Derivative encoding for parallel magnetic resonance imaging^{a)}

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Purpose: To introduce a linear shift-invariant relationship between the partial derivatives of k space signals acquired using multichannel receive coils and to demonstrate that k space derivatives can be used for image unwrapping.

Methods: Fourier transform of k space derivatives contains information on the spatial origins of aliased pixels; therefore, images can be reconstructed by k space derivatives. Fully sampled phantom and brain images acquired at 3 T using a standard eight channel receive coil were used to validate the k space derivatives theorem by unwrapping aliased images.

Results: Derivative encoding leads to new methods for parallel imaging reconstruction in both k space and image domains. Noise amplification in sensitivity encoding image reconstruction, which is considered to produce the optimal SNR, can be further reduced using k space derivative encoding without making any assumptions on the characteristics of the images to be reconstructed.

Conclusions: This work demonstrated that the partial derivative of the k space signal acquired from one coil with respect to one direction can be expressed as a sum of partial derivatives of signals from multiple coils with respect to the perpendicular k space direction(s). This relationship between the partial derivatives of k space signals is linear and shift-invariant in the Cartesian coordinate system. © 2011 American Association of Physicists in Medicine. [DOI: 10.1118/1.3633908]

Key words: parallel imaging, magnetic resonance imaging (MRI), derivative encoding, image reconstruction, noise reduction

I. INTRODUCTION

It is well-known in the MRI literature that a “derivative image” can be generated directly by taking the inverse Fourier transform of the product of $i2\pi k$ and the observed k space signal $s(k)$. In structural imaging, derivative images are useful for extracting important information about the boundaries in an object because derivative images enhance the edges of the object in the direction of the derivative.¹ In susceptibility-weighted imaging, there is much interest in quantifying susceptibility effects by paramagnetic or diamagnetic contrast agents and turning dark contrast due to signal dephasing into positive contrast. One method to achieve these goals is to generate phase gradient maps using derivatives of real and imaginary spin density images.²

Similar to image domain derivatives, the first order k space derivatives can be calculated by taking the forward Fourier transform of the product of $i2\pi x$ and spin density $\rho(x)$. Since k space derivatives contain information on spatial coordinates, they should be useful for spatial encoding. In parallel magnetic resonance imaging spatially separated multiple receive coils replace part of field gradient encoding for increased imaging speed and/or reduced image artifacts.^{3–5} The sensitivity profiles of the multiple receive coils can be characterized using a polynomial function of spatial coordinates. For example, in SENSE reconstruction, spatial polynomial functions are used to fit coil sensitivity profiles for extrapolation and noise removal.⁶ Because the product of a spatial ordinate (or its power functions) and spin density is proportional to the corresponding

Fourier transform of k space partial derivative, there should be a relationship among k space partial derivatives originated from the spatial polynomial representation of coil sensitivity profiles. Recently, we introduced a k space derivatives theorem⁷ that relates partial derivatives with respect to the under-sampled k space direction to partial derivatives with respect to the perpendicular, fully sampled k space direction(s). We showed that autocalibrated k space partial derivatives can be used for parallel imaging reconstruction in both k space and image domains.

Here, we first give a theoretical justification for the k space derivatives theorem by expressing coil sensitivity profiles as spatial polynomial series and applying the Fourier derivative theorem.⁸ Then, we validate the k space differential equations using fully sampled water phantom and brain images acquired using an eight channel receive coil. By calibrating k space partial derivatives using coil sensitivity profiles, it was found that the additional encoding constraints derived from k space partial derivatives can reduce noise amplification in SENSE reconstruction without regularization or making any assumptions on the characteristics of the images to be reconstructed.

II. THEORY

II.A. k Space derivatives theorem

For simplicity, we consider a 2D slice. Its k space signals are sampled in the Cartesian coordinate system with

undersampling in the k_y direction only. The k space signals received by coil j ($s_j(k_x, k_y)$) are the Fourier transform of a weighted spin density $\rho(x, y)$ multiplied by coil sensitivity $c_j(x, y)$

$$s_j(k_x, k_y) = \iint c_j(x, y) \rho(x, y) e^{-i2\pi(k_x x + k_y y)} dx dy. \quad (1)$$

The 1D inverse Fourier transform of Eq. (1) with respect to k_x is given by

$$h_j(x, k_y) = \int c_j(x, y) \rho(x, y) e^{-i2\pi k_y y} dy. \quad (2)$$

The sensitivity profile $c_j(x, y)$ can be expressed by spatial polynomial series

$$c_j(x, y) = \sum_{n=0}^{\infty} \chi_{jn}(x) y^n. \quad (3)$$

Substituting Eq. (3) into Eq. (2), the hybrid signal $h_j(x, k_y)$ becomes

$$h_j(x, k_y) = \sum_{n=0}^{\infty} \chi_{jn}(x) \int y^n \rho(x, y) e^{-i2\pi k_y y} dy. \quad (4)$$

From the (discrete) Fourier derivative theorem and Eq. (4)

$$h_j(x, k_y) = \sum_{n=0}^{\infty} \frac{\chi_{jn}(x)}{(i2\pi)^n} \frac{\partial^n}{\partial k_y^n} h(x, k_y), \quad (5)$$

where $h(x, k_y)$ is the hybrid signal from a hypothetical coil with uniform sensitivity. Equation (5) states that the hybrid signal received by coil j is a linear combination of k space partial derivatives of hybrid signal received with uniform sensitivity and *vice versa*

$$\frac{\partial^n}{\partial k_y^n} h(x, k_y) = \sum_{j=1}^J \kappa_{jn}(x) h_j(x, k_y), \quad (6)$$

where J is the total number of receive coil elements. Combining Eq. (6) with Eq. (5) and its partial derivatives with respect to k_y , we obtain

$$\frac{\partial^n}{\partial k_y^n} h_j(x, k_y) = \sum_{j=1}^J \xi_{j'jn}(x) h_j(x, k_y). \quad (7)$$

κ and ξ in Eqs. (6) and (7) denote the x -dependence of the corresponding linear coefficients. Expressing $\xi_{j'jn}(x)$ as polynomial series and then Fourier transforming Eq. (7) with respect to x ,

$$\frac{\partial^n}{\partial k_y^n} s_j(k_x, k_y) = \sum_{j=1}^J \sum_{m=0}^{M-1} \sigma_{j'jmn} \frac{\partial^m}{\partial k_x^m} s_j(k_x, k_y), \quad (8)$$

where $\sigma_{j'jmn}$ denotes linear shift-invariant coefficients. Equation (8) is the k space derivatives theorem which relates partial derivatives of k space data with respect to the undersampled k_y direction to the fully sampled k_x direction.⁷ It is straightforward to show that a similar partial derivatives relationship also exists for three-dimensional k space data sets in the Cartesian coordinate system.

II.B. Calibration of linear coefficients

The linear coefficients $\xi_{j'jn}(x)$ and $\sigma_{j'jmn}$ are characteristics of coil sensitivity profiles regardless of the image content, and they can be calibrated using either embedded ACS lines or separate coil sensitivity maps.

II.B.1. Calibration of linear coefficients $\xi_{j'jn}(x)$ in Eq. (7)

In Eq. (7), k space partial derivatives with respect to the undersampled k_y direction are related to the fully sampled $h_j(x, k_y)$ lines. As illustrated in Fig. 1(a), one may use a few groups of closely spaced autocalibrating k_y lines to derive partial derivatives with respect to k_y using k space polynomial regression. Then the k space polynomial coefficients can be converted into corresponding partial derivatives $\frac{\partial^n}{\partial k_y^n} h_j(x, k_y)$ after multiplication by $n!$. Alternatively, a k space core fully sampled in both k_x and k_y directions [Fig. 1(b)] as in GRAPPA (Refs. 9–12) can be used to calculate k space partial derivatives with respect to both k_x and k_y using the Fourier derivative theorem.⁸ That is, $\frac{\partial^n}{\partial k_y^n} h_j(x, k_y)$ is calculated using the fully sampled k space core [Fig. 1(b)] by: (i) performing an 1D inverse Fourier transform with respect to k_y to obtain $h_j(x, k_y)$; (ii) multiplying $h_j(x, k_y)$ by y^n ; (iii) performing an 1D forward Fourier transform with respect to y and 1D inverse Fourier transform with respect to k_x to obtain $\frac{\partial^n}{\partial k_y^n} h_j(x, k_y)$. Subsequently, $\xi_{j'jn}(x)$ in Eq. (7) is decomposed into a finite series of model functions with spatially invariant coefficients to remove the x -dependence in $\xi_{j'jn}(x)$. These spatially invariant coefficients of the model function are determined using linear regression^{12,13} and then applied to all undersampled hybrid data to synthesize the k space partial derivatives with respect to k_y at all data points $\frac{\partial^n}{\partial k_y^n} h_j(x, k_y)$.

Fig. 1. (a) Schematic illustration depicting the positions of autocalibrating signals using k space polynomial regression. The polynomial coefficients were converted into corresponding partial derivatives ($\frac{\partial^n}{\partial k_y^n} h_j(x, k_y)$ or $\frac{\partial^n}{\partial k_y^n} s_j(x, k_y)$) after multiplication by the factorial $n!$. (b) Schematic illustration depicting positions of autocalibrating signals using a fully sampled ACS core.

II.B.2. Calibration of linear shift-invariant coefficients $\sigma'_{j'mn}$ in Eq. (8)

In Eq. (8), k space partial derivatives with respect to the undersampled k_y direction are related to the fully sampled $\frac{\partial^n}{\partial k_y^n} s_j(k_x, k_y)$ lines. Either one of the two methods depicted in Fig. 1 can be utilized to obtain k space partial derivatives with respect to the undersampled k_y direction. The GRAPPA k space autocalibration procedure^{9,10} is modified for autocalibration of the linear shift-invariant coefficients $\sigma'_{j'mn}$ based on Eq. (8) as schematically shown in Fig. 2(a). Note that unlike GRAPPA, Eq. (8) requires the use of partial derivatives with respect to k_x at each (k_x, k_y) coordinate to calibrate and synthesize the partial derivatives with respect to k_y at the same (k_x, k_y) coordinate.

II.C. Image reconstruction

II.C.1. Image reconstruction in k space domain

For k space data acquired with an acceleration factor of R , unaliased image for each coil can be reconstructed from undersampled $\frac{\partial^n}{\partial k_y^n} h_{j'}(x, k_y)$ or $\frac{\partial^n}{\partial k_y^n} s_{j'}(k_x, k_y)$ ($n \in \{0, R-1\}$) in a point-by-point manner after inverse Fourier transform with respect to k_y . According to the Fourier derivatives theorem, we have

$$\int \frac{\partial^n}{\partial k_y^n} h_{j'}(x, k_y) e^{i2\pi k_y y} dk_y = (-i2\pi y)^n c_{j'}(x, y) \rho(x, y). \quad (9)$$

Equation (9) indicates that aliased pixels generated from inverse Fourier transform of the undersampled k space partial derivatives contain information on the spatial origins (y) of the aliased pixels, therefore, providing encoding constraints for image unwrapping. For $R=2$ and $n \in \{0, 1\}$, Eq. (9) in explicit matrix form is given in Fig. 2(b). Generally, an aliased pixel generated from Eq. (9) $p_{j'n(m,n)}$ ($m \in \{1, M\}$ and $n \in \{1, N/R\}$) is related to its component pixels $p_{j'(m,\tilde{n})}$ ($\tilde{n} = n + (\phi - 1)N/R$, $\phi \in \{1, R\}$) by

$$\mathbf{p}_{j'm} = \mathbf{J} \mathbf{p}_{j'm}, \quad (10)$$

where $\mathbf{p}_{j'm}$ is a $R \times 1$ column vector consisting of the $0 \sim (R-1)$ th order partial derivative image pixels acquired by coil j' at (m, n) , $\mathbf{p}_{j'm}$ is a $R \times 1$ column vector consisting of the R unwrapped image pixels at $(m, \tilde{n})$ for coil j' , and $\mathbf{J}$ is a $R \times R$ encoding matrix with

$$\mathbf{J}_{n\tilde{n}} = (-i2\pi(n_0/N + \tilde{n}))^n. \quad (11)$$

where n_0 is a constant used to center $y=0$ at $-N/(2R) - 1$. For even-numbered R , $n_0 = -N/2 - 1 \pm N/(2R)$; for odd-numbered R , $n_0 = -N/2 - 1$ or $-(N-1)/2$. From Eq. (10), the unaliased image for coil j' is reconstructed by

$$\mathbf{p}_{j'm} = \mathbf{J}^{-1} \mathbf{p}_{j'm}. \quad (12)$$

Instead of using Eq. (12) for pixelwise image unwrapping, numerical integration in k space (e.g., by the Runge-Kutta method) of partial derivatives $\frac{\partial^n}{\partial k_y^n} h_{j'}(x, k_y)$ or $\frac{\partial^n}{\partial k_y^n} s_{j'}(k_x, k_y)$ can also be used to directly calculate the skipped k space data points followed by coil-by-coil Fourier transform.

II.C.2. Image domain reconstruction

For parallel imaging reconstruction in the image domain when coil sensitivity profiles are available,⁶ the sensitivity encoding matrix is expanded by adding constraints derived from partial derivatives encoding (i.e., $y_1^n c_{j'}(x, y_1) \rho(x, y_1) + y_2^n c_{j'}(x, y_2) \rho(x, y_2) + \dots$ ($n \neq 0$)). Generally, for SENSE reconstruction with 1st- n th order partial derivative constraints, the original SENSE encoding matrix $\mathbf{C}_{mn}$ is expanded into a $((n+1)J \times R)$ matrix $\tilde{\mathbf{E}}_{mn}$ with its elements given by

$$\tilde{\mathbf{E}}_{m\tilde{n}(n+1)j'} = \mathbf{J}_{n\tilde{n}} c_{j'}(m, \tilde{n}). \quad (13)$$

The weighted least square solution of the unwrapped image is

$$\mathbf{p}_{m\tilde{n}} = (\tilde{\mathbf{E}}_{mn}^H \Gamma_m^{-1} \tilde{\mathbf{E}}_{mn})^{-1} \tilde{\mathbf{E}}_{mn}^H \Gamma_m^{-1} \mathbf{p}_{mn}, \quad (14)$$

where $\mathbf{p}_{mn}$ is the $(n+1)J \times 1$ column vector of aliased 0- n th order partial derivative image pixels, $\mathbf{p}_{m\tilde{n}}$ is the $R \times 1$

FIG. 2. (a) A flow chart depicting the calibrating procedures based on Eq. (8). (b) Eqs. (10)–(12) in explicit matrix form for $R=2$ and $n \in \{0, 1\}$.

column vector of the unwrapped image pixels, and Γ_m is the weighted noise covariance matrix.

For $R = 2$ and $n = 1$, the explicit encoding matrix is schematically illustrated in Fig. 3. In Fig. 3 Γ_m was omitted for clarity. A detailed analysis of noise propagation in image domain reconstruction using k space partial derivatives is given in Appendix.

III. METHODS

Image reconstruction and analysis were performed using MATLAB (version 7.9.0 R2009b, TheMathWorks, Inc., Natick, MA, USA). Brain imaging experiments on healthy volunteers and water phantoms were performed to test the k space derivatives theorem and the proposed derivative encoding methods using a standard GE's 3.0 Tesla Excite scanner (General Electric, Waukesha, WI) equipped with eight receive channels. Standard eight-channel receive-only head coils and a transmit body coil were used for image data acquisition. Because using fully sampled images is the gold standard for assessing the quality of image reconstruction, all acquired images were fully sampled along both the readout (k_x) and phase-encoding (k_y) directions. Gradient echo and MPAGE (3D Flash with IR preparation) images were acquired. Gradient echo imaging parameters are: TR = 0.3 s, TE = 6.5–15 ms, Flip angle = 40°, Image matrix size = 256 × 256, Slice thickness = 5 mm, FOV = 220–240 mm × 220–240 mm, and Number of averages = 1. For gradient echo imaging of the water phantom, additional k space lines were acquired adjacent to five central k_y lines for auto-calibration [see Fig. 1(a)]. The distance between the ACS lines is $(60/256)\Delta k_y$. Total number of ACS lines = 20. The standard RF chopping method was applied to shift the zero frequency stripe to $-\text{FOV}_y/2$ which was digitally removed. MPAGE imaging parameters are: TR = 2.9 s, TI = 725 ms, TE = 3.1 ms, Flip angle = 6°, Number of slices = 128, Matrix size = 256 × 256, Slice thickness = 1.3 mm, FOV = 256 mm × 256 mm, and Number of averages = 1. To quantify k space domain reconstruction performance, the relative mean square reconstruction error¹³ between the magnitude of the fully sampled reference image and that of the reconstructed accelerated image, normalized by the magnitude of the fully sampled reference image, was calculated.

For SENSE reconstruction, coil sensitivity profiles were estimated by dividing the sum-of-square image into each individual coil image to remove image contrast after polynomial fitting and Hanning filtering.⁶ The extra spatial

variation in $\sqrt{\sum_{j=1}^J |c_j|^2}$ introduced into individual coil sensitivity maps were ignored for the purpose of calibrating $\xi_{j'n}(x)$ by Eq. (7) since phantom $\sqrt{\sum_{j=1}^J |c_j|^2}$ images acquired at 3 T by the eight channel receive coils used in this study were found to be fairly homogeneous. The SENSE g factor maps were calculated for integer Rs according to Eq. (23) in Ref. 6. The g factor maps for accelerated image reconstructed using SENSE + derivative encoding constraints were calculated using Eq. (A3) ($n = 1$, see Appendix).

IV. RESULTS

IV.A. Proof of concept

Gradient echo imaging studies were performed to validate the basic principles described in theory. The fully sampled image (256 × 256) of a water phantom was shown in Fig. 4(a), left. A typical 3rd order k space polynomial fit is shown in Fig. 4(b). The overall relative mean square polynomial regression error (256 × 5 × 8 fits) is 3.8×10^{-6} . The zero and first order partial derivatives $\frac{\partial}{\partial k_y} h_j(x, k_y)$ were calculated for all odd-numbered k_y lines which were used for image reconstruction. The quality of the least square regression in autocalibration was assessed in Table I.

Then, the undersampled $\frac{\partial}{\partial k_y} h_j(x, k_y)$ maps were inversely Fourier transformed with respect to k_y . Two sets of aliased images were obtained: a conventional one described by $[c_j(x, y_1)\rho(x, y_1) + c_j(x, y_2)\rho(x, y_2)]$ and a new one described by $-i2\pi[y_1c_j(x, y_1)\rho(x, y_1) + y_2c_j(x, y_2)\rho(x, y_2)]$ [see Fig. 2(b)]. The unaliased individual coil images were reconstructed in k space using Eq. (12). The unwrapped sum-of-square image was shown in Fig. 4(a), middle. The difference image (8×) was shown in Fig. 4(a), right. The overall relative mean square difference between Figs. 4(a), left and 4(a), middle is 2.0×10^{-4} .

IV.B. Image domain reconstruction with k space partial derivative constraints

A spatially independent weighting factor ($w_{mn1} = \text{const.}$, see Appendix) was found empirically which gives the maximum reduction of the largest SENSE g factor. The results were listed in Table II. For $R = 2$ –8, adding the partial derivative constraints ($n = 1$) leads to smaller SENSE g factors. Similar results were obtained when a small number of embedded ACS lines were used to calibrate $\xi_{j'n}(x)$. For easy

$$\begin{array}{l}
 \text{Pixel from a} \\
 \text{conventional} \\
 \text{aliased image} \\
 \\
 \text{Pixel from} \\
 \text{Fourier} \\
 \text{transform} \\
 \text{of 1st order } k_y \\
 \text{derivatives}
 \end{array}
 \left\{ \begin{array}{l} \text{coil\#1} \\ \text{coil\#2} \\ \text{coil\#1} \\ \text{coil\#2} \end{array} \right\} = \begin{pmatrix} c_1(x, y_1) & c_1(x, y_2) \\ c_2(x, y_1) & c_2(x, y_2) \\ -i2\pi y_1 c_1(x, y_1) & -i2\pi y_2 c_1(x, y_2) \\ -i2\pi y_1 c_2(x, y_1) & -i2\pi y_2 c_2(x, y_2) \end{pmatrix} \begin{pmatrix} \rho(x, y_1) \\ \rho(x, y_2) \end{pmatrix}$$

$|y_1 - y_2| = \text{FOV}/2$

FIG. 3. The expanded SENSE encoding matrix equation for $R = 2$ and $n \in \{0, 1\}$. The Γ_m matrix in Eq. (14) was omitted for clarity.

FIG. 4. (a) Left: magnitude gradient echo image of a water phantom reconstructed from a fully sampled k space dataset using the conventional sum-of-squares method. Middle: accelerated image ($R=2$) reconstructed using Eq. (12) with 20 ACS lines. Right: difference image: (Left-Middle) $\times 8$. (b) Third order polynomial regression analysis of the five k_y points centered at $h_1(64.64)$. *: acquired real k space data; o: acquired imaginary k space data; solid lines: third order k space polynomial fits.

visualization of reduction in noise amplification by k space partial derivative constraints, Fig. 5 gives a comparison for $R=6$. Figure 5(a) shows a fully sampled sum-of-square MPRAGE image. Accelerated image reconstructed using SENSE ($R=6$, mask threshold = 0.1 [Fig. 5(b)] shows large noise amplification as expected. Its g factor map is shown in Fig. 5(d). For comparison, the accelerated image reconstructed using SENSE + partial derivative constraints $[y_1 c_j(x, y_1) \rho(x, y_1) + \dots + y_6 c_j(x, y_6) \rho(x, y_6)]$ ($R=6$, $w_{m1}=0.25$) was shown in Fig. 5(c). Its g factor map was shown in Fig. 5(e).

V. DISCUSSION

To the best of our knowledge there are no previous reports in the MR literature that used k space derivatives. The usefulness of k space partial derivatives in parallel imaging reconstruction, as demonstrated in this work, is

TABLE I. Relative mean square error ($\times 10^{-3}$) in autocalibration of $\frac{\partial}{\partial k_y} h_j(x, k_y)$.^a

Coil No.	1	2	3	4	5	6	7	8
$n=0$	0.0026	0.0019	0.0021	0.0020	0.0019	0.0018	0.0020	0.0027
$n=1$	0.2022	0.0568	0.0452	0.0948	0.0797	0.0516	0.1410	0.4246

^aFor image reconstruction shown in Fig. 4. Gradient echo image parameters: matrix size = 256×256 , Number of ACS lines = 20, $R=2$.

TABLE II. Calculated maximum SENSE g factors^a.

R	2	3	4	5	6	7	8
SENSE	1.10	1.59	2.53	6.40	23.1	38.7	60.6
SENSE + derivative constraints ($n=1$)	1.08	1.56	2.41	4.72	8.92	14.3	16.9

^aMPRAGE image (fully sampled matrix size = 256×256); mask threshold = 0.1; Number of k_y lines from estimated coil sensitivity profiles used for calibrating $\xi_{j1n}(x) = 56 \sim 64$, $w_{m1}(R) = 0.12(2)$, $0.12(3)$, $0.062(4)$, $0.36(5)$, $0.25(6)$, $0.33(7)$, $0.17(8)$.

resulted from combining polynomial structure of coil sensitivity maps with the Fourier derivative theorem.⁸ Since k space partial derivatives contain information on the spatial origins of aliased pixels they can provide additional encoding for image unwrapping. We showed that derivative encoding can be applied to parallel imaging reconstruction in both k space and image domains. It should be noted that the k space derivatives used here are discrete samples of continuous derivatives, not derivatives approximated by numerical differentiation.

Equation (8) states that $\frac{\partial}{\partial k_y} s_j(k_x, k_y)$ can be determined by J fully sampled 1D k_y lines. It is therefore possible to use Eq. (8) for reconstructing accelerated radially sampled images. Previously, extrapolation in k space was proposed for image artifact reduction.^{15–17} For example, k -space extrapolation¹⁷ using the GRAPPA-operator method has been shown to significantly reduce artifact in undersampled PROPELLER images.^{18,19} Equation (8) may also be used to extrapolate k space data points from a fully sampled core in a fashion similar to linear prediction in NMR spectroscopy.²⁰ To avoid excessive noise amplification only limited extrapolation steps could be used; therefore, the extrapolation procedure is expected to have little value to conventional high resolution imaging. For spectroscopic imaging of weak metabolite signals where low spatial resolution (such as 8×8) is often used, however, acceleration by k space extrapolation can be significant. For example, from a 4×4 2D k space core, extrapolation by $\Delta k_{x,y}$ and $2\Delta k_{x,y}$ would triple the number of k space data while acceleration by conventional undersampling is not feasible with very small data sizes. A recent study showed that a similar goal can be accomplished in the image domain when high resolution coil sensitivity profiles are available.²¹

Combining Taylor expansion, $s_j(k_x, k_y + n\Delta k_y) = \sum_{n'=0}^{\infty} \frac{\partial^{n'}}{\partial k_y^{n'}} s_j(k_x, k_y) (n\Delta k_y)^{n'} / n'!$ and Eq. (8) we can also represent $s_j(k_x, k_y + n\Delta k_y)$ by a linear combination of $\partial^m / \partial k_x^m s_j(k_x, k_y)$. When Δk_y is chosen to be the Nyquist interval in the k_y direction and $m=0$, this line of reasoning leads to the familiar 1D GRAPPA equation (Eq. (1) in Ref. 22). The GRAPPA method derives its synthesis kernel mainly from k space data along the undersampled k_y direction.^{9,10} In contrast, imaging reconstruction by Eqs. (7) and (8) mainly uses information along the fully sampled k_x direction. After calculating the skipped data points, the GRAPPA method generates individual coil images by the unitary Fourier transform. The Fourier transform per se does not

FIG. 5. (a) Magnitude MPRAGE brain image reconstructed from a fully sampled k space data set using the conventional sum-of-squares method. (b) Accelerated image ($R = 6$) reconstructed using SENSE. (c) Accelerated image ($R = 6$) reconstructed using SENSE and first order partial derivative constraints. Equation (7) ($n = 1$) and coil sensitivity maps were used for calibrating $\xi_j^n(x)$ and calculation of partial derivatives for the under-sampled k_y lines. $w_{mn1} = 0.25$. (d) SENSE g factor map for (b) calculated using the standard g factor equation [Ref. (6)], the estimated coil sensitivity maps and $\Psi = \text{diag}\{1\}$. (e) SENSE + partial derivative constraints g factor map for (c) calculated using Eq. (A3).

introduce noise amplification. In contrast, k space domain reconstruction based on Eq. (12) requires a second nonunitary step. Numerous improvements on the original GRAPPA methods have been reported in the literature (e.g., Refs. 23–27). Many of these improvements can be readily applied to the derivative encoding methods described here as well. A thorough optimization and comparison of different k space methods is beyond the scope of this report, however.

When *a priori* assumptions of image content are not made, SENSE has been thought to produce the optimal SNR. Of theoretical interest, Eq. (8) shows that there exist additional constraints rooted in the spatial polynomial structure of coil sensitivity maps that can further reduce noise amplification in SENSE without any knowledge of image content. Because of the polynomial structure of sensitivity profiles, Eq. (8) allows one to recombine the acquired data in k space to generate the additional image domain constraints ($y_1^n c_j(x, y_1) \rho(x, y_1) + y_2^n c_j(x, y_2) \rho(x, y_2) + \dots$ ($n \neq 0$)). In essence, the known spatial polynomial relationship among $c_j(x, y_1)$, $c_j(x, y_2)$, etc. adds additional information to SENSE reconstruction, leading to overall noise reduction with proper weighting (w_{mn1}). Previously, noise amplification in SENSE reconstruction has been addressed using regularization methods,^{28,29} which leads to intrinsic loss in spatial resolution. The derivative constraints are different from regularization since calculation of k space partial derivatives makes no assumptions on the image to be reconstructed. As shown by Table II, the additional constraint derived from the first order partial derivatives can significantly reduce noise amplification for high R/J ratios. Only the first order partial derivatives ($\frac{\partial}{\partial k_y} h_j(x, k_y)$) were investigated here. Constraints derived from higher order partial derivatives ($n > 1$) may further reduce noise amplification. It should be pointed out

that these additional constraints are not independent measurements because they are the result of linearly combining the undersampled data. As a result, the conditioning of the enlarged SENSE encoding matrix is improved. For a demonstration of principle, we only imposed a diagonal matrix W with a constant w_{mn1} on the first order partial derivatives constraints. A more sophisticated W may further and significantly improve the results. For example, w_{mn1} may be optimized using $\frac{\partial \langle g_{mn} \rangle}{\partial w_{mn1}} = 0$ for each (m, n) , where $\langle g_{mn} \rangle$ is g_{mn} averaged over $\phi \in \{1, R\}$.

The singular noise covariance matrix given by Eq. (A1) could be diagonalized using eigen value decomposition. The linear dependence in p_{mn} can be therefore removed using the eigen vectors of the singular noise covariance matrix Ω_m . g_{mn} can then be calculated using the whitened encoding matrix according to Ref. 28. This procedure, however, does not necessarily minimize g_{mn} . For the examples given in Table II, the eigen vectors of Ω_m were found to increase the maximum g factors.

Without utilizing common characteristics of anatomical images (for example, their sparse representation in a known and fixed mathematical transform domain³⁰) k space partial derivatives constraints ($n = 1$) alone can significantly reduce SENSE noise amplification for high R/J ratios. It should be possible to combine derivative encoding with existing methods that take advantages of *a priori* knowledge of images to be reconstructed.^{28–31} For example, it should be possible to combine k space partial derivatives constraints with Tikhonov regularization to reduce the magnitude of regularization parameters and hence the loss of spatial resolution. Combining derivative encoding with the virtual coil concept³² to further expand the SENSE encoding matrix is also possible.

VI. CONCLUSIONS

The partial derivatives of the k space signals from one coil with respect to one k space direction can be expressed as a sum of partial derivatives of signals from multiple coils with respect to the perpendicular k space direction(s). This k space derivatives theorem led to new k space domain reconstruction methods. For image domain reconstruction, it is possible to leverage the spatial polynomial structure of coil sensitivity profiles to reduce noise amplification in SENSE reconstruction without imposing any assumptions on the characteristics of the image to be reconstructed.

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APPENDIX: NOISE PROPAGATION IN IMAGE DOMAIN RECONSTRUCTION

For an arbitrary linear transformation $\mathbf{y} = \mathbf{A}\mathbf{x}$, the covariance matrix of $\mathbf{y}$ is given by $\text{Cov}(\mathbf{y}) = \mathbf{A}\text{Cov}(\mathbf{x})\mathbf{A}^H$. The noise covariance of $\mathbf{p}_{mn}$ therefore can be calculated using the $J \times J$ receiver noise covariance matrix Ψ and a $J \times J$ matrix ξ_{mn} which transforms a $J \times 1$ column vector of measured hybrid signals from the J coils with x -ordinate at m into a $J \times 1$ column vector of n th partial derivative of the corresponding hybrid signals. The $(n+1)J \times (n+1)J$ noise covariance matrix of the 0- n th order aliased partial derivative images ($\mathbf{p}_{mn}$) is given by

$$\Omega_m = \begin{pmatrix} \xi_{m0}\Psi\xi_{m0}^H & \xi_{m0}\Psi\xi_{m1}^H & \xi_{m0}\Psi\xi_{m2}^H & \bullet \\ \xi_{m1}\Psi\xi_{m0}^H & \xi_{m1}\Psi\xi_{m1}^H & \xi_{m1}\Psi\xi_{m2}^H & \bullet \\ \xi_{m2}\Psi\xi_{m0}^H & \xi_{m2}\Psi\xi_{m1}^H & \xi_{m2}\Psi\xi_{m2}^H & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} \quad (\text{A1})$$

Note that Ω_m is a rank J singular positive semidefinite matrix and shift-invariant with respect to k_y or y (i.e., $\tilde{n}$).

Using Eq. (14) to reconstruct the SENSE image with partial derivative constraints, the noise covariance matrix of the reconstructed $\mathbf{p}_{m\tilde{n}}$ is obtained using $\text{Cov}(\mathbf{y}) = \mathbf{A}\text{Cov}(\mathbf{x})\mathbf{A}^H$

$$\tilde{\mathbf{G}}_{m\tilde{n}} = (\tilde{\mathbf{E}}_{mn}^H \Gamma_m^{-1} \tilde{\mathbf{E}}_{mn})^{-1} \tilde{\mathbf{E}}_{mn}^H \Gamma_m^{-1} \Omega_m \Gamma_m^{-1} \tilde{\mathbf{E}}_{mn} (\tilde{\mathbf{E}}_{mn}^H \Gamma_m^{-1} \tilde{\mathbf{E}}_{mn})^{-1} \quad (\text{A2})$$

The geometric factor matrix $\mathbf{g}_{m\tilde{n}}$ for the reconstructed SENSE image with partial derivative constraints is therefore

$$\mathbf{g}_{m\tilde{n}} = \{\text{diag}(\tilde{\mathbf{G}}_{m\tilde{n}}) \text{diag}(\mathbf{C}_{m\tilde{n}}^H \Psi^{-1} \mathbf{C}_{m\tilde{n}})\}^{1/2} \quad (\text{A3})$$

For simplicity, $\Gamma_m^{-1} \equiv [\Psi^{-1}0; 0\mathbf{W}]$, $\mathbf{W} = \text{diag}\{\mathbf{w}_{mn}\}$. $\mathbf{w}_{mn}$ denotes the weight imposed on the n th order partial derivative constraints. Note that Eq. (A3) is reduced to Eq. (23) in Ref. 6 when $\mathbf{w}_{mn} = 0$. Equations (A1)–(A3) were verified by Monte Carlo simulation. The g factor maps by Eq. (A3)

Fig. 6. Verification of Eq. (A3) by numerical Monte Carlo simulation. A fully sampled 2D gradient echo image of a sphere was reconstructed using Eq. (14) with undersampling along k_y . Fully sampled matrix size = 256×256 . $J = 8$. $R = 6$. $n = 1$. $\mathbf{w}_{mn} = 0.1$. The 1D g factor maps for a single x line were shown. x ordinate = 128. Red line: g factors calculated by Eq. (A3). Blue line: g factors were calculated according to $\text{SNR}^{\text{full}} / (R^{0.5} \text{SNR}^{\text{red}})$ as given in Ref. 6. For the purpose of Monte Carlo simulation, artificial Gaussian noise was added to the k space data acquired by each individual channel. Number of noise realizations = 200.

were found to be the same as $\text{SNR ratios} / R^{1/2}$ calculated empirically when the number of noise realizations in Monte Carlo simulation was sufficiently large (see Fig. 6).

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