Finite-time synchronization of coupled neural networks via discontinuous controllers

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Received: 13 December 2010 / Revised: 10 April 2011 / Accepted: 20 June 2011 / Published online: 8 July 2011
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Abstract This paper investigates finite-time synchronization of an array of coupled neural networks via discontinuous controllers. Based on Lyapunov function method and the discontinuous version of finite-time stability theory, some sufficient criteria for finite-time synchronization are obtained. Furthermore, we propose switched control and adaptive tuning parameter strategies in order to reduce the settling time. In addition, pinning control scheme via a single controller is also studied in this paper. With the hypothesis that the coupling network topology contains a directed spanning tree and each of the strongly connected components is detail-balanced, we prove that finite-time synchronization can be achieved via pinning control. Finally, some illustrative examples are given to show the validity of the theoretical results.

Keywords Neural networks · Finite-time synchronization · Discontinuous control · Pinning control

Introduction

In the last decades, the study of artificial neural networks has received considerable attention due to its broad applications in signal processing, pattern recognition, fixed-point computation, associative memories and optimization problems (Gupta et al. 2003). There are many papers concerning stability analysis and periodic oscillation of recurrent neural networks (Cao and Song 2006; Lu and Chen 2003). On the other hand, since neural network systems may also exhibit oscillation or chaotic behaviors, synchronization of coupled neural networks (Chen et al. 2004; Cao and Lu 2006; Cao et al. 2008) has attracted tremendous attention by many researchers from a wide range of disciplines. The study on synchronization of coupled neural networks is also an important step for both understanding brain science and designing coupled neural networks for real world applications.

Since the pioneering works of Wu and Chua (1995) where chaos synchronization in an array of cellular neural network systems was investigated, many results on synchronization in various coupled systems were reported, (see, e.g., Lu and Chen 2004; Yu et al. 2010; Li et al. 2007; Wu and Chen 2008; Lu and Ho 2010) and references therein. Most of the previous works on synchronization of coupled neural networks adopted linear feedback controller and can only realize exponential or asymptotical convergence rate. However, finite-time convergence is more important in practical application. The advantage of finite-time synchronization include faster convergence rate and robustness against uncertainties. As is well known, some discontinuous dynamical systems are more likely to achieve finite-time convergence. Discontinuous controllers are sometimes intentionally designed to realize finite-time regulation and stabilization. For instance, in Cortés (2006),

This work was jointly supported by the National Natural Science Foundation of China under Grant No. 60874088 and No. 11072059, the Specialized Research Fund for the Doctoral Program of Higher Education under Grant 20070286003, and the Natural Science Foundation of Jiangsu Province of China under Grant No. BK2009271.

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two discontinuous control algorithms that achieve consensus of multi-agent systems in finite time were designed. In Hui et al. (2009, 2010), the authors addressed finite-time semistability for discontinuous dynamical systems and applied this theory to finite-time consensus of nonlinear multi-agent systems with switching topologies. However, there are few papers using discontinuous controllers to synchronize coupled neural networks.

Motivated by the above discussions, in this paper, we consider finite-time synchronization of an array of coupled neural networks via discontinuous controllers. The main contribution of this paper is of five fold. Firstly, the previous works concerning the synchronization problems of coupled neural networks mainly focused on exponential or asymptotical synchronization. Finite-time synchronization of coupled neural networks were rarely reported in the existed literature. Under the proposed control algorithms in this paper, finite-time synchronization can be achieved. Secondly, finite-time synchronization issue via continuous controllers was discussed in some papers (Chen and Lü 2009; Yang and Cao 2010). However, there are relatively few works about how to realize finite-time synchronization via discontinuous controllers. In this paper, discontinuous control algorithms based on sign function are analyzed by employing the notion of Filippov solutions and Lie derivative. Thirdly, switched control and adaptive tuning parameter strategies are proposed to reduce the settling time. Intuitively, one can imagine that the convergence rate might be very slow when the synchronization error is small if continuous controller is adopted. However, it is not the case for the discontinuous controller and it will have its advantage when the synchronization error is very small. Therefore, when the synchronization error is large, we might use the continuous controller and then switch to the discontinuous controller when the synchronization error becomes small. Fourthly, compared with the results in (Chen and Lü 2009) where finite-time synchronization of complex dynamical networks via nonlinear coupling and pinning control is investigated, the network topology is not required to be undirected and the synchronization can be globally reached in this paper. Fifthly, in this paper, it is proved that Filippov solutions of coupled neural networks globally exist while some previous results only proved the solution existence of a single neural network.

The remaining part of this paper is organized as follows. In “Preliminaries” some necessary notations and lemmas about graph theory and discontinuous dynamical systems are firstly introduced. In “Finite-time synchronization via discontinuous controller” discontinuous

controllers for finite-time synchronization of coupled neural networks are proposed and analyzed. In “Some control strategies to reduce the settling time” we propose some novel control strategies to reduce the settling time. “Nonlinear coupling and pinning control scheme” focuses on pinning control scheme. Illustrative examples are presented to show the effectiveness and validity of the theoretical results in “Illustrative examples”. In the last section we present “Conclusions” for this paper.

Preliminaries

Graph theory

The coupled neural network systems can be modeled by a graph. In what follows, we introduce some simple notions of graph theory. Let $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order N with the set of nodes $\mathcal{N} = \{1, 2, \dots, N\}$, set of edges $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$, and an adjacency matrix $\mathcal{A} = [a_{ij}]$. If $(j, i) \in \mathcal{E}$, then $a_{ij} > 0$, else $a_{ij} = 0$. In addition, we assume $a_{ii} = 0$ for $i = 1, 2, \dots, N$. The set of neighbors of node i is denoted by $N_i = \{j \in \mathcal{N} : (j, i) \in \mathcal{E}\}$.

A directed graph has a directed spanning tree if there exists at least one node that has a directed path to all the other nodes. The Laplacian matrix L with respect to graph $\mathcal{G}$ is $L = [l_{ij}]$ with $l_{ii} = \sum_{j \in N_i} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. It is easy to verify that L has at least one zero eigenvalue with a corresponding eigenvector $\mathbf{1}_N$, where $\mathbf{1}_N = (1, 1, \dots, 1)^T$. A digraph $\mathcal{G}$ is called balanced provided the in-degree and out-degree of each node are the same, i.e., $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$ for all $i \in \mathcal{N}$.

Suppose that graph $\mathcal{G}$ is strongly connected, then its corresponding Laplacian matrix L is irreducible. Let $\xi = (\xi_1, \xi_2, \dots, \xi_N)^T$ be a left eigenvector of L corresponding to the zero eigenvalue, then one can obtain that $\xi_i > 0$ for $i \in \mathcal{N}$ by the Perron-Frobenius theorem (Horn and Johnson 1990). Since ξ is a left eigenvector of L , $\xi^T L = 0$, i.e., $\sum_{j=1}^N \xi_i a_{ij} = \sum_{j=1}^N \xi_j a_{ji}$. Denote $\Xi = \text{diag}(\xi)$ and let $B = [\xi_i a_{ij}] = \Xi \mathcal{A}$, one can readily obtain that $\mathcal{G}(B)$ is balanced. Furthermore, A weighted directed graph $\mathcal{G}$ is said to be detail-balanced in weights (or detail-balanced) (Wang and Xiao 2010) if there exist some scalars $\omega_i > 0$, $i \in \mathcal{N}$, such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{N}$. It is easy to see that $\omega = (\omega_1, \omega_2, \dots, \omega_N)^T$ is exactly the left eigenvector of L corresponding to the zero eigenvalue.

Lemma 1 (Olfati-Saber and Murray 2004) *Given an undirected graph $\mathcal{G}$ with adjacency matrix $\mathcal{A} = [a_{ij}]$ and Laplacian matrix L , equality*

$$x^T L x = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} (x_i - x_j)^2$$

holds for arbitrary $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Lemma 2 (Proposition 1 (Chen et al. 2007)) *If $A = [a_{ij}] \in \mathbb{R}^{m \times m}$ is an irreducible matrix with $\text{Rank}(A) = m - 1$ and satisfying $a_{ij} = a_{ji} \geq 0$, if $i \neq j$, and $\sum_{j=1}^m a_{ij} = 0$, for $i = 1, 2, \dots, m$. Let $\tilde{A} = [\tilde{a}_{ij}]$ with $\tilde{a}_{11} = a_{11} - \varepsilon$ ($\varepsilon > 0$) and $\tilde{a}_{ij} = a_{ij}$ otherwise. Then all eigenvalues of the matrix $\tilde{A}$ are negative, i.e., $\tilde{A}$ is negative definite. Furthermore, for any nonnegative column vector b with compatible dimensions, if $b \neq 0$, then $A - \text{diag}(b)$ is negative definite.*

Lemma 3 (Wu 2005) *If graph $\mathcal{G}$ contains a directed spanning tree, then with some proper permutation T , the Laplacian matrix L can be reduced to the Frobenius normal form*

$$T^T L T = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{k1} & L_{k2} & \cdots & L_{kk} \end{bmatrix} \quad (1)$$

where L_{ii} , $i = 2, \dots, k$, are irreducible, each $L_{ii} \in \mathbb{R}^{r_i \times r_i}$ has at least one row with positive row sum, L_{11} is irreducible or is a zero matrix of dimension one and $r_1 + \dots + r_k = N$.

Lemma 4 (Liu et al. 2009) *Suppose that digraph $\mathcal{G}$ contains a directed spanning tree and the Laplacian matrix $L \in \mathbb{R}^{N \times N}$ has the form (1), then the left eigenvector ξ of L corresponding to eigenvalue 0 has the following property: $\xi = (\hat{\xi}^T, 0, \dots, 0)^T$, where $\hat{\xi} = (\hat{\xi}_1, \dots, \hat{\xi}_{r_1})^T$ and $\hat{\xi}_i > 0$, $i = 1, 2, \dots, r_1$.*

Some notions and tools for discontinuous dynamical systems

Set valued maps and Filippov solutions

Definition 1 (Cortés 2008) *Suppose $E \subset \mathbb{R}^n$. Map $x \mapsto \mathcal{F}(x)$ is called a set-valued map from $E \rightarrow \mathfrak{B}(\mathbb{R}^n)$ where $\mathfrak{B}(\mathbb{R}^n)$ is the set consisting of all the subsets of $\mathbb{R}^n$, if to each point of the set E , there corresponds an nonempty set $\mathcal{F}(x) \subset \mathbb{R}^n$.*

In what follows, we introduce the concept of Filippov set-valued maps and Filippov solutions. Consider the following system

$$\dot{x}(t) = f(x(t)) \quad (2)$$

where $f(\cdot)$ is not necessarily continuous.

Definition 2 (Cortés 2008) *Filippov set-valued map is defined as*

$$\mathcal{F}(x) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} K[f(B(x, \delta) \setminus N)],$$

where $K(E)$ is the closure of the convex hull of set E , $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$, and $\mu(N)$ is the Lebesgue measure of set N . A solution in the sense of Filippov of the Cauchy problem of (2) with initial conditions $x(0) = x_0$ is an absolutely continuous function $x(t)$, $t \in [0, T]$, which satisfies $x(0) = x_0$ and differential inclusion:

$$\dot{x}(t) \in \mathcal{F}(x(t)), \quad \text{a.e. } t \in [0, T].$$

Some useful properties of Filippov set-valued maps are presented as follows.

Lemma 5 (Cortés 2008)

- (1) *If $X : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is continuous at $x \in \mathbb{R}^d$, then $\mathcal{F}[X](x) = \{X(x)\}$.*
- (2) *If function $X_1, X_2 : \mathbb{R}^d \rightarrow \mathbb{R}^m$ are locally bounded at $x \in \mathbb{R}^d$, then*

$$\mathcal{F}[X_1 + X_2](x) \subseteq \mathcal{F}[X_1](x) + \mathcal{F}[X_2](x),$$

Moreover, if either X_1 or X_2 is continuous at x , the equality holds.

- (3) *If $X_1 : \mathbb{R}^d \rightarrow \mathbb{R}^m$ and $X_2 : \mathbb{R}^d \rightarrow \mathbb{R}^n$ are locally bounded at $x \in \mathbb{R}^d$, then*

$$\mathcal{F}[(X_1, X_2)^T](x) \subseteq \mathcal{F}[X_1](x) \times \mathcal{F}[X_2](x).$$

Moreover, if either X_1 or X_2 is continuous at x , then equality holds.

- (4) *If $X : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is locally bounded at $x \in \mathbb{R}^d$ and $Z : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ is continuous at $x \in \mathbb{R}^d$, then*

$$\mathcal{F}[ZX](x) = Z(x)\mathcal{F}[X](x).$$

Lie derivative

Definition 3 (Cortés 2008) *Given a locally Lipschitz function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and a set-valued map $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$, the set-valued Lie derivative $\mathcal{L}_{\mathcal{F}}f : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R})$ of f with respect to $\mathcal{F}$ at x is defined as*

$$\mathcal{L}_{\mathcal{F}}f(x) = \{a \in \mathbb{R} : \text{there exists } v \in \mathcal{F}(x) \text{ such that } \zeta^T v = a \text{ for all } \zeta \in \partial f(x)\}.$$

If f is continuously differentiable at x , then

$$\mathcal{L}f(x) = \{(\nabla f(x))^T v : v \in \mathcal{F}(x)\}.$$

The following lemma allows us to study how the function f evolves along the solutions of a differential inclusion without having to obtain the solutions explicitly.

Lemma 6 (Proposition 10 (Cortés 2008)) *Let $x : [0, t_1] \rightarrow \mathbb{R}^d$ be a solution of the differential inclusion $\dot{x}(t) \in \mathcal{F}(x(t))$, and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be locally Lipschitz and regular. Then, the following statements hold:*

- (i) *The composition $t \mapsto f(x(t))$ is differentiable at almost all $t \in [0, t_1]$.*
- (ii) *The derivative of $t \mapsto f(x(t))$ satisfies*

$$\frac{d}{dt}(f(x(t))) \in \mathcal{L}f(x(t)) \quad \text{for a.e. } t \in [0, t_1].$$

Finite-time synchronization via discontinuous controller

In this paper, we consider a neural network model described as follows:

$$\dot{x}(t) = -Bx(t) + Tg(x(t)) + J(t) \quad (3)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons, $B = \text{diag}\{b_1, b_2, \dots, b_n\}$ is a positive diagonal matrix with b_i modeling self-inhibition of i th neuron, $T = [t_{ij}] \in \mathbb{R}^{n \times n}$ is the interconnection matrix, $J(t) = (J_1(t), J_2(t), \dots, J_n(t))^T \in \mathbb{R}^n$ with $\|J(t)\| \leq \bar{J}$ is a bounded external input, and $g(x) = (g_1(x_1), g_2(x_2), \dots, g_n(x_n))^T$ with $g_i(\cdot)$ modeling the input-output activation of the i th neuron.

To study synchronization, one may consider the following model of N linearly coupled neural networks:

$$\begin{aligned} \dot{y}_i(t) = & -By_i(t) + Tg(y_i(t)) + J(t) + c \sum_{j=1}^N a_{ij}(y_j - y_i) \\ & + u_i(t), \\ i = & 1, 2, \dots, N, \end{aligned} \quad (4)$$

where $u_i(t) \in \mathbb{R}^n$ is the control input, $\mathcal{A} = [a_{ij}]$ represents the adjacency matrix of the coupled neural networks and $c > 0$ denotes the coupling strength. For a vector $x = (x_1, x_2, \dots, x_m)^T \in \mathbb{R}^m$, denote $\text{sign}(x) = (\text{sign}(x_1), \text{sign}(x_2), \dots, \text{sign}(x_m))^T$. In order to make the coupled

neural networks synchronize the isolated neural network (3), we propose a control scheme designed as follows:

$$u_i(t) = -d_i(y_i - x) - \beta \text{sign}(y_i - x), \quad i = 1, 2, \dots, N, \quad (5)$$

where $d_i > 0$ and $\beta > 0$ are positive control gains.

Let I_N represent the identity matrix of dimension N and $\mathbf{1}_n$ denote the n -dimensional column vector with all entries equal to one. Under the above control algorithm (5), system (3) and (4) can be represented as follows:

$$\begin{cases} \dot{x}(t) = -Bx(t) + Tg(x(t)) + J(t), \\ \dot{y}(t) = -(I_N \otimes B)y + (I_N \otimes T)G(y) + I_N \otimes J(t) \\ \quad - c(L \otimes I_n)y - (D \otimes I_n)(y - \mathbf{1}_N \otimes x) \\ \quad - \beta \text{sign}(y - \mathbf{1}_N \otimes x). \end{cases} \quad (6)$$

where $y(t) = (y_1^T(t), y_2^T(t), \dots, y_N^T(t))^T$, $G(y) = (g(y_1)^T, g(y_2)^T, \dots, g(y_N)^T)^T$, L denotes the Laplacian matrix associated with adjacency matrix $\mathcal{A}$, $D = \text{diag}\{d_1, d_2, \dots, d_N\}$.

One can easily give the definition of Filippov solutions of the initial value problem (IVP) of system (6) and hence we omit it here.

Assumption 1 Suppose that the activation function $g(\cdot)$ satisfies the following hypothesis:

For arbitrary $u, v \in \mathbb{R}$, there exists a constant $l_i > 0$ such that

$$|g_i(u) - g_i(v)| \leq l_i |u - v|, \quad i = 1, 2, \dots, n. \quad (7)$$

Global existence of Filippov solutions of coupled neural networks

In order to study the synchronization problem of the coupled neural networks, the global existence of Filippov solutions of IVP of system (6) should be guaranteed *a priori*. Based on Viability Theorem (see Lu and Chen 2005; Aubin 1991), we only need to prove the solution is bounded in every bounded interval.

Theorem 1 Suppose that the activation functions $g(\cdot)$ satisfy Assumption 1, then there exists at least one solution defined on $[0, +\infty)$ of the IVP of system (6) in the sense of Filippov.

Proof Under condition (7), it is easy to see that $\|g(x)\| \leq K_1 \|x\| + K_2$ where $K_1 = \max_{1 \leq i \leq n} \{l_i\}$ and $K_2 = \|g(0)\|$. Let y_{ij} be the j th entry of vector y_i , we have

$$\begin{aligned}
& \left\| -(I_{N+1} \otimes B) \begin{bmatrix} x \\ y \end{bmatrix} + (I_{N+1} \otimes T) \begin{bmatrix} g(x) \\ G(y) \end{bmatrix} + I_{N+1} \otimes J(t) - \begin{bmatrix} 0 & 0 \\ 0 & cL \otimes I_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right. \\
& \quad \left. - \begin{bmatrix} 0 \\ (D \otimes I_n)(y - \mathbf{1}_N \otimes x) \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \mathcal{F}[\text{sign}(y - \mathbf{1}_N \otimes x)]((x^T, y^T)^T) \end{bmatrix} \right\| \\
& \leq \max_{1 \leq i \leq n} \{b_i\} \|(x^T(t), y^T(t))^T\| + \|T\| \left(K_1 \left(\|x\| + \sum_{i=1}^N \|y_i\| \right) + (N+1)K_2 \right) + \bar{J} + c\|L \otimes I_n\| \|y\| \\
& \quad + \max_{1 \leq i \leq N} \{d_i\} \left(\sum_{i=1}^N \sum_{j=1}^n |y_{ij} - x_j| \right) + Nn\beta \\
& \leq \max_{1 \leq i \leq n} \{b_i\} \|(x^T(t), y^T(t))^T\| + \|T\| \left(K_1 \left(\sum_{j=1}^n |x_j| + \sum_{i=1}^N \sum_{j=1}^n |y_{ij}| \right) + (N+1)K_2 \right) + \bar{J} \\
& \quad + c\|L \otimes I_n\| \|y\| + \max_{1 \leq i \leq N} \{d_i\} \left(\sum_{i=1}^N \sum_{j=1}^n |y_{ij}| + N \sum_{j=1}^n |x_j| \right) + Nn\beta \\
& \leq P \|(x^T(t), y^T(t))^T\| + Q,
\end{aligned}$$

where

$$\begin{aligned}
P &= \max_{1 \leq i \leq n} \{b_i\} + CK_1 \|T\| + c\|L \otimes I_n\| + CN \max_{1 \leq i \leq N} \{d_i\}, \\
Q &= (N+1)K_2 \|T\| + \bar{J} + Nn\beta,
\end{aligned}$$

and C is some positive constant such that $\sum_{j=1}^n |x_j| + \sum_{i=1}^N \sum_{j=1}^n |y_{ij}| = \|(x^T(t), y^T(t))^T\|_1 \leq C \|(x^T(t), y^T(t))^T\|$. Denote $z(t) = (x^T(t), y^T(t))^T$, since $z(t)$ is absolutely continuous, one has

$$z(t) = z(0) + \int_0^t \dot{z}(s) ds.$$

It follows that

$$\begin{aligned}
\|z(t)\| &\leq \|z(0)\| + \int_0^t \|\dot{z}(s)\| ds \leq \|z(0)\| + Qt \\
&\quad + P \int_0^t \|z(s)\| ds.
\end{aligned}$$

By the Gronwall inequality, it directly follows that

$$\|z(t)\| \leq (\|z(0)\| + Qt)e^{Pt}.$$

Therefore, since $z(t)$ remains bounded in any bounded time interval, it is defined on $[0, +\infty)$. $\square$

Remark 1 In order to study the synchronization issue of coupled neural networks (6), the global existence of Filippov solutions of IVP ought to be addressed a priori. However, in Liu and Cao (2011), only the global existence of Filippov solutions of the isolated neural network is

discussed. A question might naturally arise, whether the Filippov solutions of the coupled neural networks still exist. In Theorem 1, we have clearly stated that the Filippov solutions of IVP of the coupled neural networks globally exist.

Synchronization in finite time

In this section, we consider the synchronization problem of the coupled neural network system (6). The definition of synchronization of this discontinuous system is firstly given as follows.

Definition 4 Consider coupled system (6), if for each Filippov solution of the IVP which is defined on $[0, +\infty)$, there exists a positive constant T , such that $\|y_i(t) - x(t)\| = 0$ for $t \geq T$, then we say the coupled system (6) is finite-time globally (completely) synchronized.

Remark 2 We do not investigate the uniqueness of Filippov solutions of IVP of (6) in this paper. Because Filippov solution includes set-valued function, it is difficult to guarantee the uniqueness and there are very few papers concerning this issue. The above notion of synchronization does not involve the uniqueness of Filippov solutions. If there exist more than one Filippov solutions, synchronization is said to be achieved in the sense that every solution satisfies the notion of synchronization.

To consider finite time convergence of discontinuous dynamical systems, the following lemmas are needed.

Lemma 7 (Property 2 (Forti et al. 2006)) Suppose that function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is C -regular, and that $x(t) : [0, +\infty) \rightarrow \mathbb{R}^n$ is absolutely continuous on any compact interval of $[0, +\infty)$. Let $v(t) = V(x(t))$, if there exists a continuous function $\gamma : (0, +\infty) \rightarrow \mathbb{R}$, with $\gamma(\sigma) > 0$ for $\sigma \in (0, +\infty)$, such that

$$\dot{v}(t) \leq -\gamma(v(t))$$

for all $t \geq 0$ such that $v(t) > 0$ and $v(t)$ is differentiable at t and $\gamma(\cdot)$ satisfies the condition

$$\int_0^{v(0)} \frac{1}{\gamma(\sigma)} d\sigma = t^* < +\infty.$$

Then, we have $v(t) = 0$ for $t \geq t^*$. In particular, if $\gamma(\sigma) = -K \sigma^\mu$ for all $\sigma \in (0, +\infty)$, where $\mu \in (0, 1)$ and $K > 0$, then the settling time is estimated by

$$t^* = \frac{v^{1-\mu}(0)}{K(1-\mu)}.$$

Lemma 8 (Hardy et al. 1988) If $\xi_1, \xi_2, \dots, \xi_n \geq 0$ and $0 < p \leq 1$, then

$$\left(\sum_{i=1}^n \xi_i \right)^p \leq \sum_{i=1}^n \xi_i^p.$$

Theorem 2 Suppose that the activation function $g(\cdot)$ satisfies Assumption 1 and there exists a positive constant ε such that the following inequality holds:

$$-(I_N \otimes B) - (c \frac{L+L^T}{2} + D) \otimes I_n + \varepsilon \|T\|^2 I_{Nn} + \frac{1}{4\varepsilon} I_N \otimes l^2 \leq 0 \quad (8)$$

where $l = \text{diag}\{l_1, l_2, \dots, l_n\}$. Then the coupled neural networks (6) realize synchronization in finite time $t^* = \sqrt{2}V(0)^{1/2}/\beta$ where $V(t) = 1/2e^T(t)e(t)$.

Proof Let $e_i = y_i - x, i = 1, 2, \dots, N$ and $e = (e_1^T, e_2^T, \dots, e_N^T)^T$, also note that $(L \otimes I_n)(\mathbf{1}_N \otimes x) = (L\mathbf{1}_N) \otimes (I_n x) = 0$, then the error system of (6) can be written as

$$\begin{aligned} \dot{e} &= -(I_N \otimes B)e + (I_N \otimes T)G(e) \\ &\quad - c(L \otimes I_n)e - (D \otimes I_n)e - \beta \text{sign}(e) \triangleq f(e) \end{aligned} \quad (9)$$

where $G(e) = G(y) - \mathbf{1}_N \otimes g(x)$ with a little abuse of notation. Take the Lyapunov candidate

$$V(e) = \frac{1}{2} e^T e,$$

then let us compute the Lie derivative of V with respect to $\mathcal{F}[f]$ at e which is denoted by $\mathcal{L}_{\mathcal{F}}V(e)$. It is easy to see that $\mathcal{F}[\text{sign}(e)](e) = (\text{Sign}(e_{11}), \dots, \text{Sign}(e_{1n}), \dots, \text{Sign}(e_{N1}), \dots, \text{Sign}(e_{Nn}))^T \triangleq \text{Sign}(e)$ where

$$\text{Sign}(x) = \begin{cases} -1, & x < 0 \\ [-1, 1], & x = 0 \\ 1, & x > 0 \end{cases}$$

and e_{ij} denotes the j th entry of vector e_i . Also note that

$$\begin{aligned} e^T \mathcal{F}[\text{sign}(e)](e) &= e^T \text{Sign}(e) = \sum_{i=1}^N \sum_{j=1}^n e_{ij} \text{Sign}(e_{ij}) \\ &= \sum_{i=1}^N \sum_{j=1}^n |e_{ij}|. \end{aligned} \quad (10)$$

According to Definition 3, we have

$$\begin{aligned} \mathcal{L}_{\mathcal{F}}V(e) &= (\nabla V(e))^T \mathcal{F}[f](e) \\ &= e^T (-(I_N \otimes B)e + (I_N \otimes T)G(e) \\ &\quad - ((cL + D) \otimes I_n)e) - \beta \sum_{i=1}^N \sum_{j=1}^n |e_{ij}|. \end{aligned} \quad (11)$$

Obviously, $V(e)$ is locally Lipschitz and regular. By Lemma 6, we can obtain that $\frac{d}{dt}(V(e(t))) \in \mathcal{L}_{\mathcal{F}}V(e(t))$ for a.e. $t \geq 0$. Note that $\mathcal{L}_{\mathcal{F}}V(e(t))$ is indeed singleton valued and therefore for a.e. $t \geq 0$, we have

$$\begin{aligned} \frac{d}{dt}(V(e(t))) &= e^T (-(I_N \otimes B)e + (I_N \otimes T)G(e) \\ &\quad - ((cL + D) \otimes I_n)e) - \beta \sum_{i=1}^N \sum_{j=1}^n |e_{ij}| \\ &\leq -e^T (I_N \otimes B + (cL + D) \otimes I_n)e \\ &\quad + \varepsilon e^T (I_N \otimes T)^T (I_N \otimes T)e \\ &\quad + \frac{1}{4\varepsilon} G(e)^T G(e) - \beta \sum_{i=1}^N \sum_{j=1}^n |e_{ij}| \\ &\leq -e^T (I_N \otimes B + (cL + D) \otimes I_n)e \\ &\quad + \varepsilon \|T\|^2 e^T e + \frac{1}{4\varepsilon} e^T (I_N \otimes l^2)e \\ &\quad - \beta \sum_{i=1}^N \sum_{j=1}^n |e_{ij}|. \end{aligned} \quad (12)$$

Due to condition (8), we have

$$\dot{V}(t) \leq -\beta \sum_{i=1}^N \sum_{j=1}^n |e_{ij}|. \quad (13)$$

By Lemma 8, we can obtain that

$$\dot{V}(t) \leq -\beta \left(\sum_{i=1}^N \sum_{j=1}^n e_{ij}^2 \right)^{1/2} = -\sqrt{2}\beta V(t)^{1/2}. \quad (14)$$

Finally, by Lemma 7, one can conclude that $e(t) = 0$ for $t \geq t^*$ where $t^* = \sqrt{2}V(0)^{1/2}/\beta$. $\square$

Some control strategies to reduce the settling time

In this section, we will discuss how to reduce the settling time via some control strategies. A switched control scheme between continuous and discontinuous controllers is proposed to cut down the settling time. Moreover, an adaptive tuning parameter approach is also given to achieve faster convergence speed.

First, let us recall some results on finite-time synchronization via continuous controllers (Yang and Cao 2010). For $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, denote $\text{sig}^\alpha(x) = (\text{sign}(x_1)|x_1|^\alpha, \text{sign}(x_2)|x_2|^\alpha, \dots, \text{sign}(x_n)|x_n|^\alpha)^T$, $0 < \alpha < 1$. If we adopt the following continuous feedback controller

$$u_i(t) = -d_i(y_i - x) - \beta \text{sig}^\alpha(y_i - x), \quad i = 1, 2, \dots, N, \quad (15)$$

under certain conditions, finite-time synchronization can be realized and the settling time can be estimated by $t^* = (2V(0))^{(1-\alpha)/2} / (\beta(1-\alpha))$ where $V(t) = 1/2 e^T(t)e(t)$. The minimum value of $t^*(\alpha)$ is closely related to the initial error $V(0)$. If $V(0) > e^2/2$, when $\alpha = 1 - 2/\ln(2V(0))$, $t_{\min}^* = e \ln(2V(0)) / (2\beta)$. If $V(0) \leq e^2/2$, then $t^*(\alpha)$ is strictly increasing when $\alpha \in (0, 1)$. Combined with the results in Theorem 2, one can conclude that when $\alpha = 0$ (the corresponding controller is discontinuous), $t_{\min}^* = \sqrt{2}V(0)^{1/2} / \beta$ (see Remark 3 (Yang and Cao 2010) for details). From the above discussion, we can find that when the initial error of the system is very small such that $V(0) \leq e^2/2$, it is better to use the discontinuous controller (5). In what follows, it is assumed that $V(0) > e^2/2$. We will propose some control strategies to reduce the settling time.

Reduce settling time via switched control

It is easy to see that under condition (8), $V(t)$ is monotonically decreasing. The initial error might be very large, and it is better to use continuous controller (15). Then, after some time, the synchronization error becomes smaller and it is better to use discontinuous controller (5). Therefore, it is reasonable to propose the following switched control strategy, i.e., to switch the parameter α according to the synchronization error:

$$\alpha(t) = \begin{cases} 1 - \frac{2}{\ln(2V(0))}, & \text{when } V(t) > \frac{1}{2}e^2, \\ 0, & \text{when } V(t) \leq \frac{1}{2}e^2. \end{cases} \quad (16)$$

Theorem 3 Suppose that the activation function $g(\cdot)$ satisfies Assumption 1 and condition (8) holds. Then, the coupled neural networks (3, 4) reach synchronization in finite time under the switched control scheme (16) and the estimated settling time is

$$t^* = \frac{e \ln(2V(0)) - \ln(2V(0))e^{\frac{2}{\ln(2V(0))}} + 2e}{2\beta}$$

where $V(t) = 1/2 e^T(t)e(t)$.

Proof By a similar discussion as in the proof of Theorem 2, we arrive at

$$\dot{V}(t) \leq -\beta(2V(t))^{\frac{1+\alpha(t)}{2}} \triangleq \gamma(V(t)).$$

According to Lemma 7, finite-time synchronization can be achieved and the settling time can be estimated by

$$\begin{aligned} t^* &= \int_0^{V(0)} \frac{1}{\gamma(\sigma)} d\sigma = \int_0^{\frac{e^2}{2}} \frac{d\sigma}{\beta(2\sigma)^{\frac{1}{2}}} + \int_{\frac{e^2}{2}}^{V(0)} \frac{d\sigma}{\beta(2\sigma)^{1-\frac{1}{\ln(2V(0))}}} \\ &= \frac{e \ln(2V(0)) - \ln(2V(0))e^{\frac{2}{\ln(2V(0))}} + 2e}{2\beta}. \end{aligned}$$

□

Remark 3 Note that this settling time is smaller than the optimal case ($t_{\min}^* = e \ln(2V(0)) / (2\beta)$) when the parameter α is fixed since $xe^{\frac{2}{x}} > 2e$ and strictly increasing when $x \geq 2$. When the initial synchronization error increases, the difference of settling time between the switched control and the optimal fixed parameter case also increases. It gives us a guidance on how to design the controller to reduce the total time cost to reach synchronization.

Remark 4 Intuitively, we can imagine that the convergence speed might be very slow when the synchronization error is very small if continuous controller (15) is adopted since $\text{sign}(x)|x|^\alpha \rightarrow 0$ ($0 < \alpha < 1$) when $x \rightarrow 0$. However, it is not the case for the sign function $\text{sign}(x)$. The discontinuous controller will have its advantage when the synchronization error is very small. Therefore, we propose the above switched control strategy. When the synchronization error is large, we use the continuous controller and then switch to the discontinuous controller when the synchronization error becomes small.

An adaptive tuning parameter approach

In what follows, we propose an adaptive tuning parameter approach. The control parameter α is adjusted according to the synchronization error as follows:

$$\alpha(t) = \begin{cases} 1 - \frac{2k}{\ln(2V(t))} \quad (0 < k < 1), & \text{when } V(t) > \frac{1}{2}e^2, \\ 0, & \text{when } V(t) \leq \frac{1}{2}e^2. \end{cases} \quad (17)$$

Theorem 4 Suppose that the activation function $g(\cdot)$ satisfies Assumption 1 and condition (8) holds. Then, the coupled neural networks (3, 4) reach synchronization in

finite time under the switched control scheme (17) and the estimated settling time is

$$t^* = \frac{e^k(\ln(2V(0)) - 2) + 2e}{2\beta}$$

where $V(t) = 1/2e^T e$.

Proof The proof is similar to that of Theorem 3 and hence is omitted here. $\square$

Remark 5 If the initial error $V(0)$ is relatively large, then the settling time is approximately $1/e^{1-k}$ of the minimum settling time of the fixed parameter case. If k is chosen sufficiently small, faster convergence rate might be achieved by utilizing the proposed adaptive tuning parameter approach. The two control strategies proposed in this section may also be applied to finite-time synchronization of complex networks and finite-time consensus of multi-agent systems.

Nonlinear coupling and pinning control scheme

In Chen and Lü (2009), finite-time synchronization of complex dynamical networks via nonlinear coupling and

Under control algorithm (19), the global existence of Filippov solutions of the initial value problem of the coupled system can be similarly analyzed as in Theorem 1. In the following we prove that this pinning control scheme will work when the coupling network topology $\mathcal{G}$ is strongly connected and detail-balanced.

Theorem 5 Suppose that the coupling network topology $\mathcal{G}$ is strongly connected and detail-balanced. If the activation function $g(\cdot)$ satisfies Assumption 1 and there exists a positive constant ε such that the following inequality holds:

$$\Xi \otimes \left(B + \varepsilon \|T\|^2 I_n + \frac{1}{4\varepsilon} l^2 \right) - (\Xi(cL + \tilde{D})) \otimes I_n \leq 0 \quad (20)$$

where $l = \text{diag}\{l_1, l_2, \dots, l_n\}$ and $\tilde{D} = \text{diag}\{d_1, 0, 0, \dots, 0\}$. Then under control algorithm (19), the coupled neural networks (18) will synchronize the isolated neural network (3) in finite time. $\square$

Proof Consider the following error system

$$\dot{e}_i = f_i(e) := \begin{cases} -Be_i + Tg(e_i) + c \sum_{j=1}^N a_{ij}(e_j - e_i + \sigma \text{sign}(e_j - e_i)) \\ \quad -d_1 e_1 - \beta \text{sign}(e_1), & \text{if } i = 1; \\ -Be_i + Tg(e_i) + c \sum_{j=1}^N a_{ij}(e_j - e_i + \sigma \text{sign}(e_j - e_i)), & \text{otherwise} \end{cases} \quad (21)$$

pinning control was investigated. It was indicated there that sometimes suitable nonlinear coupling can improve the network performance. In what follows, we also consider the synchronization problem of an array of nonlinearly coupled neural networks via pinning control. The nonlinearly coupled neural networks are formulated by

$$\begin{aligned} \dot{y}_i(t) &= -By_i(t) + Tg(y_i(t)) + J(t) \\ &\quad + c \sum_{j=1}^N a_{ij}(y_j - y_i + \sigma \text{sign}(y_j - y_i)) + u_i(t), \quad i \\ &= 1, 2, \dots, N. \end{aligned} \quad (18)$$

For simplicity, we only consider pinning control algorithm with a single controller (Chen et al. 2007) when the network topology is strongly connected. Pinning control scheme with more than one controllers can be similarly analyzed. The pinning control scheme is given as follows:

$$u_i(t) = \begin{cases} -d_1(y_1 - x) - \beta \text{sign}(y_1 - x), & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases} \quad (19)$$

where $g(e_i) = g(y_i) - g(x)$ with a little abuse of notation. Denote $f(e) = (f_1(e)^T, f_2(e)^T, \dots, f_N(e)^T)^T$. By Lemma 5, it follows that

$$\begin{aligned} &\mathcal{F} \left[\begin{pmatrix} c\sigma \sum_{j=1}^N a_{1j} \text{sign}(e_j - e_1) - \beta \text{sign}(e_1) \\ c\sigma \sum_{j=1}^N a_{2j} \text{sign}(e_j - e_2) \\ \vdots \\ c\sigma \sum_{j=1}^N a_{Nj} \text{sign}(e_j - e_N) \end{pmatrix} (e) \right] \\ &\subseteq \begin{pmatrix} \mathcal{F}[c\sigma \sum_{j=1}^N a_{1j} \text{sign}(e_j - e_1) - \beta \text{sign}(e_1)](e) \\ \mathcal{F}[c\sigma \sum_{j=1}^N a_{2j} \text{sign}(e_j - e_2)](e) \\ \vdots \\ \mathcal{F}[c\sigma \sum_{j=1}^N a_{Nj} \text{sign}(e_j - e_N)](e) \end{pmatrix} \\ &\subseteq \begin{pmatrix} c\sigma \sum_{j=1}^N a_{1j} \text{Sign}(e_j - e_1) - \beta \text{Sign}(e_1) \\ c\sigma \sum_{j=1}^N a_{2j} \text{Sign}(e_j - e_2) \\ \vdots \\ c\sigma \sum_{j=1}^N a_{Nj} \text{Sign}(e_j - e_N) \end{pmatrix} \end{aligned}$$

where $\text{Sign}(e_j - e_i) = (\text{Sign}(e_{j1} - e_{i1}), \dots, \text{Sign}(e_{jn} - e_{in}))^T$. Take the Lyapunov function

$$V(t) = \frac{1}{2} \sum_{i=1}^N \xi_i e_i^T e_i = \frac{1}{2} e^T (\Xi \otimes I_n) e$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_N)^T$ is a left eigenvector of L corresponding to the zero eigenvalue. Since graph $\mathcal{G}$ is detail-balanced, i.e., $\xi_i a_{ij} = \xi_j a_{ji}$ for all $i, j = 1, 2, \dots, N$, we have

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N \xi_i a_{ij} e_i^T \text{Sign}(e_j - e_i) \\ &= \sum_{i=1}^N \sum_{j=1}^N \xi_j a_{ji} e_j^T \text{Sign}(e_i - e_j) \\ &= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \xi_i a_{ij} (e_j - e_i)^T \text{Sign}(e_j - e_i) \\ &= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^n \xi_i a_{ij} |e_{jk} - e_{ik}|. \end{aligned} \quad (22)$$

Computing the Lie derivative of V with respect to $\mathcal{F}[f]$ at e , it is not hard to obtain

$$\begin{aligned} \mathcal{L}_{\mathcal{F}} V(e) &\subseteq -\sum_{i=1}^N \xi_i e_i^T B e_i + \sum_{i=1}^N \xi_i e_i^T T g(e_i) \\ &\quad - c e^T ((\Xi L) \otimes I_n) e - c \sigma \sum_{i=1}^N \sum_{j=1}^N \xi_i a_{ij} e_i^T \text{Sign}(e_j - e_i) \\ &\quad - d_1 \xi_1 e_1^T e_1 - \xi_1 \beta e_1^T \text{Sign}(e_1) \\ &= -\sum_{i=1}^N \xi_i e_i^T B e_i + \sum_{i=1}^N \xi_i e_i^T T g(e_i) - c e^T ((\Xi L) \otimes I_n) e \\ &\quad - \frac{1}{2} c \sigma \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^n \xi_i a_{ij} |e_{jk} - e_{ik}| \\ &\quad - d_1 \xi_1 e_1^T e_1 - \xi_1 \beta \sum_{k=1}^n |e_{1k}|. \end{aligned} \quad (23)$$

Note that $\mathcal{L}_{\mathcal{F}} V(e)$ is in fact singleton valued. By employing Lemma 8 and 1 and following a similar discussion as in the proof of Theorem 2, we have

$$\begin{aligned} \dot{V}(t) &\leq -\frac{1}{2} c \sigma \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^n \xi_i a_{ij} |e_{jk} - e_{ik}| - \xi_1 \beta \sum_{k=1}^n |e_{1k}| \\ &\leq -\frac{1}{2} c \sigma \left(\sum_{k=1}^n \left(\sum_{i=1}^N \sum_{j=1}^N (\xi_i a_{ij})^2 (e_{jk} - e_{ik})^2 + \left(\frac{2\xi_1 \beta}{c\sigma} \right)^2 e_{1k}^2 \right) \right)^{1/2} \\ &= -\frac{1}{2} c \sigma (e^T ((\bar{L} + \bar{D}) \otimes I_n) e)^{1/2} \end{aligned} \quad (24)$$

where $\bar{L}$ is the Laplacian matrix associated with the adjacency matrix $\bar{A} = [(\xi_i a_{ij})^2]$ and $\bar{D} = \text{diag}\{(2\xi_1 \beta / (c\sigma))^2, 0, \dots, 0\}$. Note that $\bar{L} + \bar{D}$ is positive definite according to Lemma 2 and hence the minimum eigenvalue $\lambda_{\min}(\bar{L} + \bar{D}) > 0$. Then, it directly follows that

$$\begin{aligned} \dot{V}(t) &\leq -\frac{1}{2} c \sigma (\lambda_{\min}(\bar{L} + \bar{D}))^{1/2} (e^T e)^{1/2} \\ &\leq -\frac{1}{2} c \sigma \left(\frac{2\lambda_{\min}(\bar{L} + \bar{D})}{\max_{1 \leq i \leq N} \{\xi_i\}} \right)^{1/2} V(t)^{1/2} \end{aligned} \quad (25)$$

Based on Lemma 7, we can obtain that $V(t) = 0$ for $t \geq t^*$ where $t^* = 2V(0)^{1/2}/K$ with $K = 1/2c\sigma(2\lambda_{\min}(\bar{L} + \bar{D})/\max_{1 \leq i \leq N} \{\xi_i\})^{1/2}$. Hence, under control algorithm (19), the coupled neural networks (18) synchronize the isolated neural network (3) in finite time t^* .

Remark 6 In Theorem 3.5 Chen and Lü (2009) also investigated finite-time synchronization of complex dynamical networks via nonlinear coupling and pinning control. However, the network topology is required to be undirected and the synchronization can not be globally reached (the initial states of the coupled networks are required in a bounded area). In Theorem 2, we only suppose that the network topology is strongly connected and detail-balanced. Also, we do not assume that the initial states of the coupled networks are in a bounded area and global synchronization can be achieved.

In what follows we extend the results to the case when coupling network topology contains a directed spanning tree and each strongly connected component is detail-balanced. For simplicity, we suppose that digraph $\mathcal{G}$ contains a directed spanning tree with two strongly connected components. Then, the Laplacian matrix L has the following form:

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \quad (26)$$

where $L_{22} \in \mathbb{R}^{r_2 \times r_2}$ is irreducible and $L_{11} \in \mathbb{R}^{r_1 \times r_1}$ is irreducible or an zero matrix of dimension one, $L_{21} \neq 0$ and $r_1 + r_2 = N$.

Theorem 6 Suppose that the coupling network topology $\mathcal{G}$ contains a directed spanning tree (the corresponding Laplacian matrix has form (26)) and each of the two strongly connected components (denoted by $\mathcal{G}_1$ and $\mathcal{G}_2$) are detail-balanced. If the activation function $g(\cdot)$ satisfies Assumption 1 and there exists a positive constant ε such that the following two inequality holds:

$$\hat{\Xi} \otimes \left(-B + \varepsilon \|T\|^2 I_n + \frac{1}{4\varepsilon} I^2 \right) - (\hat{\Xi}(cL_{11} + D_1)) \otimes I_n \leq 0 \quad (27)$$

$$\Omega \otimes \left(-B + \varepsilon \|T\|^2 I_n + \frac{1}{4\varepsilon} I^2 \right) - (c\Omega((\tilde{L}_{22} + D_2)) \otimes I_n) \leq 0 \quad (28)$$

where $\hat{\Xi} = \text{diag}\{\hat{\xi}_1, \hat{\xi}_2, \dots, \hat{\xi}_{r_1}\}$ ($\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \dots, \hat{\xi}_{r_1}, 0, \dots, 0)^T$ is the left eigenvector of L associated with eigenvalue 0) and $D_1 = \text{diag}\{d_1, 0, 0, \dots, 0\}$. $\tilde{L}_{22}$ is the Laplacian matrix corresponding to the strongly connected component $\mathcal{G}_2$, $\Omega = \text{diag}\{\omega_{r_1+1}, \omega_{r_1+2}, \dots, \omega_N\}$ with $\omega_i a_{ij} = \omega_j a_{ji}$ for $i, j = r_1 + 1, \dots, N$ ($(\omega_{r_1+1}, \omega_{r_1+2}, \dots, \omega_N)^T$ is indeed the left eigenvector of $\tilde{L}_{22}$ associated with eigenvalue 0), $D_2 = \text{diag}\{\sum_{j=1}^{r_1} a_{r_1+1,j}, \sum_{j=1}^{r_1} a_{r_1+2,j}, \dots, \sum_{j=1}^{r_1} a_{N,j}\}$. Then under control scheme (19), the coupled neural networks (18) will synchronize the isolated neural network (3) in finite time.

Proof To avoid unnecessary duplication, here we only show the sketch of the proof, omitting the related details. First, consider the error system (21) and construct the Lyapunov function

$$V(t) = \frac{1}{2} \sum_{i=1}^{r_1} \hat{\xi}_i e_i^T e_i. \quad (29)$$

Then, follow a similar discussion as in the proof of Theorem 5, one can conclude that under condition (27), there exists a positive constant t_1^* such that $e_1 = e_2 = \dots = e_{r_1} = 0$ when $t \geq t_1^*$. When $t \geq t_1^*$, the error system becomes:

$$\begin{aligned} \dot{e}_i &= -Be_i + Tg(e_i) \\ &+ c \left(\sum_{j=r_1+1}^N a_{ij}(e_j - e_i) + \sigma \sum_{j=r_1+1}^N a_{ij} \text{sign}(e_j - e_i) \right) \\ &- c \left(\sum_{j=1}^{r_1} a_{ij} e_j + \sigma \sum_{j=1}^{r_1} a_{ij} \text{sign}(e_j) \right), \\ i &= r_1 + 1, r_1 + 2, \dots, N. \end{aligned} \quad (30)$$

Construct the following Lyapunov function

$$V(t) = \frac{1}{2} \sum_{i=r_1+1}^N \omega_i e_i^T e_i, \quad (31)$$

Then by a similar process as in the proof of Theorem 5, one can conclude that under condition (28), there exists a positive constant t_2^* such that $e_{r_1+1} = e_{r_1+2} = \dots = e_N = 0$ for $t \geq t_2^*$. The expression of t_2^* is not hard to be computed and hence is omitted here. $\square$

Remark 7 In Theorem 5 and Theorem 6, each of the strongly connected components of the directed network is required to be detail-balanced. Of course, a general directed topology with a rooted spanning tree is more suitable

for realistic networks. However, compared with undirected network topology considered in Chen and Lü (2009), detail-balanced topology is already a relatively weak condition. Since only a portion of coupled neural networks have access to the state information of the target neural network when we consider pinning control scheme, it is more difficult to design a pinning controller to achieve finite-time synchronization. Note that the result in Theorem 2 is also applicable for a general directed network without the assumption of detail-balanced topology. Therefore, when the condition of detail-balanced topology does not hold, the control strategies proposed in “Finite-time synchronization via discontinuous controller” and “Some control strategies to reduce the settling time” are still valid.

Remark 8 In Theorem 2, we do not make any additional assumptions on graph $\mathcal{G}$ and therefore the coupling network topology is very general. However, the main disadvantage of control algorithm (5) is that every coupled neural network is required to have access to the state of the target neural network. In order to utilize pinning control algorithm (19) with a single controller to achieve finite-time synchronization, the assumption of containing a directed spanning tree is necessary. However, the detail-balanced network topology might be restrictive. How to remove this assumption is a meaningful yet challenging problem.

Illustrative examples

Example 1 Consider a 3D cellular neural network (CNN)

$$\dot{x}(t) = -Bx(t) + Tg(x(t)) + J(t) \quad (32)$$

with

$$T = \begin{bmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{bmatrix}$$

$B = I_3$, $J(t) = (0, 0, 0)^T$, and $g_i(s) = (|s + 1| - |s - 1|)/2$, $i = 1, 2, 3$. It can be readily computed that $l_i = 1$, $i = 1, 2, 3$. As is indicated in (Zou and Nossek 1993), the above neural network has a double-scrolling chaotic attractor (see Fig. 1). The coupling network topology with $N = 10$ is illustrated in Fig. 2 and we assume that the digraph has 0–1 weights for simplicity. By Theorem 2, taking $c = 1$ and $d_i = d > 0$, it can be easily computed that $d > \lambda_{\max}(-((L + L^T) \otimes I_3)/2 + (\|T\| - 1)I_{30}) = 6.2562$ suffices to ensure condition (8). Let $\beta = 1$, $x(0) = (0.1, 0.3, 0.2)^T$ and the initial states of each of the coupled neural networks $y_i(0)$ are randomly selected from $[-6, 6]^3$. In order to show the effectiveness of our proposed control strategies in “Some control

Fig. 1 Phase trajectory of chaotic CNN (32)

Fig. 2 The coupling network topology in Example 1

Fig. 3 The synchronization error of optimal fixed parameter case. The colored lines represent the variation of the synchronization error of each state variable when the fixed parameter α is properly selected

strategies to reduce the settling time”, the case with parameter α fixed is compared with the case where the adaptive parameter tuning method (17) with $k = 0.1$ is adopted. When the parameter α is fixed, we choose $\alpha =$

Fig. 4 The synchronization error of the case where adaptive parameter tuning strategy is adopted. The colored lines represent the variation of the synchronization error of each state variable when the parameter α varies according to the adaptive law (17)

$1 - 2/\ln(2V(0))$ (where $V(t) = 1/2e^T(t)e(t)$) which results in the minimal settling time. The synchronization error of each state variable ($e_{ij} = y_{ij} - x_j, i = 1, 2, \dots, N, j = 1, 2, 3$) of the above two cases is depicted in Figs. 3 and 4 respectively. In addition, The evolution of the total synchronization error ($e = \sqrt{\sum_{i=1}^N e_i^T e_i}$) of these two cases is also shown in Fig. 5. From Fig. 5, it can be observed that by utilizing the adaptive parameter tuning strategy, about half of the settling time can be reduced compared with the optimal fixed parameter case. We can also find that the convergence speed is very slow when the synchronization error is very small if the continuous controller is adopted. This phenomenon is in accordance with our intuition as is discussed in Remark 4.

Example 2 We still use the chaotic CNN in Example 1 as the objective neural network. The coupling network topology contains a directed spanning tree with two detail-balanced strongly connected components as illustrated in Fig. 6. It is not hard to be computed that $\hat{\Xi} = \text{diag}\{5, 1, 2\}$ and $\Omega = \text{diag}\{4, 6, 5, 10\}$.

In what follows, discontinuous pinning control scheme (19) is employed to realize finite-time synchronization. We can choose the coupling strength $c = 45$ and the feedback control gain $d_1 = 42$ to guarantee condition (27) and (28). Let $\beta = \sigma = 1$. The synchronization error of the state variables is depicted in Fig. 7. From the simulation, we also find that the coupling strength and feedback control gain determined by condition (27) and (28) are relatively

Fig. 5 Evolution of total synchronization error versus time for the optimal fixed parameter case and the case where adaptive parameter tuning strategy is adopted respectively

Fig. 6 The coupling network topology in Example 2

Fig. 7 The synchronization error of state variable for Example 2

large, a much smaller c and d_1 might still work. Therefore, to give a more delicate estimation of the coupling strength and feedback control gain by employing some advanced methods is a meaningful task.

Conclusions

In this paper, we have discussed finite-time synchronization problem for coupled neural network systems. Discontinuous controllers are designed to achieve finite-time synchronization. Furthermore, switched control and adaptive tuning parameter strategies are proposed in order to reduce the settling time. In addition, A pinning control scheme is also designed to realize finite-time synchronization. However, we make an additional assumption that each of the strongly connected components should be detail-balanced. How to remove this assumption is still a challenging task. Moreover, our proposed control algorithm might not be suitable for delayed neural network

systems. Therefore, to design controllers to reach finite-time synchronization of coupled neural networks with delays still needs to be further investigated. Another interesting yet challenging problem is to study finite-time synchronization of coupled neural networks whose activation functions are discontinuous via discontinuous controllers. These problems are our future research directions.

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