the new HOT is straightforward to implement in reconstruction algorithms. In the next section, we will define the interior SPECT problem in terms of uniformly attenuated and truncated Radon transform data. In the third section, we will present our major theoretical results in the HOT minimization framework. In the fourth section, we will describe numerical results. In the last section, we will discuss relevant issues and conclude the paper.

2. Interior SPECT problem

Without loss of generality, we assume the following conditions throughout this paper.

Condition (1) An object image $f_0(x)$ is compactly supported on a disk $\Omega_A = \{x = (x_1, x_2) \in \mathbb{R}^2: |x| < A\}$, where A is a positive constant. Furthermore, $f_0(x)$ is a piecewise smooth function; that is, Ω_A can be partitioned into finitely many sub-domains $\{D_j\}_{j=1}^{N_0}$, such that $f_0(x)$ is smooth with bounded derivatives in each D_j .

Condition (2) An internal ROI is a smaller disk $\Omega_a = \{x = (x_1, x_2) \in \mathbb{R}^2: |x| < a\}$, as shown in figure 1, where a is a positive constant and $a < A$.

Condition (3) Attenuated projections through the ROI

$$R_{\mu}f_0(s, \theta) = \int_{-\infty}^{\infty} f_0(s\theta + t\theta^{\perp}) e^{-\mu t} dt, \quad -a < s < a, \quad \theta \in S^1, \quad (2.1)$$

are available, where μ is a constant attenuation coefficient and $\theta = (\cos \varphi, \sin \varphi)$, $\theta^{\perp} = (-\sin \varphi, \cos \varphi)$, $0 \leq \varphi < 2\pi$.

The interior SPECT with uniformly attenuated local projection data is to find an image $f(x)$ such that

Condition (4) $f(x)$ is a piecewise smooth function and compactly supported on the disk Ω_A ;

Condition (5) $R_{\mu}f(s, \theta) = R_{\mu}f_0(s, \theta)$, $-a < s < a$, $\theta \in S^1$.

It is well known that under conditions (4) and (5) the interior problem does not have a unique solution [16, 25]. The following theorem characterizes the structure of solutions to the interior SPECT problem.

Theorem 1. Any image $f(x)$ satisfying conditions (4) and (5) can be written as $f(x) = f_0(x) + u(x)$ for $x \in \mathbb{R}^2$, where $u(x)$ is an analytic function in the disk Ω_a , and

$$R_{\mu}u(s, \theta) = 0, \quad -a < s < a, \quad \theta \in S^1.$$

We call such an image $f(x)$ a candidate image, and correspondingly $u(x)$ an ambiguity image.

Proof. Let $u(x) = f(x) - f_0(x)$. Clearly, $u(x)$ is a piecewise smooth function and compactly supported on the disk Ω_A by conditions (1) and (4), and

$$R_{\mu}u(s, \theta) = 0, \quad -a < s < a, \quad \theta \in S^1. \quad (2.2)$$

By Tretiak and Metz's inversion formula, which was derived from the shift property (5) and corollary 2 in [26], we have

$$u(x) = \frac{1}{4\pi^2} \int_0^{2\pi} e^{-\mu x \cdot \theta^{\perp}} \int_{a < |s| \leq A} \frac{\cos(\mu(s - x \cdot \theta))}{s - x \cdot \theta} \frac{\partial R_{\mu}u(s, \theta)}{\partial s} ds d\varphi. \quad (2.3)$$

The term $\frac{\cos(\mu(s - x \cdot \theta))}{s - x \cdot \theta} e^{-\mu_0 x \cdot \theta^{\perp}}$ in the integral is analytic, which can be expressed in a power series. Therefore, $u(x)$ can also be expressed in a power series, that is,

$$u(x) = \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} c_{k_1, k_2} \chi_1^{k_1} \chi_2^{k_2}. \quad (2.4)$$

Hence, the function $u(x)$ is an analytic function in Ω_a . Examples of non-zero ambiguity images can be found in [16].

From now onwards, let $u(x)$ always represent an ambiguity image unless otherwise stated. We will rely on the HOT minimization proposed in [24] to solve the interior SPECT

problem with uniformly attenuated local projection data under the assumption that $f_0(x)$ is piecewise polynomial in a ROI Ω_a .

3. Theoretical analysis

If an object image $f_0(x)$ is piecewise polynomial in ROI Ω_a , we can prove that $f_0(x)$ is the only candidate image that minimizes the HOT. First, let us prove that if an ambiguity image is polynomial in Ω_a , then it must be zero. This result will be formally stated as theorem 2. In order to prove theorem 2, we will need lemmas 1–3.

Lemma 1 ([24]). *Suppose that a is a positive constant. If*

- (a) $g(z)$ is an analytic function in $\mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty)$;
- (b) $p(x)$ is a polynomial function;
- (c) $g(x) = \frac{1}{\pi} \text{PV} \int_{|t| < a} \frac{p(t)}{x-t} dt$, for $x \in (-a, a)$,

then $\lim_{y \rightarrow 0^+} \text{Im}(g(x + iy)) = p(x)$, for $x \in (-\infty, -a) \cup (a, +\infty)$.

Lemma 2. *Assume that a is a positive constant. If*

- (a) $g(z)$ is an analytic function in $\mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty)$;
- (b) $p(x)$ is a polynomial function;
- (d) $g(x) = \frac{1}{\pi} \text{PV} \int_{|t| < a} \frac{\cosh(\mu(x-t)) p(t)}{x-t} dt$, for $x \in (-a, a)$,

then we have

$$\lim_{y \rightarrow 0^+} \text{Im}(g(x + iy)) = p(x),$$

for $x \in (-\infty, -a) \cup (a, +\infty)$.

Proof. $g(z)$ can be rewritten as

$$g(z) = g_1(z) + g_2(z), \quad \text{for } z \in \mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty), \quad (3.1)$$

where

- (i) $g_1(z)$ is an analytic function in $\mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty)$, and

$$g_1(x) = \frac{1}{\pi} \text{PV} \int_{|t| < a} \frac{p(t)}{x-t} dt, \quad \text{for } x \in (-a, a); \quad (3.2)$$

- (ii) $g_2(z)$ is an analytic function in $\mathbb{C}$, and

$$g_2(z) = \frac{1}{\pi} \int_{|t| < a} \frac{[\cosh(\mu(z-t)) - 1] p(t)}{z-t} dt, \quad \text{for } z \in \mathbb{C}. \quad (3.3)$$

Using lemma 1, we obtain

$$\lim_{y \rightarrow 0+} \operatorname{Im} (g_1 (x+iy)) = p(x), \quad \text{for } x \in (-\infty, -a) \cup (a, +\infty). \quad (3.4)$$

Because of the continuity of $g_2(z)$ in $\mathbb{C}$, it is clear that

$$\lim_{y \rightarrow 0+} \operatorname{Im} (g_2 (x+iy)) = \operatorname{Im} (g_2 (x)) = 0, \quad \text{for } x \in (-\infty, +\infty). \quad (3.5)$$

Therefore, we have

$$\lim_{y \rightarrow 0+} \operatorname{Im} (g(x+iy)) = \lim_{y \rightarrow 0+} \operatorname{Im} (g_1 (x+iy)) + \lim_{y \rightarrow 0+} \operatorname{Im} (g_2 (x+iy)) = p(x),$$

for

$$x \in (-\infty, -a) \cup (a, +\infty), \quad (3.6)$$

which completes the proof.

Lemma 3. Assume that a and A are positive constants with $a < A$. If a single variable function $v(x) \in L^\infty(\mathbb{R})$ satisfies

- (e) $v(x)$ is compactly supported in $[-A, A]$,
- (f) $v(x) = p(x)$ for $x \in (-a, a)$, where $p(x)$ is a polynomial function,
- (g) $H_\mu v(x) = 0$ for $x \in (-a, a)$, where $H_\mu v(x)$ is the generalized Hilbert transform of $v(x)$, that is

$$H_\mu v(x) = \frac{1}{\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{\cosh(\mu(x-s)) v(s)}{x-s} ds;$$

then $v(x) = 0$.

Proof. The function ([27], p 42, lemma 4.2.4)

$$g(z) = \frac{1}{\pi} \int_{|t| \geq a} \frac{\cosh(\mu(t-z)) v(t)}{t-z} dt \quad (3.7)$$

is analytic on $\mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty)$. Then, $g(z)$ can be rewritten as

$$g(z) = g_1(z) + g_2(z), \quad \text{for } z \in \mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty), \quad (3.8)$$

where

- (i) $g_1(z)$ is an analytic function in $\mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty)$, and

$$g_1(z) = \frac{1}{\pi} \int_{|t| \geq a} \frac{v(t)}{t-z} dt, \quad \text{for } z \in \mathbb{C} \setminus (-\infty, -a] \cup [a, +\infty); \quad (3.9)$$

- (ii) $g_2(z)$ is an analytical function in $\mathbb{C}$, and

$$g_2(z) = \frac{1}{\pi} \int_{|t| \geq a} \frac{[\cosh(\mu(t-z)) - 1] v(t)}{t-z} dt, \quad \text{for } z \in \mathbb{C}. \quad (3.10)$$

With $g_1(z)$, we have for $y > 0$

$$\operatorname{Im}(g_1(x+iy)) = \frac{1}{\pi} \int_{|t| \geq a} \frac{y}{(t-x)^2 + y^2} v(t) dt = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} \tilde{v}(t) dt, \quad (3.11)$$

where

$$\tilde{v}(x) = \begin{cases} v(x), & x \in (-\infty, -a) \cup (a, \infty) \\ 0, & x \in [-a, a]. \end{cases} \quad (3.12)$$

Applying theorem 13 in [28] to (3.11) ([27], p 40, the proof of corollary 4.1.2), we obtain

$$\lim_{y \rightarrow 0+} \operatorname{Im}(g_1(x+iy)) = \tilde{v}(x) = v(x), \quad \text{for a.e. } x \in (-\infty, -a) \cup (a, \infty). \quad (3.13)$$

Using (3.5), we have

$$\begin{aligned} \lim_{y \rightarrow 0+} \operatorname{Im}(g(x+iy)) &= \lim_{y \rightarrow 0+} \operatorname{Im}(g_1(x+iy)) \\ &= v(x) \quad \text{for a.e. } x \in (-\infty, -a) \cup (a, \infty). \end{aligned} \quad (3.14)$$

On the other hand, for $x \in (-a, a)$,

$$\begin{aligned} g(x) &= \frac{1}{\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{\cosh(\mu(t-x)) v(t)}{t-x} dt - \frac{1}{\pi} \operatorname{PV} \int_{|t| < a} \frac{\cosh(\mu(t-x)) v(t)}{t-x} dt \\ &= -H_{\mu} v(x) + \frac{1}{\pi} \operatorname{PV} \int_{|t| < a} \frac{\cosh(\mu(t-x)) p(t)}{x-t} dt \\ &= \frac{1}{\pi} \operatorname{PV} \int_{|t| < a} \frac{\cosh(\mu(t-x)) p(t)}{x-t} dt. \end{aligned} \quad (3.15)$$

Using lemma 2, we obtain

$$\lim_{y \rightarrow 0+} \operatorname{Im}(g(x+iy)) = p(x), \quad x \in (-\infty, -a) \cup (a, +\infty). \quad (3.16)$$

Combining equations (3.14) and (3.16), we have

$$v(x) = p(x), \quad \text{for a.e. } x \in (-\infty, \infty). \quad (3.17)$$

Condition (e) and equation (3.17) imply

$$p(x) \equiv 0, \quad x \in (-\infty, -A) \cup (A, \infty). \quad (3.18)$$

Because $p(x)$ is a polynomial, it follows that $p(x) \equiv 0$. Therefore,

$$v(x) = 0, \quad \text{for a.e. } x \in (-\infty, \infty). \quad (3.19)$$

Theorem 2. If an artifact image $u(x)$ satisfies

- (a) $u(x) = p(x)$ with $x \in \Omega_a$, where $p(x)$ is a 2D polynomial function;
- (b) $R_\mu u(s, \theta) = 0$ with $s \in (-a, a)$, $\theta \in S^1$,

then $u(x) = 0$.

Proof. As illustrated in figure 2, for an arbitrary $\varphi_0 \in [0, \pi)$, let L_{θ_0} denote the line through the origin and tilted at $\theta_0 = (\cos \varphi_0, \sin \varphi_0)$. When $u(x)$ is restricted to the line L_{θ_0} , it can be expressed as

$$u_{\theta_0}(t) = u(t(\cos \varphi_0, \sin \varphi_0)), \quad t \in (-\infty, \infty). \quad (3.20)$$

By the relationship between the differentiated backprojection of the attenuated projection data and the generalized Hilbert transform of the image [5], we have

$$H_\mu u_{\theta_0}(t) = -\frac{1}{2\pi} \int_{\varphi_0 - \frac{\pi}{2}}^{\varphi_0 + \frac{\pi}{2}} \frac{\partial R_\mu u(s, \theta)}{\partial s} \Big|_{s=x_{\theta_0}^t} e^{-x_{\theta_0}^t \cdot \theta^\perp} d\varphi, \quad (3.21)$$

where

$$x_{\theta_0}^t = t(\cos \varphi_0, \sin \varphi_0), \quad \theta = (\cos \varphi, \sin \varphi), \quad \theta^\perp = (-\sin \varphi, \cos \varphi), \quad (3.22)$$

$$H_\mu u_{\theta_0}(t) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{\cosh(\mu(t-s)) u_{\theta_0}(s)}{t-s} ds. \quad (3.23)$$

By (a) and equation (3.20), we have

$$u_{\theta_0}(t) = p(t(\cos \varphi_0, \sin \varphi_0)), \quad \text{for } t \in (-a, a), \quad (3.24)$$

where $p(t(\cos \varphi_0, \sin \varphi_0))$ is a polynomial function with respect to t . By (b) and equation (3.21), we have

$$H_\mu u_{\theta_0}(t) = 0, \quad \text{for } t \in (-a, a). \quad (3.25)$$

Using lemma 3, we obtain

$$u_{\theta_0}(t) = 0. \quad (3.26)$$

Therefore,

$$u(x) = 0, \quad (3.27)$$

which completes the proof.

Lemma 4 [23]. Suppose that $f_0(x)$ is piecewise constant in ROI Ω_a ; that is, Ω_a can be decomposed into finitely many sub-domains $\{\Omega_i\}_{i=1}^m$ (figure 3) such that

$$f_0(x) = f_i(x) = c_i, \quad \text{for } x \in \Omega_i, \quad 1 \leq i \leq m, \quad (3.28)$$

and each sub-domain Ω_i is adjacent to its neighboring sub-domains Ω_j with piecewise smooth boundaries $\Gamma_{i,j}$, $j \in N_i$. For any candidate image $f(x) = f_0(x) + u(x)$, where $u(x)$ is an arbitrary ambiguity image, let

$$\text{TV}(f) = \sup \left\{ \int_{\Omega_a} f \operatorname{div} \phi \, dx : \phi \in C_0^1(\Omega_a)^2, |\phi| \leq 1 \text{ in } \Omega_a \right\}, \quad (3.29)$$

where $C_0^1(\Omega_a)^2 = C_0^1(\Omega_a) \times C_0^1(\Omega_a)$; then,

$$\text{TV}(f) = \sum_{i=1}^m \sum_{j>i, j \in N_i} |c_i - c_j| |\Gamma_{i,j}| + \int_{\Omega_a} \sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2} \, dx_1 \, dx_2. \quad (3.30)$$

Under the assumption that $f_0(x)$ is piecewise constant in ROI Ω_a , through the TV minimization [21] we can establish the uniqueness of interior SPECT as follows.

Theorem 3. Suppose that $f_0(x)$ is piecewise constant in a ROI Ω_a , as defined by equation (3.28). If $h(x)$ is a candidate image and $\text{TV}(h) = \min_{f=f_0+u} \text{TV}(f)$, then $h(x) = f_0(x)$ for $x \in \Omega_a$.

Proof. Let

$$h(x) = f_0(x) + u_1(x), \quad (3.31)$$

where $u_1(x)$ is an ambiguity image. By lemma 4, we have

$$\text{TV}(h) = \sum_{i=1}^m \sum_{j>i, j \in N_i} |c_i - c_j| |\Gamma_{i,j}| + \int_{\Omega_a} \sqrt{\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2} \, dx_1 \, dx_2. \quad (3.32)$$

Since

$$\text{TV}(h) = \min_{f=f_0+u} \text{TV}(f), \quad (3.33)$$

and

$$\min_{f=f_0+u} \text{TV}(f) = \text{TV}(f_0) = \sum_{i=1}^m \sum_{j>i, j \in N_i} |c_i - c_j| |\Gamma_{i,j}|, \quad (3.34)$$

we have

$$\int_{\Omega_a} \sqrt{\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2} dx_1 dx_2 = 0; \quad (3.35)$$

that is

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial x_2} = 0, \quad \text{in } \Omega_a. \quad (3.36)$$

Therefore, there exists a constant c such that

$$u_1(x) = c, \quad \text{in } \Omega_a; \quad (3.37)$$

that is, $u_1(x)$ is a zeroth-order polynomial in Ω_a . By theorem 2, we have

$$u_1(x) = 0, \quad \text{in } \Omega_a. \quad (3.38)$$

Thus, we have

$$h(x) = f_0(x) \quad \text{for } x \in \Omega_a. \quad (3.39)$$

In [24], we introduced an expression for high-order TV (HOT). Here, we will provide an alternative expression. We define the $(n+1)$ th ($n \geq 1$) order TV as the limit of the following sum:

$$\text{HOT}_{n+1}(f) = \limsup_{\left(\max_{1 \leq k \leq M} \text{diam}\{Q_k\}\right) \rightarrow 0} \sum_{k=1}^M I_k^{n+1}(f), \quad (3.40)$$

where $\{Q_k\}_{k=1}^M$ is an arbitrary partition of Ω_a , $\text{diam}(Q_k)$ is the diameter of Q_k , and

$$I_{k,1}(f) = \min \{I_{k,1}(f), I_{k,2}^{n+1}(f)\}, \quad (3.41)$$

$$I_{k,1}(f) = \sup \left\{ \int_{Q_k} f \operatorname{div} g \, dx = (gl)_{l=1}^2 \in C_0^\infty(Q_k)^2, |g| \leq 1 \text{ in } Q_k \right\}, \quad (3.42)$$

$$\operatorname{div} g = \sum_{l=1}^2 \frac{\partial gl}{\partial x_l}, \quad |g| = \sqrt{\sum_{l=1}^2 |gl|^2}; \quad (3.43)$$

$$I_{k,2}^{n+1}(f) = \sup \left\{ \int_{Q_k} f \operatorname{div}_{n+1} \bar{g} \, dx : \bar{g} = (\bar{g}_r)_{r=0}^{n+1} \in C_0^\infty(Q_k)^{n+2}, |\bar{g}| \leq 1 \text{ in } Q_k \right\}, \quad (3.44)$$

$$\operatorname{div}_{n+1} \bar{g} = \sum_{r=0}^{n+1} \frac{\partial^{n+1} \bar{g}_r}{\partial x_1^r \partial x_2^{n+1-r}}, \quad |\bar{g}| = \sqrt{\sum_{r=0}^{n+1} |\bar{g}_r|^2}. \quad (3.45)$$

Under the assumption that $f_0(x)$ is a piecewise n th ($n \geq 1$) order polynomial function in an ROI Ω_a , through the HOT minimization we can establish the uniqueness of interior SPECT, just as we have done for interior CT [24].

First, we will derive an explicit formula of $\operatorname{HOT}_{n+1}(f)$ for any candidate image f under the assumption that $f_0(x)$ is a piecewise n th ($n \geq 1$) order polynomial function in Ω_a .

Lemma 5. Suppose that $f_0(x)$ is a piecewise n th ($n \geq 1$) order polynomial function in Ω_a ; that is, Ω_a can be decomposed into finitely many sub-domains $\{\Omega_i\}_{i=1}^m$ (figure 3) such that

$$f_0(x) = f_i(x), \quad \text{for } x \in \Omega_i, \quad 1 \leq i \leq m, \quad (3.46)$$

where $f_i(x)$ is an n th order polynomial function, and each sub-domain Ω_i is adjacent to its neighboring sub-domains Ω_j with piecewise smooth boundaries $\Gamma_{i,j}$, $j \in N_i$. Then, for any candidate image $f(x) = f_0(x) + u(x)$, we have

$$\operatorname{HOT}_{n+1}(f) = \sum_{i=1}^m \sum_{j>i, j \in N_i} \int_{\Gamma_{i,j}} |f_i - f_j| \, ds + \int_{\Omega_a} \min \left\{ \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2}, \sqrt{\left(\frac{\partial f}{\partial x_1} \right)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2} \right\} \, dx, \quad (3.47)$$

where the second term is a Lebesgue integral.

Proof. Note that $f(x) = f_0(x) + u(x)$, where $u(x)$ is an analytic function. Let $\{\Omega_k\}_{k=1}^M$ be an arbitrary partition of Ω_a . First, let us consider Q_k that covers a common boundary $\Gamma_{i,j}$ of a pair of neighboring sub-domains Ω_i and Ω_j , and is contained in $\Omega_i \cup \Omega_j$ (figure 4(a)). The normal vector of the curve $\Gamma_{i,j}$ pointing from Ω_j towards Ω_i is denoted by $\theta^{i,j} = (\theta_1^{i,j}, \theta_2^{i,j})$. For $g = (g_l)_{l=1}^2 \in C_0^\infty(Q_k)^2$, we have

$$\int_{Q_k} f \operatorname{div} g \, dx = \int_{Q_k \cap \Omega_i} f \operatorname{div} g \, dx + \int_{Q_k \cap \Omega_j} f \operatorname{div} g \, dx. \quad (3.48)$$

Performing integration by parts for the two terms on the right-hand side of equation (3.48) respectively and utilizing the fact that $g(x) = (0, 0)$ near the boundary of Q_k , we have

$$\int_{Q_k \cap \Omega_i} f \operatorname{div} g dx = - \int_{\Gamma_{i,j} \cap Q_k} (f_i(x) + u(x)) (g_1 \theta_1^{i,j} + g_2 \theta_2^{i,j}) ds - \int_{Q_k \cap \Omega_i} \sum_{l=1}^2 \frac{\partial f}{\partial x_l} g_l dx, \quad (3.49)$$

$$\int_{Q_k \cap \Omega_i} f \operatorname{div} g dx = \int_{\Gamma_{i,j} \cap Q_k} (f_j(x) + u(x)) (g_1 \theta_1^{i,j} + g_2 \theta_2^{i,j}) ds - \int_{Q_k \cap \Omega_i} \sum_{l=1}^2 \frac{\partial f}{\partial x_l} g_l dx. \quad (3.50)$$

Inserting equations (3.49) and (3.50) into equation (3.48), we obtain

$$\int_{Q_k} f \operatorname{div} g dx = \int_{\Gamma_{i,j} \cap Q_k} (f_j - f_i) (g_1 \theta_1^{i,j} + g_2 \theta_2^{i,j}) ds - \int_{Q_k \cap \Omega_i} \sum_{l=1}^2 \frac{\partial f}{\partial x_l} g_l dx - \int_{Q_k \cap \Omega_j} \sum_{l=1}^2 \frac{\partial f}{\partial x_l} g_l dx. \quad (3.51)$$

Furthermore, we have

$$\sup_{g=(g_l)_{l=1}^2 \in C_0^\infty(Q_k)^2, |g| \leq 1 \text{ in } Q_k} \int_{\Gamma_{i,j} \cap Q_k} (f_j - f_i) (g_1 \theta_1^{i,j} + g_2 \theta_2^{i,j}) ds = \int_{\Gamma_{i,j} \cap Q_k} |f_j - f_i| ds, \quad (3.52)$$

$$\int_{Q_k \cap \Omega_i} \sum_{l=1}^2 \frac{\partial f}{\partial x_l} g_l dx + \int_{Q_k \cap \Omega_j} \sum_{l=1}^2 \frac{\partial f}{\partial x_l} g_l dx = O(1) |Q_k \cap \Omega_i| + O(1) |Q_k \cap \Omega_j| = O(1) |Q_k|, \text{ for } g=(g_l)_{l=1}^2 \in C_0^\infty(Q_k)^2, \quad |g| \leq 1 \text{ in } Q_k. \quad (3.53)$$

where $O(1)$ represents a quantity bounded by a constant which depends only on $f(x)$. Combining equations (3.51)–(3.53), we obtain

$$I_{k,1}(f) = \int_{\Gamma_{i,j} \cap Q_k} |f_j - f_i| ds + O(1) |Q_k|. \quad (3.54)$$

On the other hand, for $\bar{g} = \left(\bar{g}_r \right)_{r=0}^{n+1} \in C_0^\infty(Q_k)^{n+2}$, repeatedly performing the 2D integration by parts and utilizing the fact that $\bar{g}(x) = (0, \dots, 0)$ near the boundary of Q_k , we have

$$\begin{aligned}
\int_{Q_k} f \operatorname{div}_{n+1} \bar{g} \, dx &= \int_{Q_k \cap \Omega_i} f \operatorname{div}_{n+1} \bar{g} \, dx + \int_{Q_k \cap \Omega_j} f \operatorname{div}_{n+1} \bar{g} \, dx \\
&= \sum_{l=1}^{n+1} (-1)^{l-1} \int_{\Gamma_{i,j} \cap Q_k} \left[\sum_{r=0}^{l-1} \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^r \partial x_2^{l-1-r}} \frac{\partial^{n+1-l} \bar{g}_r}{\partial x_2^{n+1-l}} \theta_2^{i,j} + \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^{l-1}} \sum_{r=l}^{n+1} \frac{\partial^{n+1-l} \bar{g}_r}{\partial x_1^{r-l} \partial x_2^{n+1-r}} \theta_1^{i,j} \right] ds \\
&\quad + (-1)^{n+1} \int_{Q_k \cap \Omega_i} \sum_{r=0}^{n+1} \frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \bar{g} \, dx \\
&= \sum_{l=1}^{n+1} (-1)^{l-1} \int_{\Gamma_{i,j} \cap Q_k} \left[\sum_{r=0}^{l-1} \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^r \partial x_2^{l-1-r}} \frac{\partial^{n+1-l} \bar{g}_r}{\partial x_2^{n+1-l}} \theta_2^{i,j} + \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^{l-1}} \sum_{r=l}^{n+1} \frac{\partial^{n+1-l} \bar{g}_r}{\partial x_1^{r-l} \partial x_2^{n+1-r}} \theta_1^{i,j} \right] ds \\
&\quad + (-1)^n \int_{\Gamma_{i,j} \cap Q_k} \left[\sum_{r=0}^n \frac{\partial^n (f_j - f_i)}{\partial x_1^r \partial x_2^{n-r}} \bar{g}_r \theta_2^{i,j} + \frac{\partial^n (f_j - f_i)}{\partial x_1^n} \bar{g}_{n+1} \theta_1^{i,j} \right] ds
\end{aligned}
\tag{3.55}$$

Furthermore, we have

$$\begin{aligned}
&(-1)^n \int_{\Gamma_{i,j} \cap Q_k} \left[\sum_{r=0}^n \frac{\partial^n (f_j - f_i)}{\partial x_1^r \partial x_2^{n-r}} \bar{g}_r \theta_2^{i,j} + \frac{\partial^n (f_j - f_i)}{\partial x_1^n} \bar{g}_{n+1} \theta_1^{i,j} \right] ds \\
&= O(1) |\Gamma_{i,j} \cap Q_k|, \quad \text{for } \bar{g} \\
&= \left(\bar{g}_r \right)_{r=0}^{n+1} \in C_0^\infty(Q_k)^{n+2}, \quad |\bar{g}| \leq 1 \text{ in } Q_k,
\end{aligned}
\tag{3.56}$$

$$\begin{aligned}
&(-1)^{n+1} \int_{Q_k \cap \Omega_i} \sum_{r=0}^{n+1} \frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \bar{g}_r \, dx \\
&\quad + (-1)^{n+1} \int_{Q_k \cap \Omega_j} \sum_{r=0}^{n+1} \frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \bar{g}_r \, dx \\
&= O(1) |Q_k \cap \Omega_i| \\
&\quad + O(1) |Q_k \cap \Omega_j| \\
&= O(1) |Q_k|, \quad \text{for } \bar{g} \\
&= \left(\bar{g}_r \right)_{r=0}^{n+1} \in C_0^\infty(Q_k)^{n+2}, \quad |\bar{g}| \leq 1 \text{ in } Q_k.
\end{aligned}
\tag{3.57}$$

Let us evaluate $I_{k,2}^{n+1}(f)$ in the following two cases.

Case 1. If there exists some $x \in \Gamma_{i,j} \cap Q_k$ such that

$$\sum_{l=1}^n \sum_{r=0}^{l-1} \left| \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^r \partial x_2^{l-1-r}} \theta_2^{i,j} \right| + \sum_{l=1}^n \left| \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^{l-1}} \theta_1^{i,j} \right| \neq 0,$$

then for an arbitrary real constant C and an arbitrary compact set $K \subset Q_k$, there exists

$$\bar{g} = \left(\bar{g}_r \right)_{r=0}^{n+1} \in C_0^\infty(Q_k)^{n+2}, \quad |\bar{g}| \leq 1 \text{ in } Q_k \leq 1 \text{ in } Q_k \text{ such that}$$

$$\begin{aligned}\frac{\partial^{n+1-l}\bar{g}_r}{\partial x_2^{n+1-l}} &= c(-1)^{l-1} \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^r \partial x_2^{l-1-r}} \theta_2^{i,j}, \quad \text{for } 1 \leq l \leq n, 0 \leq r \leq l-1, \quad x \in \Gamma_{i,j} \cap K; \\ \frac{\partial^{n+1-l}\bar{g}_r}{\partial x_1^{r-1} \partial x_2^{n+1-r}} &= c(-1)^{l-1} \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^{l-1}} \theta_1^{i,j}, \quad \text{for } 1 \leq l \leq n, l \leq r \leq n+1, \quad x \in \Gamma_{i,j} \cap K.\end{aligned}\quad (3.58)$$

Consequently,

$$\sum_{\substack{\bar{g} = (\bar{g}_r)_{r=0}^2 \\ \bar{g} \in C_0^\infty(Q_k)^3, |\bar{g}| \leq \ln Q_k}} (-1)^{l-1} \int_{\Gamma_{i,j} \cap Q_k} \times \left[\sum_{r=0}^{l-1} \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^r \partial x_2^{l-1-r}} \frac{\partial^{n+1-l}\bar{g}_r}{\partial x_2^{n+1-l}} \theta_2^{i,j} + \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^{l-1}} \sum_{r=1}^{n+1} \frac{\partial^{n+1-l}\bar{g}_r}{\partial x_1^{r-1} \partial x_2^{n+1-r}} \theta_1^{i,j} \right] ds = +\infty. \quad (3.59)$$

Combining equations (3.55)–(3.57) and (3.59), we obtain

$$I_{k,2}^{n+1}(f) = +\infty. \quad (3.60)$$

Case 2. Otherwise, $\sum_{l=1}^n \sum_{r=0}^{l-1} \left| \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^r \partial x_2^{l-1-r}} \theta_2^{i,j} \right| + \sum_{l=1}^n \left| \frac{\partial^{l-1}(f_j - f_i)}{\partial x_1^{l-1}} \theta_1^{i,j} \right| = 0$ for any $x \in \Gamma_{i,j} \cap Q_k$, and the first term on the right-hand side of equation (3.55) equals 0. Then, we have

$$I_{k,2}^{n+1}(f) = \sup_{\substack{\bar{g} = (\bar{g}_r)_{r=0}^2 \\ \bar{g} \in C_0^\infty(Q_k)^3, |\bar{g}| \leq \ln Q_k}} (-1)^n \int_{\Gamma_{i,j} \cap Q_k} \times \left[\sum_{r=0}^n \frac{\partial^n(f_j - f_i)}{\partial x_1^r \partial x_2^{n-r}} \bar{g}_r \theta_2^{i,j} + \frac{\partial^n(f_j - f_i)}{\partial x_1^n} \bar{g}_{n+1} \theta_1^{i,j} \right] ds + O(1) |Q_k|. \quad (3.61)$$

In this case, we also have

$$\int_{\Gamma_{i,j} \cap Q_k} |f_j(x) - f_i(x)| ds = 0, \quad (3.62)$$

$$I_{k,1}(f) = \int_{\Gamma_{i,j} \cap Q_k} |f_j - f_i| ds + O(1) |Q_k| = O(1) |Q_k|. \quad (3.63)$$

Combining equations (3.60), (3.61) and (3.63), we arrive at the following conclusion. If Q_k covers the common boundary $\Gamma_{i,j}$ of a pair of neighboring sub-domains Ω_i and Ω_j and is contained in $\Omega_i \cup \Omega_j$, then

$$I_k^{n+1}(f) = \min \{I_{k,1}(f), I_{k,2}^{n+1}(f)\} = I_{k,1}(f) = \int_{\Gamma_{i,j} \cap Q_k} |f_j - f_i| ds + O(1) |Q_k|. \quad (3.64)$$

Next, let us consider Q_k that is completely contained in some sub-domain Ω_i (figure 4(b)).

For $g = (g_l)_{l=1}^2 \in C_0^\infty(Q_k)^2$, performing the 2D integration by parts again, we have

$$\int_{Q_k} f \operatorname{div} g dx = - \int_{Q_k} \nabla f \cdot g dx, \quad (3.65)$$

$$I_{k,1}(f) = \int_{Q_k} |\nabla f| dx = \int_{Q_k} \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2} dx. \quad (3.66)$$

Using a similar approach, we have

$$\int_{Q_k} f \operatorname{div}_{n+1} \bar{g} dx = (-1)^{n+1} \int_{Q_k} \sum_{r=0}^{n+1} \frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \bar{g}_r dx, \quad \text{for } \bar{g} = (\bar{g}_r)_{r=0}^{n+1} \in C_0^\infty(Q_k)^{n+2}, \quad (3.67)$$

$$I_{k,2}^{n+1}(f) = \int_{Q_k} \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2} dx; \quad (3.68)$$

therefore, we have

$$I_k^{n+1}(f) = \min \left(\int_{Q_k} \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2} dx, \int_{Q_k} \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2} dx \right) \\ \int_{Q_k} \min \left\{ \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2}, \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2} \right\} dx + O(1) |Q_k|, \quad (3.69)$$

where we have used the fact that $f_0(x)$ is a piecewise polynomial function,

$\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} (0 \leq r \leq n+1)$ are continuous at $x \in \Omega_a \bigcup_{i=1}^m \bigcup_{j \in N_i} \Gamma_{i,j}$. Combining (3.64) and (3.69), we obtain

$$\begin{aligned}
& \sum_{k=1}^M I_k^{n+1}(f) \\
&= \sum_{i=1}^m \sum_{j>i, j \in N_i} \int_{\Gamma_{i,j}} |f_i - f_j| ds \\
&\quad + \int_{\Omega_a} \min \left\{ \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2}, \sqrt{\left(\frac{\partial f}{\partial x_1} \right)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2} \right\} dx \\
&\quad + O(1) \left(\sum_{i=1}^m \sum_{j>i, j \in N_i} |\Gamma_{i,j}| \right) \left(\max_{1 \leq k \leq M} \text{diam} \{Q_k\} \right) \\
&\quad + o(1) |\Omega_a|. \tag{3.70}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\text{HOT}_{n+1}(f) &= \limsup_{\left(\max_{1 \leq k \leq M} \text{diam} \{Q_k\} \right) \rightarrow 0} \sum_{k=1}^M I_k^{n+1}(f) \\
&= \sum_{i=1}^m \sum_{j>i, j \in N_i} \int_{\Gamma_{i,j}} |f_i - f_j| ds + \int_{\Omega_a} \min \left\{ \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2}, \sqrt{\left(\frac{\partial f}{\partial x_1} \right)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2} \right\} dx,
\end{aligned}$$

(3.71)

which completes the proof of lemma 5.

Note that if $n = 0$, equation (3.47) can be reduced to the conventional TV

$$\text{HOT}_1(f) = \sum_{i=1}^m \sum_{j>i, j \in N_i} \int_{\Gamma_{i,j}} |f_i - f_j| ds + \int_{\Omega_a} \sqrt{\left(\frac{\partial f}{\partial x_1} \right)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2} dx.$$

Finally, we have the following theorem on the uniqueness of interior SPECT.

Theorem 4. Suppose that $f_0(x)$ is a piecewise n th ($n \geq 1$) order polynomial function in Ω_a , as defined in lemma 5. If $h(x)$ is a candidate image and $\text{HOT}_{n+1}(h) = \min \text{HOT}_{n+1}(f)$ where $u_1(x)$ is an arbitrary ambiguity image, then $h(x) = f_0(x)$ for $x \in \Omega_a$.

Proof. Let $h(x) = f_0(x) + u(x)$ for some ambiguity image $u(x)$. By lemma 5, we have

$$\text{HOT}_{n+1}(f) \geq \sum_{i=1}^m \sum_{j>i, j \in N_i} \int_{\Gamma_{i,j}} |f_i - f_j| ds, \quad \text{for } f = f_0(x) + u_1(x), \tag{3.72}$$

where $u_1(x)$ is an arbitrary ambiguity image, and

$$\text{HOT}_{n+1}(f_0) = \sum_{i=1}^m \sum_{j>i, j \in N_i} \int_{\Gamma_{i,j}} |f_i - f_j| ds. \quad (3.73)$$

Hence

$$\min_{f=f_0+u_1} \text{HOT}_{n+1}(f) = \sum_{i=1}^m \sum_{j>i, j \in N_i} \int_{\Gamma_{i,j}} |f_i - f_j| ds. \quad (3.74)$$

Because $\text{HOT}_{n+1}(h) = \min_{f=f_0+u} \text{HOT}_{n+1}(f)$, we have

$$\int_{\Omega_a} \min \left\{ \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2}, \sqrt{\left(\frac{\partial h}{\partial x_1} \right)^2 + \left(\frac{\partial h}{\partial x_2} \right)^2} \right\} dx = 0. \quad (3.75)$$

Therefore,

$$\min \left\{ \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2}, \sqrt{\left(\frac{\partial h}{\partial x_1} \right)^2 + \left(\frac{\partial h}{\partial x_2} \right)^2} \right\} = 0, \quad \text{for } x \in \Omega_a \setminus \bigcup_{i=1}^m \bigcup_{j>i, j \in N_i} \Gamma_{i,j}. \quad (3.76)$$

That is,

$$\left(\frac{\partial h}{\partial x_1} \right)^2 + \left(\frac{\partial h}{\partial x_2} \right)^2 = 0 \text{ or } \sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2 = 0, \quad \text{for } x \in \Omega_a \setminus \bigcup_{i=1}^m \bigcup_{j>i, j \in N_i} \Gamma_{i,j}. \quad (3.77)$$

We assert that

$$\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2 = 0, \quad \text{for } x = (x_1, x_2) \in \Omega_a \setminus \bigcup_{i=1}^m \bigcup_{j>i, j \in N_i} \Gamma_{i,j}. \quad (3.78)$$

Otherwise, there must exist some $x_0 \in \Omega_a \setminus \bigcup_{i=1}^m \bigcup_{j>i, j \in N_i} \Gamma_{i,j}$ such that

$$\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2 > 0. \quad (3.79)$$

By continuity, there exists a neighborhood of x_0 denoted by Ω_{x_0} such that

$$\Omega_{x_0} \subset \Omega_a \setminus \bigcup_{i=1}^m \bigcup_{j>i, j \in N_i} \Gamma_{i,j}, \quad (3.80)$$

and

$$\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2 > 0, \quad \text{for } x \in \Omega_{x_0}. \quad (3.81)$$

By equation (3.77), we have

$$\left(\frac{\partial h}{\partial x_1} \right)^2 + \left(\frac{\partial h}{\partial x_2} \right)^2 = 0, \quad \text{for } x \in \Omega_{x_0}, \quad (3.82)$$

e.g.

$$\frac{\partial h}{\partial x_1} = \frac{\partial h}{\partial x_2} = 0, \quad \text{for } x \in \Omega_{x_0}. \quad (3.83)$$

Therefore,

$$\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} = 0, \quad \text{for } x \in \Omega_{x_0}, \quad 0 \leq r \leq n+1, \quad (3.84)$$

which leads to

$$\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2 = 0, \quad \text{for } x \in \Omega_{x_0}. \quad (3.85)$$

This is in contradiction to equation (3.81). Equation (3.78) implies that

$$\frac{\partial^{n+1} h}{\partial x_1^r \partial x_2^{n+1-r}}(x) = 0, \quad \text{for } x \in \Omega_a \setminus \bigcup_{i=1}^m \bigcup_{j>i, j \in N_i} \Gamma_{i,j}, \quad 0 \leq r \leq n+1. \quad (3.86)$$

Because $f_0(x)$ is a piecewise n th ($n \geq 1$) order polynomial function in Ω_a , it follows from equation (3.86) that

$$\frac{\partial^{n+1} u}{\partial x_1^r \partial x_2^{n+1-r}}(x) = 0, \quad \text{for } x \in \Omega_a \setminus \bigcup_{i=1}^m \bigcup_{j>i, j \in N_i} \Gamma_{i,j}, \quad 0 \leq r \leq n+1. \quad (3.87)$$

Due to the analyticity of $u(x)$ in theorem 1, we have

$$\frac{\partial^{n+1} u}{\partial x_1^r \partial x_2^{n+1-r}}(x) = 0, \quad \text{for } x \in \Omega_a, \quad 0 \leq r \leq n+1. \quad (3.88)$$

Hence, $u(x)$ is an n th ($n \geq 1$) order polynomial function in Ω_a . By theorem 2, we obtain $u(x) = 0$ and $h(x) = f_0(x)$.

To justify a numerical implementation of the HOT minimization for interior SPECT in the next section, we need the following theorem.

Theorem 5. Suppose that $f_0(x)$ is a piecewise n th ($n \geq 1$) order polynomial function in Ω_a , as defined in lemma 5. For any candidate image $f = f_0 + \mu$, define the modified $(n+1)$ th order TV as

$$\text{HOT}_{n+1}^{\text{mod}}(f) = \sum_{i=1}^m \int_{\Omega_i} \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2} dx. \quad (3.89)$$

If $h(x)$ is a candidate image and $\text{HOT}_{n+1}^{\text{mod}}(h) = \min_{f=f_0+u_1} \text{HOT}_{n+1}^{\text{mod}}(f)$ where $u_1(x)$ is an arbitrary ambiguity image, then $h(x) = f_0(x)$ for $x \in \Omega_a$.

Proof. It can be similarly proved as theorem 4. Since

$$\text{HOT}_{n+1}^{\text{mod}}(f_0) = \sum_{i=1}^m \int_{\Omega_i} \sqrt{\sum_{r=0}^{n+1} \left(\frac{\partial^{n+1} f_i}{\partial x_1^r \partial x_2^{n+1-r}} \right)^2} dx = 0, \quad (3.90)$$

the condition

$$\text{HOT}_{n+1}^{\text{mod}}(h) = \min_{f=f_0+u_1} \text{HOT}_{n+1}^{\text{mod}}(f), \quad (3.91)$$

leads to $\text{HOT}_{n+1}^{\text{mod}}(h) = 0$. Therefore, equation (3.86) holds, and the conclusion can be deduced

Compared to $\text{HOT}_{n+1}(f)$, $\text{HOT}_{n+1}^{\text{mod}}(f)$ only takes into account the $(n+1)$ th derivatives of an image f inside sub-regions but overlooks image jumps across the boundaries between neighboring regions. Hence, $\text{HOT}_{n+1}(f)$ is a more faithful generalization of the TV.

4. Numerical simulation

To verify our theoretical findings obtained in section 3, we developed a HOT minimization-based interior SPECT algorithm in an iterative framework. This algorithm is a modification of our previously reported HOT minimization-based interior tomography algorithm [24]. The major difference between this algorithm and that in [24] lies in the formulations for the steepest gradient of HOT and the ordered-subset simultaneous algebraic reconstruction technique (OSSART) [29]. Let $f_{u,v} = f(u\Delta, v\Delta)$ be a digital image reconstructed from the available local projections, where Δ represents the sampling interval, and u and v are integers. To demonstrate the computation for the steepest gradient direction, here we use the piecewise-linear case as an example. For any candidate image $f = f_0 + \mu$, with μ being analytic, we introduce an approximate discretization of $\text{HOT}_2(f)$ as follows:

$$\text{HOT}_2^{\text{dis}}(f) = \sum_{u,v} \sqrt{(D_{11}(u,v))^2 + (D_{12}(u,v))^2 + (D_{22}(u,v))^2} = \sum_{u,v} G(u,v), \quad (4.1)$$

where $D_{11}(u, v) = f_{u+1,v} + f_{u-1,v} - 2f_{u,v}$, $D_{12}(u, v) = (f_{u+1,v+1} + f_{u-1,v-1} - f_{u-1,v+1} - f_{u+1,v-1})/4$, $D_{22}(u, v) = f_{u,v+1} + f_{u,v-1} - 2f_{u,v}$ are the second-order finite differences along the coordinate directions, for $i, j = 1, 2$, respectively.

To justify the validity of $\text{HOT}_2^{\text{dis}}$, we split $\text{HOT}_2^{\text{dis}}$ into two terms:

$$\text{HOT}_2^{\text{dis}}(f) = \sum_{(u,v) \text{ across } \Gamma_{ij}, 1 \leq i \leq m, j \in N_i} G(u, v) + \sum_{(u,v) \text{ inside } \Omega_i, 1 \leq i \leq m} G(u, v) = \text{HOT}_{2,1}^{\text{dis}}(f) + \text{HOT}_{2,2}^{\text{dis}}(f), \quad (4.2)$$

where (u, v) across Γ_{ij} means 'some of the nine discretization points (u, v) , $(u+1, v)$, $(u-1, v)$, $(u+1, v+1)$, $(u-1, v-1)$, $(u-1, v+1)$, $(u+1, v-1)$, $(u, v+1)$, $(u, v-1)$ are in the region Ω_i , the others are in region Ω_j ', and (u, v) inside Ω_i means 'the nine discretization points are all in Ω_i '.

$\text{HOT}_{2,1}^{\text{dis}}(f)$ captures the image jump across the boundaries between neighboring sub-regions, and mainly depends on f_0 because of the fact $f = f_0 + \mu$ with μ being analytic, and $\text{HOT}_{2,2}^{\text{dis}}(f)$ is the discretization of $\text{HOT}_2^{\text{mod}}(f)$ and represents the second derivative variation of the image f inside sub-regions. Therefore, we have

$$\text{agr/min}_{f=f_0+\mu} \text{HOT}_2^{\text{dis}}(f) = \text{agr/min}_{f=f_0+\mu} (\text{HOT}_{2,1}^{\text{dis}}(f) + \text{HOT}_{2,2}^{\text{dis}}(f)) \simeq \text{agr/min}_{f=f_0+\mu} (\text{HOT}_{2,1}^{\text{dis}}(f_0) + \text{HOT}_{2,2}^{\text{dis}}(f)) = \text{agr/min}_{f=f_0+\mu} \text{HOT}_{2,2}^{\text{dis}}(f) \simeq \dots \quad (4.3)$$

The last approximation comes from theorem 5.

It is easy to verify that

$$\begin{aligned}
& \frac{\partial \text{HOT}_2^{\text{dis}}(f)}{\partial f_{u,v}} \\
&= \frac{D_{11}(u-1, v)}{G(u-1, v)} \\
&+ \frac{D_{11}(u+1, v)}{G(u+1, v)} \\
&+ \frac{D_{22}(u, v-1)}{G(u, v-1)} \\
&+ \frac{D_{22}(u, v+1)}{G(u, v+1)} \\
&+ \frac{1}{4} \left(\frac{D_{12}(u-1, v-1)}{G(u-1, v-1)} + \frac{D_{12}(u+1, v+1)}{G(u+1, v+1)} - \frac{D_{12}(u-1, v+1)}{G(u-1, v+1)} - \frac{D_{12}(u+1, v-1)}{G(u+1, v-1)} \right) \\
&- \frac{2(D_{11}(u, v) + D_{22}(u, v))}{G(u, v)}.
\end{aligned}$$

(4.4)

For CT scanning systems, the discrete model of projections in terms of the Radon transform can be expressed as [29]

$$Af=b, \quad (4.5)$$

where data $b = (b_1, \dots, b_M) \in R^M$ represents the measured projections and each b_m is an x-ray path, 1D vector $f = (f_1, \dots, f_N) \in R^N$ reformatted from the 2D image $f_{u,v}$ and a known non-zero matrix $A = (a_{mn})$ whose component a_{mn} is the intersection area between the m th x-ray path and n th pixel. The SART solution of equation (4.5) is [29]

$$f_n^{(k+1)} = f_n^{(k)} + \lambda_k \frac{1}{a_{+n}} \sum_{m=1}^M \frac{a_{mn}}{a_{m+}} (b_m - A_m f^{(k)}), \quad (4.6)$$

where k is the iteration number, λ is a relax parameter, and we require that

$$\begin{aligned}
a_{m+} &\equiv \sum_{n=1}^N a_{mn} \neq 0, \quad m=1, \dots, M, \\
a_{+n} &\equiv \sum_{m=1}^M a_{mn} \neq 0, \quad n=1, \dots, N.
\end{aligned} \quad (4.7)$$

Considering the attenuation Radon transform of SPECT as expressed by equations (2.1) and (4.5) can be modified as

$$\tilde{A}f=b, \quad (4.8)$$

with $\tilde{A} = (a_{mn} w_{mn}^{\mu_0})$, where $w_{mn}^{\mu_0}$ is the corresponding discrete term $e^{\mu_0 t}$ in equation (2.1). And the SART solution for SPECT can be expressed as

$$f_n^{(k+1)} = f_n^{(k)} + \lambda_k \frac{1}{a_{+n}} \sum_{m=1}^M \frac{a_{mn}}{\tilde{a}_{m+}} (b_m - \tilde{A}_m f^{(k)}), \quad (4.9)$$

$$\begin{aligned} \tilde{a}_{m+} &\equiv \sum_{n=1}^N a_{mn} w_{mn}^{\mu_0} \neq 0, \quad m=1, \dots, M, \\ a_{+n} &\equiv \sum_{m=1}^M a_{mn} \neq 0, \quad n=1, \dots, N. \end{aligned} \quad (4.10)$$

Clearly, the CT reconstruction formula is the special case $\mu_0 = 0$ of the SPECT solution. Because key details of our algorithm were already reported [21, 24], here we omit the implementation steps for simplification.

In our numerical simulation, we assumed a popular parallel-beam imaging geometry in the SPECT field. All other imaging geometry parameters and reconstruction control parameters are the same as in our previous paper [24]. At the origin, we assumed a disk-shaped compact support of radius 100 mm with a uniformly attenuation coefficient μ_0 . Inside the compact support, there is a modified and piecewise linear Shepp–Logan phantom, whose parameters were listed in table 1 in our previous work [24]. Representative reconstructed images are shown in figures 5 and 6. As seen in figures 5 and 6, the reconstructed SPECT images using our proposed HOT minimization algorithm are in excellent agreement with the truth inside the ROI. However, it can be observed that the profiles near the edges deviate substantially from the truth. Although a rigorous stability analysis has not been performed yet, it is conjectured that this phenomenon is a stability issue of interior tomography. We believe that this behavior is similar to what we analyzed for the knowledge-based interior CT [10]. The closer the distance to the peripheral region of an ROI, the less the stability of the interior reconstruction. A different numerical implementation could reduce this discrepancy. However, it cannot be completely removed unless stronger prior knowledge is incorporated for interior reconstruction.

To demonstrate the feasibility of the proposed approach, we downloaded a head SPECT image (<http://www.medsci.ox.ac.uk/optima/old-page/selected-highlights/spect>) from the Internet and modified it into a realistic image phantom (see figure 7). It represents a radioisotope distribution inside a human head slice. We assumed a disk-shaped attenuating background of radius 100 mm with a constant attenuating background μ_0 , and all other imaging geometry and reconstruction parameters being the same as before. Clearly, this realistic image phantom does not satisfy the piecewise polynomial model for the cases of $n = 0$ and $n = 1$. To improve the stability of interior SPECT, two additional constraints were incorporated into the OS-SART loop in the projection-onto-convex-sets framework. The first is the non-negativity, which means that the radioisotope distribution should not be negative. Thus, we made the negative values zero during the iterative process. The second is the compactness, which means that the radioisotope distribution should be inside the head. That is, we assumed a known head contour and forced the pixels outside the head to be zero iteratively. As a benchmark, we also implemented the piecewise constant model (case $n = 0$) for comparison analysis. The results are in figure 7. It can be seen from figure 7 that the proposed algorithm produced excellent results even if the piecewise polynomial model is not exactly satisfied. Surprisingly, the performance of the piecewise constant model ($n = 0$) is also favorable. Therefore, it is our hypothesis that $n < 3$ should be enough for practical applications.

5. Discussions and conclusion

The above-reported interior SPECT results in the case of a uniform attenuation background represent a significant step forward relative to what we reported before [6] in which a known sub-region in an ROI is required. Although some sub-region can be assumed known in certain types of SPECT applications, such an assumption is not generally valid. Our new interior SPECT theory is based on a piece-wise polynomial model of an ROI image, which is quite general, and thus opens a door towards accurate interior SPECT reconstruction without any specific information on the ROI. Clearly, this type of interior SPECT is potentially useful for numerous practical applications.

In our numerical simulation, the realistic head phantom results show that both piecewise constant and piecewise linear image models work well. In other words, the improvement in the case $n = 1$ is not so significant as compared to that in the case $n = 0$. There are basically two reasons. First, the spatial resolution of this phantom is low, and the image variation is small. Therefore, it can be well approximated by a piecewise constant model. Second, we assumed 720 views in a full scan, which implies a strong data fidelity constraint and leads to an excellent local reconstruction. To demonstrate the advantage of the HOT ($n = 1$) method over the TV method ($n = 0$), we constructed another phantom, which is the Shepp–Logan phantom superposed by a sine function (figure 8(a)). We repeated the aforementioned experiment but with 80 views and 600 iterations. Comparing the error images (figures 8(b) and (c)), it can be observed that the HOT method has a better performance than the TV method. In practical applications, we could identify more complicated examples, such as cardiac or brain images in which perfusion gradients are significant, to demonstrate the same effect, but we consider this attempt beyond the scope of this paper and will do so in the future.

An improved formulation of HOT has been introduced in this paper to perform interior SPECT reconstruction. Compared with the original formulation of HOT in [24], the new one has an explicit formula for any n th order piecewise function. This is important for digital implementation, while the original formulation [24] does not have an explicit formula with the cases $n \geq 2$. Although the new formulation of HOT is different from that in [24], they are equivalent in the sense of $C_1(n) \text{HOT}_{n+1}^{\text{old}}(f) \leq \text{HOT}_{n+1}^{\text{new}}(f) \leq C_2(n) \text{HOT}_{n+1}^{\text{old}}(f)$, where $\text{HOT}_{n+1}^{\text{old}}$ represents the HOT defined in [24], $\text{HOT}_{n+1}^{\text{new}}$ the HOT defined in this paper and $C_1(n)$ and $C_2(n)$ are constants that depend on the polynomial order n . This result has been rigorously proved by us, but we prefer not to include the details for brevity.

The current discretization in (4.1) is a numerical approximation for $\text{HOT}_2^{\text{dis}}$. The general expression of $\text{HOT}_{n+1}^{\text{dis}}$ would be similar to (4.1) and include only all the possible $(n+1)$ th order finite differences, especially $\text{HOT}_1^{\text{dis}} = \sqrt{(f_{u,v} - f_{u-1,v})^2 + (f_{u,v} - f_{u,v-1})^2}$ ($n=0$), which reduces to the conventional discretization for TV. Although the current numerical approximation scheme is straightforward and easy for implementation, it may not be the optimal one. Further work is needed to find more efficient discrete HOT schemes. Techniques such as those for finding shock wave solutions in the community of partial differential equations [30] could be helpful, where complicated discretization methods were proven useful.

Although we have established the uniqueness of interior SPECT, its stability remains a challenge to be addressed. Numerically, our simulation results have indicated a satisfactory imaging performance. There should be excellent research opportunities in this regard. The stability of interior SPECT would rely on the interior SPECT scheme to be used, prior

knowledge available in a specific application, as well as imaging requirements, an object under consideration and an underlying radioactive probe distribution therein. Our feeling is that error bounds of interior SPECT would be practically acceptable in many applications. Also, an important direction is to generalize our results in the case of a non-uniform attenuation background.

There are at least two advantages associated with interior SPECT. First of all, the size of a SPECT camera can accordingly be reduced, substantially lowering the system cost. This is especially desirable when an expensive high-resolution camera is needed or a low-cost system is intended for wider healthcare coverage. Use of a smaller field of view would also allow the camera to be closer to a subject under study, improving the signal-to-noise ratio. Furthermore, a higher resolution detector of a smaller size would improve spatial resolution of the associated interior image reconstruction. To support this point, we simulated a full SPECT reconstruction with the total number of measurement lines being the same as that for the above interior SPECT setting, and obtained results in figure 9. Since figure 9(f) corresponds to a boundary between two sub-regions, we can measure the image spatial resolution from the profile across this boundary. It is noted that the interior SPECT scheme gives ~30% resolution improvement relative to the global SPECT reconstruction, because a smaller iso-center detector is used for interior SPECT assuming the same number of total measurement lines as that in the global SPECT case. The second advantage, which can be more interesting, is that interior SPECT may greatly improve temporal resolution of SPECT with a multi-camera system architecture. This concept can be considered as an extension of the multi-source interior tomography scheme proposed in the CT field [31]. In a multi-camera interior SPECT system (MISS), multiple SPECT cameras can be arranged along a circle and collimated towards an internal ROI or VOI centered at the iso-center on the imaging plane to acquire a complete dataset. More symmetrically, these cameras can be so arranged that each of them defines one facet of a polyhedron such as a regular polyhedron. Because of our interior SPECT theory, the size of such a MISS can be made more compact than that built according to classic SPECT theory. As a result, we will have not only a higher performance-to-cost ratio but also a faster imaging speed to extract functional, molecular and cellular information tomographically.

In conclusion, we have formulated an interior SPECT problem assuming a piece-wise polynomial ROI and a constant attenuation background, proposed high-order TV measures, established the uniqueness of interior SPECT via the HOT minimization and demonstrated the feasibility of interior SPECT in the numerical simulation. Also, we have proposed to prototype a multi-camera interior SPECT system based on interior SPECT theory that may significantly improve clinical and preclinical nuclear imaging. Finally, there are a number of other research opportunities, such as the generalization of the current results into a non-uniform attenuation setting.

Acknowledgments

This work was partially supported by the National Basic Research Program of China (973 Program) (2011CB809105), the National Science Foundation of China (10990013, 60325101, 60872078), Key Laboratory of Machine Perception (Ministry of Education) of Peking University, Microsoft Research of Asia, USA NSF/MRI program (CMMI-0923297) and NIH/NIBIB grant (EB011785). The authors are grateful for the constructive comments from reviewers, which helped improve the quality of the manuscript.

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Figure 1.
Configuration of a compact support and a region-of-interest (ROI) for interior SPECT.

Figure 2.
Radial line through the origin.

Figure 3.
ROI consisting of seven sub-domains.

Figure 4.
Illustration of Q_k of (a) the first and (b) second types.

Figure 5. Interior SPECT reconstruction of the Shepp–Logan phantom modified with linear variations over sub-domains. (a) The original phantom; (b) and (c) the reconstructions using the proposed HOT minimization-based algorithm after 40 iterations with the attenuation coefficient $\mu_0 = 0$ and $\mu_0 = 0.15$, respectively. The display window for (a)–(c) is $[0.1, 0.4]$. (d) and (e) Representative profiles along the white (d) horizontal and (e) vertical lines. The horizontal axis in (d) and (e) represents the 1D coordinate, the vertical axis denotes the functional value and the black thick lines on the horizontal axis indicate the ROI.

Figure 6.
Same as in figure 5 but with 1% Gaussian noise in the measured data.

Figure 7.

Interior SPECT reconstruction of a realistic head phantom. (a) The original phantom; (b) and (c) the reconstructions using the proposed HOT minimization-based algorithm after 40 iterations with the attenuation coefficient $\mu_0 = 0.15$ for $n = 0$ and $n = 1$, respectively. The display window for (a)–(c) is $[0 \ 1.0]$. (d)–(f) The pseudo-color counterparts of (a)–(c) using a brain color map. (g) and (h) The representative profiles along the white (g) horizontal and (h) vertical lines, respectively. The horizontal axis in (g) and (h) represents the 1D coordinate, and the vertical axis denotes the functional value, where the black thick lines on the horizontal axis indicates the ROI. And the circles in all the images indicate the ROI.

Figure 8.

Advantage of the HOT minimization ($n = 1$) over the TV minimization ($n = 0$). (a) The Shepp-Logan phantom superposed by a sine function and displayed in the window $[0,1]$; (b) and (c) the absolute error images associated with the HOT and TV minimization methods, respectively, displayed within the window $[0,0.4]$. The white circles indicate the ROI.

Figure 9.

Demonstration of spatial resolution improvement with interior SPECT. (a) A high-resolution interior reconstruction in a 1024×1024 matrix using the same projection dataset and parameters as those for figure 5(c); (b) the counterpart of (a) using the same detector elements covering the whole object; (c) and (d) the magnifications of the ROIs in (a) and (b), respectively; (e) the profiles along the white lines in (c) and (d), respectively; and (f) the magnification of a portion of (e) demonstrating the spatial resolution improvement with the interior SPECT approach.