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A simple scheme for generating nearly uniform distribution of antipodally symmetric points on the unit sphere

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Abstract

A variant of the Thomson problem, which is about placing a set of points uniformly on the surface of a sphere, is that of generating uniformly distributed points on the sphere that are endowed with antipodal symmetry, i.e., if x is an element of the point set then $-x$ is also an element of that point set. Point sets with antipodal symmetry are of special importance to many scientific and engineering applications. Although this type of point sets may be generated through the minimization of a slightly modified electrostatic potential, the optimization procedure becomes unwieldy when the size of the point set increases beyond a few thousands. Therefore, it is desirable to have a deterministic scheme capable of generating this type of point set with near uniformity. In this work, we will present a simple deterministic scheme to generate nearly uniform point sets with antipodal symmetry.

Keywords

uniform distribution on sphere; antipodal symmetry; deterministic scheme

1. Introduction

A variant of the Thomson problem [17, 6], is that of generating a nearly uniform distribution of antipodally symmetric points on the sphere via a deterministic scheme. Point sets with antipodal symmetry are of special importance to many scientific and engineering applications, e.g., 3D radial MRI [14], Diffusion Tensor Imaging [3, 9, 11] or other higher order diffusion imaging techniques [18, 7, 2]. Interestingly, the use of point sets without antipodal symmetry are more prevalent such as in non-Cartesian MR trajectory [8], Material Science [19] and many other applications e.g. [15, 4, 20, 8, 1]. Although the point sets with antipodal symmetry may be generated through the minimization of a slightly modified electrostatic potential as described in [9, 5], the optimization procedure becomes unwieldy when the size of the point set increases beyond a few thousands. Therefore, it is desirable to have a deterministic scheme capable of generating uniform distribution of points on the sphere that are endowed with antipodal symmetry.

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Many deterministic schemes have been proposed for generating uniform distribution of points on the sphere without the constraint of antipodal symmetry, e.g. [15, 16, 4, 20, 8, 1, 12]. As noted in our recent work [12], spiral point sets are topologically incompatible with antipodal symmetry and the usual ad hoc strategy to come up with an antipodally symmetric point set is to collect half of the points that are located on one of the hemispheres. However, the main problem with this strategy is that the discrepancy in the uniformity of the distribution is clearly visible along the equator.

In this work, we will present a simple deterministic scheme to generate nearly uniform point sets with antipodal symmetry. The present scheme is intuitive and geometrically motivated, and it is built upon some of the concepts developed in our recent work [12] and the works of Bauer [4] and Gurney et al.[8].

2. Methods

Since the spiral curve is topologically incompatible with antipodal symmetry, we will have to devise a different strategy for generating spherical point sets with antipodal symmetry. For the sake of simplicity, we use a set of equidistant latitude circles (Figure 1), which has been used elsewhere such as the design of 3D cone trajectories presented by Gurney et al. [8]. More importantly, we should point out that the boundary conditions, the placement of the first and the last latitude circles, should be treated with care to ensure uniformity. Furthermore, the spacing between two consecutive points on the same latitude should be approximately equal to the spacing between two consecutive latitudes; this particular intuitive concept has been applied with great success in the design of spiral point sets, e.g., [12, 4].

We shall begin by stating the convention of the spherical coordinate system used in this work. The spherical coordinate on the unit sphere can be described by, (θ, ϕ) , and its transformation to Cartesian coordinate, (x, y, z) , can be accomplished by the following transformations:

$$\begin{aligned}x &= \sin(\theta) \cos(\phi), \\y &= \sin(\theta) \sin(\phi), \\z &= \cos(\theta).\end{aligned}$$

The first step in designing the antipodally symmetric spherical point set is to determine the total length of all the latitude circles, which we shall denote as L . The circumference of each latitude circle is given by:

$$2\pi \sin(\theta_i), \quad i=1, \dots, n,$$

with

$$\theta_i = \left(i - \frac{1}{2}\right) \frac{\pi/2}{n}, \quad i=1, \dots, n.$$

Here, we will treat n , the number of latitude circles, as an unknown and will determine the value of n by solving a non-algebraic equation that arises out of the criterion that the spacing between two consecutive points on the same latitude should be approximately equal to the spacing between two consecutive latitudes, see e.g., [12]. We will discuss the motivation behind our choice of θ 's. First, it is clear that the spacing between two neighboring latitude

circles is $\frac{\pi/2}{n}$ and we should note that we cover only the upper hemisphere. Second, the term, $\frac{1}{2}$, is used to ensure that the first latitude circle has an angular 'width' of $\frac{\pi/2}{n}$ and that the spacing between the last latitude circle and its antipodal counterpart is also $\frac{\pi/2}{n}$. Based on the information given above, it can be shown that L can be expressed as:

$$\begin{aligned} L &= 2\pi \sum_{i=1}^n \sin(\theta_i), \\ &= \pi \csc\left(\frac{\pi}{4n}\right). \end{aligned} \quad (1)$$

The derivation of Eq.(1) does involve some trigonometric manipulations. Please refer to Appendix A for the derivation.

The second step is to take the number of points on the upper hemisphere, denoted by K , as known. By equating, the spacing between two consecutive points on the same latitude and the spacing between two consecutive latitudes, we arrive at the following nonlinear equation of n :

$$\frac{L}{K} = \frac{\pi/2}{n},$$

or

$$n = \frac{K}{2} \sin\left(\frac{\pi}{4n}\right). \quad (2)$$

Equation (2) is nonlinear and the solution can be obtained via fixed point iteration or Newton's method such as that described in our previous works, [10, 12]. Please refer to Appendix B for further details. The initial guess for the fixed point iteration can be gleaned from the asymptotic solution of Eq.(2) for the case when n is large, which is given by:

$$n \approx \frac{K\pi}{8n}, \quad (3)$$

or

$$n \approx \sqrt{\frac{K\pi}{8}}. \quad (4)$$

Figure 2 shows the behavior of the iterative solution (rounded to the nearest integer) and the asymptotic solution as functions of the number of points, K . We shall use the integer closest to the iterative solution as the number of latitude circles and will denote it by $[n]$.

The final step is to find the number of points for each latitude. Let k_i be the number of points at latitude θ_i and we define $[x]$, Lx and $\lceil x \rceil$ as the integer closest to x , closest to but less than x and closest to but greater than x , respectively. It is clear that the ratio of each

circumference, $2\pi \sin(\theta_i)$, to L is approximately equal to k_i/K . Therefore, we may express k_i as follows:

$$k_i = \begin{cases} \left\langle \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rangle_{\alpha} & : i=1, 3, 5, \dots, \text{except } [n] \text{ if } [n] \text{ is odd} \\ \left\langle \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rangle_{\beta} & : i=2, 4, 6, \dots, \text{except } [n] \text{ if } [n] \text{ is even} \\ K - \sum_i^{[n]-1} k_i & : i=n. \end{cases} \quad (5)$$

Please note that α and β may take the value of 1, 2 and 3, and $\langle \cdot \rangle_1 \equiv \lceil \cdot \rceil$, $\langle \cdot \rangle_2 \equiv \lceil \cdot \rceil$, and $\langle \cdot \rangle_3 \equiv \lfloor \cdot \rfloor$. Although there are as many as $2^{[n]-1}$ distinct choices of k 's, we shall take a more realistic approach by evaluating only some subsets of these k 's that are common and easy to use.

These subsets are listed below:

$$k_i^1 = \begin{cases} \left\lceil \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rceil & : i=1, 3, 5, \dots, \text{except } [n] \text{ if } [n] \text{ is odd} \\ \left\lfloor \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rfloor & : i=2, 4, 6, \dots, \text{except } [n] \text{ if } [n] \text{ is even} \\ K - \sum_i^{[n]-1} k_i & : i=n. \end{cases}$$

$$k_i^2 = \begin{cases} \left\lfloor \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rfloor & : i=1, 3, 5, \dots, \text{except } [n] \text{ if } [n] \text{ is odd} \\ \left\lceil \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rceil & : i=2, 4, 6, \dots, \text{except } [n] \text{ if } [n] \text{ is even} \\ K - \sum_i^{[n]-1} k_i & : i=n. \end{cases}$$

$$k_i^3 = \begin{cases} \left\lceil \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rceil & : i=1, 2, 3, 4, \dots, \text{except } [n] \\ K - \sum_i^{[n]-1} k_i & : i=n. \end{cases}$$

$$k_i^4 = \begin{cases} \left\lfloor \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rfloor & : i=1, 2, 3, 4, \dots, \text{except } [n] \\ K - \sum_i^{[n]-1} k_i & : i=n. \end{cases}$$

$$k_i^5 = \begin{cases} \left\lfloor \frac{2\pi \sin(\theta_i)}{\pi \csc\left(\frac{\pi}{4[n]}\right)} K \right\rfloor & : i=1, 2, 3, 4, \dots, \text{except } [n] \\ K - \sum_i^{[n]-1} k_i & : i=n. \end{cases}$$

Based on numerical simulations with K ranges from inclusively 50 to 12000, we observed that k^5 turned out to be the best configuration in 10000 out of 11951 cases, followed by k^2

with 1666 cases, k^1 in 283 cases and k^4 in 2 cases. None of the k^3 's configurations took the first place in all the 11951 cases tested here. Although the interested user may choose the best configuration or k among the five methods discussed above for a given value of K , such an approach may not be practical when the user is faced with the value of K that is very large, say $K > 100000$. Therefore, we suggest using k^5 for K beyond what we have tested here because the discrepancy between k^5 and the best configuration is small when K is very large. We further note that the functions in Eq.(5) are defined to keep the total number of points on the hemisphere fixed to K and the application of the ceiling, floor or rounding functions, $\lfloor x \rfloor$, $\lceil x \rceil$ or $[x]$, helps to avoid unnecessary accumulation or 'decumulation' of points on the last latitude circle. Lastly, the components of the spherical coordinates, $(\theta_i, \phi_{i,j})$ are given by:

$$\theta_i = \left(i - \frac{1}{2}\right) \frac{\pi/2}{[n]}, \quad i=1, \dots, [n], \quad (6)$$

$$\phi_{i,j} = \left(j - \frac{1}{2}\right) \frac{2\pi}{k_i^\gamma}, \quad j=1, \dots, k_i^\gamma. \quad (7)$$

where k^γ the best configuration for a given value of K and γ may take on any number from 1 to 5. As an illustration, we show here the proposed point sets with 100 and 800 points, see Figure 3.

We also compared the proposed scheme (using the fifth method, i.e., k^5) with the gold standard, some of iteratively optimized point sets have been tabulated and available for public use [21], see Figure 4.

3. Discussion

The objective of this brief communication is to share with the reader a geometrically motivated and intuitive construction for generating spherical point set that are endowed with antipodal symmetry. It is clear that the formulation of our scheme is rigorous but it does leave some room for further experimentation such as the application of the ceiling and floor functions as prescribed above to define k 's. The simplicity of the construction comes at a cost of many discontinuities and hence the application of many ceiling and floor functions. However, the effect on the uniformity of the point set caused by these discontinuities is slight as can be gleaned from Figure 3.

Based on our numerical experience, the proposed scheme works well when K is large, i.e., beyond $K > 60$. We should point out that this limitation is not a problem for 3D radial MRI or higher order diffusion MRI. For example, 3D radial MRI requires K to be above 5000 and higher order diffusion MRI usually requires K to be between 80 to several hundreds. The point set derived from the proposed method may also be used as an initial solution, after some small perturbation on the positions of the points, for the nonlinear optimization of the modified electrostatic potential energy, [9].

It is clear from Figure 4 that the performance of the proposed scheme is comparable and with less than 0.3% of relative error compared to the gold standard as the number of points increases above 50.

Our recent analytically exact spiral scheme and the present scheme are complementary and fill a specific but different need in many scientific and engineering applications. These two works represent our attempt to keep the formulation of the construction as simple and geometric as possible. However, as noted in [16], this type of geometric construction is not applicable or generalizable to higher dimensions, which may be of concerns and of importance in other applications, see e.g.[13].

In summary, we have presented an intuitive, geometric and the first deterministic approach for generating spherical point set with antipodal symmetry.

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Appendix A

In this appendix, we will show the steps needed to arrive at Eq.(1). Equivalently, we simply have to show the following statement:

$$\sum_{i=1}^n \sin(\theta_i) = \frac{1}{2} \csc\left(\frac{\pi}{4n}\right), \quad (8)$$

$$\text{with } \theta_i = \left(i - \frac{1}{2}\right) \frac{\pi/2}{n} \text{ for } i = 1, \dots, n.$$

We will use the following trigonometric property to expand the sine functions, $\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B)$.

$$\begin{aligned} \sum_{i=1}^n \sin(\theta_i) &= \sum_{i=1}^n \sin\left(\left(i - \frac{1}{2}\right) \frac{\pi/2}{n}\right), \\ &= \cos\left(\frac{\pi}{4n}\right) \sum_{i=1}^n \sin\left(i \frac{\pi/2}{n}\right) - \sin\left(\frac{\pi}{4n}\right) \sum_{i=1}^n \cos\left(i \frac{\pi/2}{n}\right). \end{aligned} \quad (9)$$

The following trigonometric summations, which can be derived from complex exponential and geometric series, will be used to further simplify our equation above:

$$\sum_{i=1}^n \sin(ix) = \sin(nx/2) \sin((n+1)x/2) / \sin(x/2), \quad (10)$$

$$\sum_{i=1}^n \cos(ix) = \sin(nx/2) \cos((n+1)x/2) / \sin(x/2). \quad (11)$$

$$\text{with } x = \frac{\pi/2}{n}.$$

It is clear then that we have the desired expression upon substituting Eq.(10) and Eq.(11) into Eq.(9):

$$\begin{aligned}
\sum_{i=1}^n \sin(\theta_i) &= \cos\left(\frac{\pi}{4n}\right) \left(\sin\left(\frac{\pi}{4}\right) \sin\left(\frac{(n+1)\pi}{4n}\right) \csc\left(\frac{\pi}{4n}\right) \right) - \sin\left(\frac{\pi}{4n}\right) \left(\sin\left(\frac{\pi}{4}\right) \cos\left(\frac{(n+1)\pi}{4n}\right) \csc\left(\frac{\pi}{4n}\right) \right), \\
&= \csc\left(\frac{\pi}{4n}\right) \sin\left(\frac{\pi}{4}\right) \left(\cos\left(\frac{\pi}{4n}\right) \sin\left(\frac{(n+1)\pi}{4n}\right) - \cos\left(\frac{(n+1)\pi}{4n}\right) \sin\left(\frac{\pi}{4n}\right) \right), \\
&= \csc\left(\frac{\pi}{4n}\right) \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{(n+1)\pi}{4n} - \frac{\pi}{4n}\right), \\
&= \csc\left(\frac{\pi}{4n}\right) \sin^2\left(\frac{\pi}{4}\right), \\
&= \csc\left(\frac{\pi}{4n}\right) \left(\frac{1}{2} - \frac{1}{2} \cos(\pi/2) \right), \\
&= \frac{1}{2} \csc\left(\frac{\pi}{4n}\right).
\end{aligned}$$

Appendix B

In this appendix, we will provide a simple algorithm based on Newton's method for finding the fixed point of n in Eq.(2).

Let $g(n) \equiv n - \frac{K}{2} \sin\left(\frac{\pi}{4n}\right)$ and $f(n) = g(n) - n = 0$. The goal of the present algorithm is to find the root of f by performing the following iterative algorithm:

$$k(n_j) \equiv n_{j+1} = n_j - \frac{f(n_j)}{\frac{df}{dn}(n_j)}. \quad (12)$$

Eq.(12) can be simplified further and is expressed as:

$$k(n) = n - \frac{4n^2 \left(2n - K \sin\left(\frac{\pi}{4n}\right) \right)}{\pi K \cos\left(\frac{\pi}{4n}\right) + 8n^2} \quad (13)$$

Finally, the initial solution, n_0 , will assume the value of the asymptotic solution, which is $\sqrt{K\pi/8}$.

Biography

Cheng Guan Koay is currently an assistant scientist in the Department of Medical Physics, University of Wisconsin-Madison. After completing his undergraduate studies in Mathematics at Berea College in 2002 and graduate studies in Physics at the University of Wisconsin-Madison in 2005, he held a research position as an IRTA fellow at the National Institutes of Health until August, 2010. His current research focuses on diffusion Magnetic Resonance Imaging (diffusion MRI), analysis of MR signals and noise and applications of optimization techniques in biomedical problems.

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Figure 1.
Latitude and longitude on the unit sphere.

Figure 2.

The number of latitude lines, n , as a function of the number of points, K . The red line is the asymptotic solution of n and the discontinuous blue line is the iterative solution of n rounded to the nearest integer.

Figure 3. Antipodally symmetric point sets with (A) 100 and (B) 800 points on the upper hemisphere or respectively 200 and 1600 points on the whole sphere.

Figure 4.
Percent of relative error of the proposed scheme compared to that of the gold standard.