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An analytic solution to the homogeneous EIT problem on the 2D disk and its application to estimation of electrode contact impedances

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Abstract

An analytic solution of the potential distribution on a 2D homogeneous disk for electrical impedance tomography under the complete electrode model is expressed via an infinite system of linear equations. For the shunt electrode model with two electrodes, our solution coincides with the previously derived solution expressed via elliptic integral (Pidcock *et al* 1995). The Dirichlet-to-Neumann map is derived for statistical estimation via nonlinear least squares. The solution is validated in phantom experiments and applied for breast contact impedance estimation *in vivo*. Statistical hypothesis testing is used to test whether the contact impedances are the same across electrodes or all equal zero. Our solution can be especially useful for a rapid real-time test for bad surface contact in clinical setting.

Keywords

breast cancer; complete electrode model; electrical impedance tomography; Neumann-to-Dirichlet map; surface impedance

1. Introduction

Electrical impedance tomography (EIT) is a rapidly developing imaging modality with various applications including breast cancer detection and screening (Zou and Guo 2003, Hartov *et al* 2005, Borsic *et al* 2008). It is well known that EIT image reconstruction is an ill-posed problem, so some sort of regularization is required (Brown 2003, Lionheart *et al* 2005). Moreover applications of EIT to breast cancer detection is complicated by a possibility of bad electrode contact with the skin leading to possible artifacts and an elevated false positive rate. The problem of contact (or surface) impedance should be addressed in every single EIT measurement exam over the entire time of the data acquisition due to possible movement of the patient.

Simultaneous estimation of contact impedances at each electrode and full image reconstruction is almost impossible because the latter problem is ill-conditioned by itself. Thus, the idea is (a) to estimate contact impedances assuming a homogeneous medium and (b) to apply those estimates for the full EIT image reconstruction. In an alternative approach, the EIT image and electrode impedances are reconstructed simultaneously dependent on the selection of the penalty term. Such an approach has been applied using the 3D finite element (FEM) approximation (Vilhunen *et al* 2002, Heikkinen *et al* 2002, Karhunen *et al* 2010), and recently for EIT with rectangular electrode design including Saulnier *et al* (2007), Boverman *et al* (2009). Needless to say, that a robust identification of contact impedances (Z) at all electrodes in real time is still a challenging problem. Moreover, we discovered that

3D FEM simultaneous conductivity and Z-identification may be unstable because different starting points produced different final iteration values.

The goal of the present work is to develop a simple and robust method for the simultaneous estimation of the conductivity and contact impedances based on the analytic solution to 2D homogeneous Laplace partial differential equation (PDE). Following a 2D-to-3D calibration, our method enables a robust estimation of contact impedances in real time during an EIT exam.

While the solution of the Laplace PDE on a 2D homogeneous disk is a fairly easy task under the gap electrode model, the complete electrode model is challenging from a mathematical prospective because it has mixed boundary conditions (Duffy 2008). In general, mixed boundary value problem is poorly studied in mathematics in spite the fact that many engineering problems with insulation (no flux on the boundary) imply mixed boundary conditions. For example, the most comprehensive source of the PDE solutions for engineering problems includes no mention of the mixed boundary PDE (Polyanin 2002). Several authors have attempted to find the analytic solution for the 2D disk EIT with the complete electrode model. For example, Somersalo *et al* (1992) and Paulson *et al* (1992) illustrate the solution by showing an U-shaped plot of the current density on the boundary of the circle. Some solutions, expressed via elliptic integral, are presented in Pidcock *et al* (1995) with two electrodes under the shunt electrode model (contact impedance is zero).

We derive the solution to the Laplace PDE on a homogeneous 2D disk under the complete electrode model with any number of electrodes and different contact impedances in terms of an infinite system of linear equations in Section 3. This solution is applied to simultaneously estimate the conductivity and electrode impedance at each electrode in a voltage-to-current system using a Dirichlet-to-Neumann (DtN) map. Our approach is validated through phantom experiments in Section 4 and applied to *in vivo* EIT data for identification of bad breast skin contact in Section 5. The appendix contains mathematical details of the analytic derivation.

2. General formulation of the EIT problem

The goal of EIT is to reconstruct spatial conductivity $\sigma = \sigma(\mathbf{s})$ within the body by measuring currents and voltages at the electrodes located on the periphery of the body, where $\mathbf{s}$ is a spatial coordinate vector. In the current-to-voltage EIT system, currents are injected and voltages are measured for the reconstruction purpose. Similarly, in the voltage-to-current system, voltages are part of the boundary conditions and the resulting currents are measured. In this paper, we shall assume the latter system because this is what we have at Dartmouth. Sometimes the EIT problem is formulated for complex $\sigma(\mathbf{s}) + j\omega\epsilon_0\epsilon_r(\mathbf{s})$, where ϵ_0 is the permittivity of free space ($8.85 \times 10^{-12} \text{ Fm}^{-1}$), $\epsilon_r = \epsilon_r(\mathbf{s})$ is the relative permittivity of the medium, ω is the angular frequency of the applied signal, and $j = \sqrt{-1}$ is the imaginary unit, but in this paper we restrict our consideration to real values so the imaginary part is not present. In 2D, we have $\mathbf{s} = (x, y)$, and in 3D we have $\mathbf{s} = (x, y, z)$; for a homogeneous medium, $\sigma(\mathbf{s}) = \text{const}$. Throughout the paper, boldface is used to indicate vectors and matrices. In this section, we specify the forward problem when the function σ is known.

2.1. Laplace equation within the body

It is assumed that there are L electrodes on the periphery of the body covering non-overlapping surface areas E_i , $i = 1, 2, \dots, L$. Potentials $U_1, U_2, \dots, U_L$ are applied at the

electrodes such that $\sum_{i=1}^L U_i = 0$. Within the body, the potential distribution is governed by the Laplace PDE:

$$\nabla \cdot (\sigma \nabla u) = 0. \quad (1)$$

The task of the forward model is to find the potential function, u , given $\sigma = \sigma(\mathbf{s})$ using the boundary conditions (BC) specified below.

2.2. Boundary conditions

Mathematical models of EIT differ by their formulation of the boundary condition at the electrodes; equation (1) remains the same. On the boundary of the body between electrodes, it is assumed that there is no current flow,

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \quad \mathbf{s} \notin E_i, \quad (2)$$

where $\mathbf{n}$ is the normal vector on the surface of the body. According to the complete electrode model,

$$u(\mathbf{s}) + Z_i \sigma(\mathbf{s}) \frac{\partial u}{\partial \mathbf{n}} = U_i, \quad \mathbf{s} \in E_i, \quad (3)$$

where Z_i is the electrode contact (or surface) impedance, $i = 1, 2, \dots, L$. It is common to assume that the contact impedance is constant over the area of the electrode but may be different across electrodes. The supplied potential is constant over the surface of the electrode because it is made of a highly conductive metal material. Model (1) together with BC (2) and (3) correctly reflects most engineering hardware designs and is the most comprehensive forward model of the electrical impedance problem (Cheng *et al* 1989, Cheney *et al* 1999).

The shunt electrode model is a special case of the complete electrode model (3) with $Z_i = 0$, $i = 1, 2, \dots, L$. Thus, the BC for this model takes the form

$$u(\mathbf{s}) = U_i, \quad \mathbf{s} \in E_i, \quad (4)$$

A positive feature of this model is that no knowledge of the contact impedance is required, although several authors have warned that this model may be not adequate in practice.

3. The homogeneous 2D disk

In this section, we derive an analytic solution for the complete electrode model on a homogeneous 2D disk in polar coordinates with finite number of electrodes under the complete electrode model. Previously, this solution was derived for the gap electrode model with Neumann boundary conditions, Isaacson (1986). Graphs of the current distribution on the boundary of the circle under the complete electrode model were shown as illustrations in Somersalo *et al* (1992) and Paulson *et al* (1992), but no explicit solutions have been presented. In addition, the solution was derived for a very special case: unit radius, equal contact impedance, sine or cosine (symmetric) electric potential on the radius.

The potential distribution on a homogeneous disk of radius R and conductivity σ with infinitesimal height is governed by the Laplace equation in polar coordinates. As a very special case of the general PDE (1),

$$\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad (5)$$

where $0 \leq r \leq R$ is the distance from the center and $0 \leq \theta < 2\pi$ is the angle. As was mentioned above in this paper, it is assumed that all variables are real. For this case, the L non-overlapping electrodes with half-width w (radians) are located on the circle of radius R at angle locations $\{\theta_i, i = 1, 2, \dots, L\}$. Separation of variables (Isaacson 1986) yields the solution expressed in Fourier series as

$$u(r, \theta) = u_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)), \quad (6)$$

where coefficients a_n and b_n are to be found from the BC (3), which, for the 2D disk, takes the form

$$u(R, \theta) + Z_i \sigma \left. \frac{\partial u}{\partial r} \right|_{r=R} = U_i, i=1, 2, \dots, L. \quad (7)$$

Note that the left-hand side of (7) is a function of θ , but the right-hand side is constant. Constant Z_i is the electrode impedance; if $Z_i = 0$, we have the shunt model.

As was shown in the Appendix, coefficients a_n and b_n are found by integrating $\sigma \left. \frac{\partial u}{\partial r} \right|_{r=R}$ over $\theta \in (0, 2\pi)$ as a solution of an infinitely large system of linear equations:

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ u_0 \end{bmatrix} = \left(\pi \sigma R^{-1} \mathbf{N} + \sum_{i=1}^L Z_i^{-1} \mathbf{M}_i \right)^{-1} \left(\sum_{i=1}^L Z_i^{-1} U_i \mathbf{r}_i \right), \quad (8)$$

where $\mathbf{a}$ and $\mathbf{b}$ are the sought infinite dimension vectors of Fourier coefficients, $\mathbf{n} = (1, 2, 3, \dots)$, and the infinite dimension matrices and vectors defined as

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{M}_{11i} & \mathbf{M}_{12i} & \mathbf{r}_{1i} \\ \mathbf{M}_{12i}^T & \mathbf{M}_{22i} & \mathbf{r}_{2i} \\ \mathbf{r}_{1i}^T & \mathbf{r}_{2i}^T & 2w \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{r}_i = \begin{bmatrix} \mathbf{r}_{1i} \\ \mathbf{r}_{2i} \\ 2w \end{bmatrix}, \quad \begin{matrix} \mathbf{r}_{1i} = 2\mathbf{n}^{-1} \sin(w\mathbf{n}) \cos(\theta_i \mathbf{n}) \\ \mathbf{r}_{2i} = 2\mathbf{n}^{-1} \sin(w\mathbf{n}) \sin(\theta_i \mathbf{n}) \end{matrix}.$$

For a special case, when the contact impedance is the same, $Z_i = Z$ and electrodes are equidistant ($\theta_i = 2\pi i/L$), we obtain $u_0 = 0$ and

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \left(\pi \sigma Z R^{-1} \begin{bmatrix} \mathbf{n} & \mathbf{0} \\ \mathbf{0} & \mathbf{n} \end{bmatrix} + \sum_{i=1}^L \begin{bmatrix} \mathbf{M}_{11i} & \mathbf{M}_{12i} \\ \mathbf{M}_{12i}^T & \mathbf{M}_{22i} \end{bmatrix} \right)^{-1} \left(\sum_{i=1}^L U_i \begin{bmatrix} \mathbf{r}_{1i} \\ \mathbf{r}_{2i} \end{bmatrix} \right). \quad (9)$$

Obviously, when it comes to the computation, infinite dimensions should be replaced with finite dimensions for approximation. A good practice is to carry out computations with

several finite dimensions and see whether they produce close results. In our computations $n = 1, 2, \dots, N = 700$ was more than sufficient, see below.

In Figure 1, we illustrate computation of coefficients (8) and the resulting voltages and currents on the perimeter of a unit radius disk with conductivity, $\sigma = 0.2$ and four equidistant electrodes with the half-width $w = 0.25$ rad (electrodes cover 32% of the disk perimeter). Values of the supplied voltage and contact impedance are shown at each electrode. In this example, the contact impedance, Z_i , varies from electrode to electrode, $i = 1, 2, 3, 4$. The infinite system of linear equations was replaced with finite $N = 700$; the $1,401 \times 1,401$ matrix in (8) was inverted. We show the left-hand side of the BC (7) to see how satisfactory our solution is (the left- and the right-hand sides should coincide). Due to the truncation (N is finite), the current density slightly deviates from 0 between electrodes. Especially visible is the oscillation near the edges of electrode #3. Mathematically, this oscillation can be explained by the fact that function $u(R, \theta)$ has singularities at $\theta_i \pm w$ and particularly has an infinite derivative. As can be seen from Figure 1, the resulting voltages are close to the supplied voltages on all electrodes except #1 due to the high contact impedance, $Z = 1$.

The resulting current density, obtained from the coefficients computed by (8), is close to zero between electrodes and has a U-shape within the electrodes, as was also demonstrated by previous authors Wiley and Webster (1985a), Wiley and Webster (1985b), Paulson *et al* (1992), and Somersalo *et al* (1992). It is worthwhile to mention that those solutions, including the solution presented in Somersalo *et al* (1992), served mostly as an illustration of singularity of the current density at the ends of the electrodes. In particular, the solution in Somersalo *et al* (1992) was derived for a very special case assuming unit radius, equal contact impedance, sine or cosine (symmetric) electric potential on the radius. The solution presented in equation (8) is free of these limitations.

As was mentioned above, at the ends of the electrodes, the current density have singularities: it goes to positive or negative infinity, however, the integral under the current density is finite. We hypothesize that those singularities may complicate the finite-element implementation, not for potential (u) approximation but its derivative/current density which is a part of BC, see Section 4 for details.

In Figure 2, we depict two solutions with $N = 50$ and $N = 500$ to see how the number of terms in Fourier series affects the accuracy of the approximation; the same specification is used as in the previous graph. As can be seen from this graph, even a small number of terms in Fourier series satisfactory approximates the voltage solution, u ($N = 50$ and $N = 500$ yield close functions). However, as typical for Fourier series, oscillation is visible for truncated series. Especially poor is the approximation between electrodes where the density must be zero. Remarkably, even a small number of terms leads to a fair voltage approximation—the left-hand side (LHS) of equation (7) practically coincides with the supplied voltage, U_i .

3.1. Check with the Pidcock solution

As was mentioned in the Introduction, a closed-form solution of the homogeneous 2D disk EIT has been previously derived in Pidcock *et al* (1995). Specifically, the potential on the circle $r = R$ with two electrodes of finite width under the shunt electrode model was expressed in terms of an elliptical integral (section 4.2.2 case E3 in Pidcock *et al* (1995)). We use this solution with two electrodes symmetrically located at radius $R = 1$ with centers at θ_1 and $\theta_2 = \theta_1 + \pi$ and half width w (it is assumed that electrodes do not overlap, namely, $\theta_2 + w < 2\pi$ and $\theta_1 + w < \theta_2 - w$) to validate our analytic solution.

Let the potential on the first electrode be U and on the opposite electrode be $-U$, the Pidcock solution (in our notation) is expressed as

$$u_p(R, \theta) = \begin{cases} -\left(\frac{2F(\zeta_1(\theta))}{F(\pi/2)} - 1\right) U, & \text{if } \theta_1 + w \leq \theta \leq \theta_2 - w \\ \left(\frac{2F(\zeta_2(\theta))}{F(\pi/2)} - 1\right) U, & \text{elsewhere on } (0, 2\pi) \end{cases} \quad (10)$$

where

$$a = x(\theta_1 + w), \quad b = x(\theta_1 - w), \quad c = x(\theta_2 + w), \quad d = x(\theta_2 - w),$$

and

$$\begin{aligned} x(\theta) &= \frac{\cos(\theta - \theta_1)}{1 - \sin(\theta - \theta_1)}, \quad \zeta_1(\theta) = \sin^{-1} \sqrt{\frac{(b-d)(x(\theta)-a)}{(a-d)(x(\theta)-b)}}, \\ \zeta_2(\theta) &= \sin^{-1} \sqrt{\frac{(b-d)(x(\theta)-c)}{(b-c)(x(\theta)-d)}}, \quad F(z) = \int_0^z \frac{dt}{\sqrt{1 - \sin^2 t}}. \end{aligned}$$

The above integral is called the elliptic integral. We compare solution $u_p(R, \theta)$ with our Fourier series solution (6) with coefficients computed by (9) with $N = 500$, $u_0 = 0$. A mathematical proof of the identity of the two solutions is beyond this paper, the two functions are evaluated numerically and depicted in Figure 3. The solutions are practically the same.

3.2. Equidistant-equal-Z approximation

If an even number of electrodes (L) are equidistant (equally spaced) at $\theta_i = 2\pi i/L$, most elements of the matrices $\mathbf{M}$ turn zero because

$$\begin{aligned} \sum_{i=1}^L \cos\left(\frac{2\pi i}{L} 2m\right) &= \begin{cases} L, & \text{if } m = L/2, 3L/2, 2L, \dots \\ 0, & \text{otherwise} \end{cases} \\ \sum_{i=1}^L \sin\left(\frac{2\pi i}{L} 2m\right) &= 0 \text{ for all } m = 1, 2, 3, \dots \end{aligned} \quad (11)$$

Moreover, if $Z_i = Z$ we have $u_0 = 0$ and matrices $\mathbf{M}_{12}$ and $\mathbf{M}_{21}$ vanish because $\sum_{i=1}^L \mathbf{M}_{12i} = \mathbf{0}$. This means that coefficients $\mathbf{a}$ and $\mathbf{b}$ can be found separately through the matrix inversion of

$\mathbf{M}_{11} = \sum_{i=1}^L \mathbf{M}_{11i}$ and $\mathbf{M}_{22} = \sum_{i=1}^L \mathbf{M}_{22i}$. Using formulas (11), we can find an approximation to that solution setting the off-diagonal elements of matrices $\mathbf{M}_{11}$ and $\mathbf{M}_{22}$ to zero. As follows from (11), the n th diagonal elements of matrices $\mathbf{M}_{11}$ and $\mathbf{M}_{22}$ are $L(w + h_n)$ and $L(w - h_n)$ respectively, where

$$h_n = \begin{cases} \frac{\sin(2nw)}{2n}, & \text{if } n = L/2, L, 3L/2, 2L, \dots \\ 0, & \text{elsewhere.} \end{cases} \quad (12)$$

This leads to the following equidistant-equal-Z approximation:

$$a_n = \frac{2 \sin(nw)}{n(n\pi\sigma Z/R + L(w + h_n))} \sum_{i=1}^L U_i \cos\left(\frac{2\pi n}{L} i\right), \quad (13)$$

$$b_n = \frac{2 \sin(nw)}{n(n\pi\sigma Z/R + L(w - h_n))} \sum_{i=1}^L U_i \sin\left(\frac{2\pi n}{L} i\right). \quad (14)$$

These approximations are compared with the exact expression (9) in Figure 4 for $L = 4$, $R = 0.1$, $\sigma = 0.6$, and $Z = 0.025$. We found that larger values of $\sigma Z/R$ yield better approximations.

3.3. Dirichlet-to-Neumann map

In this section, we use solution (8) to derive the DtN map when voltages are supplied and the resulting currents are measured at L electrodes, as in voltage-to-current EIT hardware system.

The current flowing through the j th electrode computed as the integral of the current density is

$$J(\theta) = \sigma \frac{\partial u(r, \theta)}{\partial r} \Big|_{r=R},$$

over the electrode. Namely,

$$I_j = R \int_{\theta_j - w}^{\theta_j + w} J(\theta) d\theta = \sigma R \int_{\theta_j - w}^{\theta_j + w} \frac{\partial u(r, \theta)}{\partial r} \Big|_{r=R} d\theta.$$

Using general solution (6) and elementary trigonometry the current flowing through electrode j is expressed through Fourier coefficients as

$$I_j = \sigma \sum_{n=1}^{\infty} \sin(wn) [a_n \cos(n\theta_j) + b_n \sin(n\theta_j)].$$

Thus we arrive at the DtN map, or Generalized Ohm's law of the 2D disk, in the vector form as

$$\mathbf{I} = \sigma \mathbf{C} \mathbf{U}, \quad (15)$$

where $\mathbf{I} = (I_1, I_2, \dots, I_L)'$ and $\mathbf{U} = (U_1, U_2, \dots, U_L)'$ are columns vector of current and voltage measurements, and the elements of the $L \times L$ matrix $\mathbf{C}$ are given by

$$C_{ij} = \sum_{n=1}^{\infty} \sin(wn) [p_{1in} \cos(\theta_j n) + p_{2in} \sin(\theta_j n)] Z_i^{-1},$$

where p_{1in} and p_{2in} are defined in the Appendix. Numerical issues and statistical testing with simultaneous estimation of $L + 1$ parameters $\sigma, Z_1, \dots, Z_L$ using (15) are discussed in Section 4.

When electrodes are equidistant and have the same contact impedance, we can use simplified formulas for Fourier coefficients (13) and (14) that lead to an approximate DtN matrix with elements

$$\mathbf{C} \approx 2 \sum_{n=1}^{\infty} \mathbf{T}_n \frac{\sin^2(nw)}{n}, \quad (16)$$

where $\mathbf{T}_n$ is the $L \times L$ matrix with the (i, j) th element defined as

$$\frac{\cos\left(\frac{2\pi i}{L}n\right) \cos\left(\frac{2\pi j}{L}n\right)}{n\pi\sigma Z/R+L(w+h_n)} + \frac{\sin\left(\frac{2\pi i}{L}n\right) \sin\left(\frac{2\pi j}{L}n\right)}{n\pi\sigma Z/R+L(w-h_n)}.$$

This approximation may be used for a quick simultaneous assessment of the background conductivity and constant contact impedance. It is easy to show that matrix $\mathbf{C}$ is a Magic Toeplitz matrix (symmetric circulant), Gray (2006), Demidenko (2006).

3.4. Computation of the voltage drop due to contact impedance

A nonzero contact impedance results in a voltage drop at the electrode. We assess the average drop by integrating the difference $U_i - u(R, \theta)$ over the electrode $(\theta_i, -w, \theta_i + w)$ using equation (7) as

$$\Delta u_i = \frac{1}{2w} \int_{\theta_i-w}^{\theta_i+w} Z_i \sigma \frac{\partial u}{\partial r} \Big|_{r=R} d\theta = \frac{Z_i \sigma}{wR} \sum_{n=1}^{\infty} \sin(nw) [a_n \cos(n\theta_i) + b_n \sin(n\theta_i)],$$

where Fourier coefficients a_n and b_n are defined by equation (8). Besides Z_i , one can report $100 |\Delta u_i / U_i|$ as the percent voltage change due to contact impedance, this option is illustrated in Figure 2.

4. Phantom experiments

In this section, we describe how our analytic solution has been validated through laboratory phantom experiments. A cylindrical tank of inner diameter $D = 8.6$ cm and 16 electrodes in each plane (see Figure 5) was filled with saline of approximate conductivity $0.1 (\Omega\text{m})^{-1}$. The round electrodes are 1 cm in diameter, so that $w = 1/D = 0.116$ radians. The goal was to simultaneously estimate the conductivity of the medium along with $L = 16$ contact impedances for 20 frequencies between 10 kHz and 10 MHz. In our experiment, only 16 electrodes at the bottom of the tank were used with the centers at 1.8 cm height; homogeneous saline was filled to a height of 3.6 cm.

Seventeen parameters, $\sigma, Z_1, \dots, Z_{16}$ are estimated using the nonlinear least squares based on the fit of 15 vectors of currents $\mathbf{I}_1, \dots, \mathbf{I}_{15}$ as the result of supplied 15 full trigonometric patterns of voltages $\mathbf{V}_1, \dots, \mathbf{V}_{15}$ (Isaacson 1986)

$$\sum_{i=1}^{15} \|\mathbf{I}_i - \sigma \mathbf{C}(\sigma, Z_1, \dots, Z_{16}) \mathbf{U}_i\|^2,$$

where the DtN map $\mathbf{C} = \mathbf{C}(\sigma, Z_1, \dots, Z_{16})$ is defined by equation (15).

The 3D finite element (FEM) approximation with the number of elements/volumes 10^6 failed to simultaneously estimate conductivity and the 16 contact impedances (the details of FEM implementation can be found in Borsic *et al.* 2008). In Figure 6, we depict the final iterations of the nonlinear least squares problem with 17 parameters depending on the choice

of the starting values for two frequencies. Triangle depicts final Z_s when iterations started from a constant Z for all electrodes and circle depicts final Z_s when iterations started from a neighborhood of constant Z (Ωm^2 , the values above 0.05 shown as flat segments, such as at electrodes 8, 9, 10 in the left plot). Clearly, estimation of contact impedances based on the 3D FEM approximation is unstable. We tried both the finite difference and the exact Jacobian computation for Z ; in the former case, there was no convergence; in the latter case, convergence was unstable. This suggests that previously reported simultaneous reconstruction of electrode impedances and conductivity based on the FEM approximation (Vilhunen *et al* 2002, Heikkinen *et al* 2002) may be prone to instability.

For estimation of contact impedances based on our analytic solution, we started with a calibration of the 2D parameters using the 3D FEM approximation (Borsic *et al* 2008), assuming that $Z_i = Z$ (the two-parameter nonlinear sum of squares problem). While 2D and 3D conductivities are proportionally dependent, we found that the contact impedances are linearly dependent on the log scale. Of course, this dependence is problem specific. For example, one should adjust *in vitro* 2D to 3D dependence when the EIT measurements are done *in vivo*.

The result of the nonlinear least squares estimation of 16 electrode impedances, after the 2D to 3D transformation, are seen in Figure 7. The Z -values slightly change especially for small and large frequencies. The percent voltage drop for each electrode is computed as the median $100 |\Delta u_i / U_i|$ over 15 full trigonometric voltage patterns, $i = 1, 2, \dots, L$. This plot tells how the presence of electrode impedance changes the supplied voltage. Unlike the FEM approximation, the results of the analytic reconstruction had no numerical problems with nonlinear least squares—different starting points lead to the same final iteration values for σ and the Z_s .

4.1. Statistical hypothesis testing

With almost any problem of contact impedance estimation, there are two important questions or hypotheses to test:

1. Equal- Z hypothesis: Are the contact impedances the same? In statistical terms, this is equivalent to testing the null hypothesis $H_0 : Z_i = Z$ for $i = 1, 2, \dots, L$.
2. Zero- Z hypothesis: Does the complete electrode model reduce to the shunt model? This is equivalent to the null hypothesis $H_0 : Z_i = 0$ for all $i = 1, 2, \dots, L$.

It is important to understand that, while Z_i differ from each other by visual examination as in Figure 7, this may not contradict the equal- Z or even zero- Z hypotheses because of the presence of noise.

The likelihood ratio test is routinely used in the framework of nonlinear regression. For hypothesis #1, let the residual sum of squares (sum of squared residuals) with equal and not equal contact impedances be S_2 and S_{17} , respectively (the subindex indicates the number of estimated parameters in the nonlinear least squares). Then according to the asymptotic theory of the likelihood ratio test (Seber and Wild 1989),

$$(K \times L - 17) \ln \frac{S_2}{S_{17}} \approx \chi^2(17 - 2), \quad (17)$$

where $K \times L = 240$ is the total number of measurements ($K = 15$ trigonometric patterns on $L = 16$ electrodes) and the right-hand side denotes the chi-square distribution with 15 degrees of freedom (d.f.). Equation (17) means that the quantity at left has an approximate chi-square distribution under the null hypothesis. The value at left is larger than the critical

value of the chi-distribution with 15 d.f. and the probability of 95% indicates that the null hypothesis should be rejected, otherwise it cannot be rejected with error 5%.

For our saline tank example, S_2 and S_{17} are about 2×10^{-7} and 1×10^{-7} , respectively, which gives the left-hand side value about 140 which is much larger than the critical value of the chi-distribution of about 25. Thus, we conclude that the hypothesis that contact impedances are the same should be rejected.

To test the zero-Z hypothesis, we run the nonlinear least squares with one parameter σ setting Z to a very small number. It is not surprising that this hypothesis is rejected as well because it is more stringent. The complete electrode model is statistically better than the shunt electrode model.

4.2. Bad contact experiments

One of the major applications of our 2D analytic solution is reconstruction of electrode contact impedances, especially important for application to breast cancer detection. In order to understand whether our approach can identify bad contacts, we have conducted the following tank experiment. As previously, the tank shown in Figure 5 was filled with saline but now electrodes 9 and 14 were insulated with electrical tape. The 2D analytic solution was applied to reconstruct 16 contact impedances. The results of the estimation by nonlinear least squares with 17 parameters are presented graphically in Figure 8. Electrodes 9 and 14 have practically infinite contact impedance, no current flows through these electrodes.

5. Patient breast Z reconstruction

In this section, we demonstrate the ability of the analytic solution to estimate electrode contact impedance for *in vivo* breast EIT measurements. The woman is in the lying position with 16 electrodes attached on the periphery of the breast under examination (Hartov *et al* 2001, Kerner *et al* 2002, Hartov *et al* 2005). We illustrate the electrode positions in Figure 9 using the mammogram image of the left breast of the normal patient. The breast is fairly homogeneous and was classified by the radiologist as 'fatty'. There are up to four levels of electrodes of diameter 1 cm located in the same plane level. Only electrodes in the same plane are engaged in each EIT measurement set—this hardware design somewhat justifies our 2D approach. The first level is approximately 4 cm from the chest wall, the second level is at 7.25 cm, the third level is at 10.50 cm, etc.

The results of the contact impedance reconstruction for three levels are presented in Figure 10. Clearly, there is a contact problem with electrodes 6 and 7 for measurements in plane level #1. Interestingly, the same electrodes have larger contact impedance in levels #2 and #3, but not as profoundly so. It is plausible that the electrodes in that area were not pressed firmly enough against the breast.

6. Summary and discussion

Estimation of surface-contact impedances is a burning issue in clinical EIT applications. Especially important is the problem of bad electrode contact which may affect every single EIT exam; moreover it may occur during the exam due to patient movement. It is difficult if not impossible to estimate surface impedances simultaneously with the traditional image reconstruction because the latter is an ill-conditioned problem by itself.

We have derived an analytic solution to the 2D homogeneous disk Laplace PDE under complete electrode model and have shown its validity for estimating contact impedances at each electrode in timely fashion. Application of the 2D solution is justified for EIT hardware

with a planar electrode location, as in the Dartmouth's EIT design. The 2D analytic approach has its limitation because some sort of calibrations of 2D to 3D contact impedance is required. However, when it comes to bad contact identification, the conversion from 2D to 3D is less of an issue because bad contact impedances are considered on the relative scale with respect to other electrodes. Our analytic solution does not replace 3D FEM approximation but compliments it with fast electrode contact impedance estimates needed for full 3D EIT image reconstruction.

Our approach has been validated in phantom experiments and applied to *in vivo* breast EIT data. The discovered numerical instability of the contact impedance estimation based on the 3D finite element approximation, previously suggested by other authors, underscores the value of our approach. The analytic solution may be especially useful for real-time surface-contact diagnostics in clinical settings.

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Appendix. Derivation of the solution for the 2D EIT problem

Our goal is to find the Fourier coefficients a_n, b_n from (7). First, using (7), we express the derivative at the boundary as

$$\sigma \frac{\partial u}{\partial r} \Big|_{r=R} = \begin{cases} U_i/Z_i - u(R, \theta)/Z_i, & \text{if } |\theta - \theta_i| < w \\ 0, & \text{elsewhere.} \end{cases} \quad (18)$$

Second, we multiply (18) by $\cos(k\theta)$ with $k = 1, 2, \dots$ and integrate both sides over $(0, 2\pi)$ noting that

$$\frac{\partial u}{\partial r} \Big|_{r=R} = \frac{1}{R} \sum_{n=1}^{\infty} n(a_n \cos(n\theta) + b_n \sin(n\theta)).$$

It is easy to see that the integral of the left-hand side of (18) times $\cos(k\theta)$, as the integral of the above Fourier series, is $\sigma \pi n a_n / R$ due to orthogonality of cosine and sine ($k = n$). The right-hand side of (18) times $\cos(k\theta)$ takes the form

$$\sum_{i=1}^L U_i / Z_i \int_{\theta_i-w}^{\theta_i+w} \cos(k\theta) d\theta - \sum_{i=1}^L \sum_{n=1}^{\infty} \left[a_n \int_{\theta_i-w}^{\theta_i+w} C_{kn1}(\theta) d\theta + b_n \int_{\theta_i-w}^{\theta_i+w} C_{kn2}(\theta) d\theta \right] / Z_i,$$

where

$$C_{kn1}(\theta) = \cos(k\theta) \cos(n\theta), \quad C_{kn2}(\theta) = \cos(k\theta) \sin(n\theta).$$

From trigonometry, $\int_{\theta_i-w}^{\theta_i+w} \cos(k\theta) d\theta = \frac{2}{k} \sin(wk) \cos(\theta_i k)$, and after some tedious algebra we arrive at an infinite system of linear equations for a_n, b_n with the (k, n) th elements of the infinite dimension Fourier matrices $\mathbf{M}_{11i}$ and $\mathbf{M}_{12i}$ for $i = 1, 2, \dots, L$, given by

$$\begin{aligned}\mathbf{M}_{11ikn} &= \int_{\theta_i-w}^{\theta_i+w} \cos(k\theta) \cos(n\theta) d\theta = \begin{cases} w + \frac{1}{2k} \cos(2k\theta_i) \sin(2kw), & \text{if } k=n, \\ \frac{c_{1kn}}{n-k} + \frac{s_{1kn}}{n+k}, & \text{if } k \neq n, \end{cases} \\ \mathbf{M}_{12ikn} &= \int_{\theta_i-w}^{\theta_i+w} \cos(k\theta) \sin(n\theta) d\theta = \begin{cases} \frac{1}{2k} \sin(2k\theta_i) \sin(2kw), & \text{if } k=n \\ \frac{c_{2kn}}{n+k} + \frac{s_{2kn}}{n-k}, & \text{if } k \neq n, \end{cases}\end{aligned}$$

where

$$\begin{aligned}c_{1kn} &= \cos((n-k)\theta_i) \sin((n-k)w), & s_{1kn} &= \cos((n+k)\theta_i) \sin((n+k)w), \\ c_{2kn} &= \sin((n+k)\theta_i) \sin((n+k)w), & s_{2kn} &= \sin((n-k)\theta_i) \sin((n-k)w).\end{aligned}$$

Third, we multiply both sides of (18) by $\sin(k\theta)$ and similarly integrate over $(0, 2\pi)$ using the

fact that $\int_{\theta_i-w}^{\theta_i+w} \sin(k\theta) d\theta = \frac{2}{k} \sin(wk) \sin(\theta_i k)$. The coefficients at a_n, b_n are

$$\mathbf{M}_{21ikn} = \mathbf{M}_{12ink}, \mathbf{M}_{22ikn} = \begin{cases} w - \frac{1}{2k} \cos(2k\theta_i) \sin(2kw), & \text{if } n=k \\ \frac{c_{1kn}}{n-k} - \frac{s_{1kn}}{n+k}, & \text{if } n \neq k \end{cases}.$$

Denote by $\mathbf{a}$ and $\mathbf{b}$ the infinite dimension vectors with components a_n and b_n respectively.

Also, let $\mathbf{r}_{1i}$ and $\mathbf{r}_{2i}$ be the infinite dimension vectors with components

$$r_{1in} = \frac{2}{n} \sin(wn) \cos(\theta_i n), r_{2in} = \frac{2}{n} \sin(wn) \sin(\theta_i n).$$

We express terms $(\pi\sigma/R)n$ at coefficients a_n and b_n as a diagonal matrix $(\pi\sigma R)\mathbf{n}$, where $\mathbf{n} = \text{diag}(1, 2, 3, \dots)$. To derive an equation containing u_0 we integrate both sides of (18) over $(0, 2\pi)$ which yields equation

$$\mathbf{a}' \sum_{i=1}^L Z_i^{-1} \mathbf{r}_{1i} + \mathbf{b}' \sum_{i=1}^L Z_i^{-1} \mathbf{r}_{2i} + 2wu_0 \sum_{i=1}^L 1/Z_i = 2w \sum_{i=1}^L U_i/Z_i,$$

so that the complete system of equations to be solved for $\mathbf{a}, \mathbf{b}$ and u_0 is as follows:

$$\begin{aligned}\left(\sum_{i=1}^L Z_i^{-1} \mathbf{M}_{11i} \right) \mathbf{a} + \left(\sum_{i=1}^L Z_i^{-1} \mathbf{M}_{12i} \right) \mathbf{b} + (\pi\sigma/R) \mathbf{n} \mathbf{a} + \mathbf{u}_0 \sum_{i=1}^L Z_i^{-1} \mathbf{r}_{1i} &= \sum_{i=1}^L Z_i^{-1} \mathbf{U}_i \mathbf{r}_{1i} \\ \left(\sum_{i=1}^L Z_i^{-1} \mathbf{M}_{21i} \right) \mathbf{a} + \left(\sum_{i=1}^L Z_i^{-1} \mathbf{M}_{22i} \right) \mathbf{b} + (\pi\sigma/R) \mathbf{n} \mathbf{b} + \mathbf{u}_0 \sum_{i=1}^L Z_i^{-1} \mathbf{r}_{2i} &= \sum_{i=1}^L Z_i^{-1} \mathbf{U}_i \mathbf{r}_{2i} \\ \mathbf{a}' \sum_{i=1}^L Z_i^{-1} \mathbf{r}_{1i} + \mathbf{b}' \sum_{i=1}^L Z_i^{-1} \mathbf{r}_{2i} + 2wu_0 \sum_{i=1}^L Z_i^{-1} &= 2w \sum_{i=1}^L Z_i^{-1} U_i.\end{aligned}$$

We prove that if electrodes are equidistant, $\theta_i = (2\pi i)/L$, have the same contact impedance, $Z_i = Z$, the third equation yields $u_0 = 0$. Indeed, the right-hand side of the third equation turns

zero because the sum of supplied voltages is zero. Thus it suffices to prove that $\sum_{i=1}^L r_{1in} = 0$ and $\sum_{i=1}^L r_{2in} = 0$, which follows from the fact that

$$\sum_{i=1}^L \cos \frac{2\pi i}{L} n = 0 = \sum_{i=1}^L \sin \frac{2\pi i}{L} n = 0$$

for each $n = 1, 2, \dots$

Obviously, the solution of the above system can be expressed as linear combination of measured voltages as

$$a_n = \sum_{i=1}^L U_i Z_i^{-1} p_{1in}, b_n = \sum_{i=1}^L U_i Z_i^{-1} p_{2in}$$

where p_{1in} and p_{2in} are elements of the following vectors

$$\begin{aligned} \mathbf{p}_{1i} &= \mathbf{M}^{11} \mathbf{r}_{1i} + \mathbf{M}^{12} \mathbf{r}_{2i} + 2w \mathbf{M}^{13}, \\ \mathbf{p}_{2i} &= \mathbf{M}^{21} \mathbf{r}_{1i} + \mathbf{M}^{22} \mathbf{r}_{2i} + 2w \mathbf{M}^{23}, \end{aligned}$$

and matrices $\mathbf{M}^{pq}$ are the block-partitioned matrices of the inverse matrix in (8) corresponding to $\mathbf{a}$, $\mathbf{b}$ and u_0 .

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Figure 1.

The 3D view of voltage and current distribution on a 2D homogeneous disk with four electrodes and different electrode impedances. The current density has an U-shape within electrode and is close to zero between electrodes. The applied and resulting voltages are different due to the presence of the nonzero electrode impedances. Due to large number of terms ($N = 700$), the supplied voltage and BC left-hand side are practically coincide at the electrodes and current density vanishes between electrodes.

Figure 2.

Current density and voltage approximation using Fourier series solution (8) for a four-electrode homogeneous disk as shown in Figure 1 with small ($N = 50$) and large ($N = 500$) number of terms. Even a fairly small number of terms gives a satisfactory approximation to the voltage solution, u . Percent voltage drop due to contact impedance is shown at each electrode, see Section 3.4 for detail.

Figure 3. Comparison of the Pidcock et al. (1995) solution (10) with our Fourier series solution of potential distribution on 2D homogeneous disk with two symmetrical electrodes located at $\theta_1 = \pi/2$ and $\theta_2 = 3/2\pi$ of width $\pi/4$ (the electrodes depicted as bold segments at zero voltage, $N = 500$). The supplied voltages at the first and second electrodes are $U_1 = 0.5$ and $U_2 = -0.5$. The two voltage function values are the same.

Figure 4.

Comparison of exact Fourier coefficients a_n and b_n computed by formula (9) using the first $N = 500$ terms with their approximations for equal contact impedance and equidistant electrodes computed by formulas (13) and (14).

Cylinder tank for phantom experiments

FEM mesh

Figure 5.

Left: A cylinder tank with 16 electrodes in each plane used in homogeneous tank experiments (shown empty). The diameter of the tank is 8.6 cm. Saline of conductivity $0.1 (\Omega\text{m})^{-1}$ was filled at the height 3.6 from the bottom. The voltages are applied at polymer conductive buffer cylinders connected to round electrodes of diameter 1 cm at the bottom level, with the center 1.8 cm from the bottom. Right: The FEM mesh with 10^6 elements to approximate the 3D EIT PDE solution.

Figure 6. Estimation of 16 Z_s (Ωm^2) using the 3D FEM approximation for the homogeneous EIT cylinder problem. Triangle and circle show the final iterations of the nonlinear least squares starting from constant and random Z as initial values. The bar on the top indicates that the values are above 0.05.

Figure 7. Estimation of 16 contact impedances (Ωm^2) and respective voltage drop for 20 frequencies using the analytical solution. Overall the effect of contact impedances is small although there are some differences across electrodes.

Figure 8.

The electrodes #9 and #14 were insulated with electrical tape. Our analytic solution detects these electrodes—the contact impedance is practically infinite compared with the other electrodes on almost the entire range of current frequencies.

Figure 9.

Dartmouth breast EIT hardware with three levels of electrodes. There are 16 equidistant electrode in each level. A real mammogram is used for this illustration.

Figure 10.

Reconstruction of breast electrode contact impedances (Ωm^2) in three plane levels shown in Figure 9. Bad contacts at electrodes 6 and 7 in plane level #1 are identified. The values on the top of the y-axis are truncated.