Gel'fand–Graev's reconstruction formula in the 3D real space

Yangbo Ye^{a)}

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242 and School of Mathematics, Shandong University, Jinan, Shandong 250100, China

Hengyong Yu^{b)}

Department of Radiology, Division of Radiologic Sciences, Wake Forest University Health Sciences, Winston-Salem, North Carolina 27157 and VT-WFU School of Biomedical Engineering and Sciences, Wake Forest University Health Sciences, Winston-Salem, North Carolina 27157

Ge Wang^{c)}

Biomedical Imaging Division, VT-WFU School of Biomedical Engineering and Sciences, Virginia Tech, Blacksburg, Virginia 24061 and Biomedical Imaging Division, VT-WFU School of Biomedical Engineering and Sciences, Wake Forest University Health Sciences, Winston-Salem, North Carolina 27157

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Purpose: Gel'fand and Graev performed classical work on the inversion of integral transforms in different spaces [Gel'fand and Graev, *Funct. Anal. Appl.* 25(1) 1–5 (1991)]. This paper discusses their key results for further research and development.

Methods: The Gel'fand–Graev inversion formula reveals a fundamental relationship between projection data and the Hilbert transform of an image to be reconstructed. This differential backprojection (DBP)/backprojection filtration (BPF) approach was rediscovered in the CT field, and applied in important applications such as reconstruction from truncated projections, interior tomography, and limited-angle tomography. Here the authors present the Gel'fand–Graev inversion formula in a 3D setting assuming the 1D x-ray transform.

Results: The pseudodifferential operator is a powerful theoretical tool. There is a fundamental mathematical link between the Gel'fand–Graev formula and the DBP (or BPF) approach in the case of the 1D x-ray transform in a 3D real space.

Conclusions: This paper shows the power of mathematics for tomographic imaging and the value of a pure theoretical finding, which may appear quite irrelevant to daily healthcare at the first glance. © 2011 American Association of Physicists in Medicine. [DOI: 10.1118/1.3577765]

Key words: x-ray transform, computed tomography (CT), image reconstruction, pseudo-differential operator, differential backprojection (DBP), backprojection filtration (BPF)

I. INTRODUCTION

In a series of papers^{1–3} and a classical book,⁴ Gel'fand, Graev, and others proved inversion formulas for integral transforms of various dimensions in different spaces. In particular, Gel'fand–Graev's inversion formula reveals a fundamental relationship, which expresses the Hilbert transform along a chord in the field of view in terms of projection data. This relationship was rediscovered in recent years; see Refs. 5–10. It has wide applications in the CT field, including reconstruction from truncated projections,^{11,12} differential backprojection (DBP)/backprojection filtration (BPF),¹³ interior tomography,^{14–18} and limited-angle tomography.¹⁹ For a survey of these results, please see Refs. 20–22. In this paper, we will focus on Ref. 1 and present their inversion formula and proof for the 1D x-ray transform in a 3D real space in an engineer-friendly language.

II. NOTATION

We follow closely the notations in Ref. 1 and consider the 1D x-ray transform in a 3D real space $\mathbb{R}^3$. Set $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 - \{O\}$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$. Denote by $H_{1,3}$ the set of all straight lines in $\mathbb{R}^3$. Then, any

$(\alpha, \beta) \in (\mathbb{R}^3 - \{O\}) \times \mathbb{R}^3$ defines a line $h \in H_{1,3}$ by $\alpha t + \beta, t \in \mathbb{R}$ (Fig. 1). Two $(\alpha, \beta), (\alpha', \beta') \in (\mathbb{R}^3 - \{O\}) \times \mathbb{R}^3$ define the same h if and only if

$$\alpha' = \alpha\lambda, \beta' = \beta + \alpha t_0 \text{ for some } \lambda \in \mathbb{R} - \{O\}, t_0 \in \mathbb{R}. \quad (1)$$

Let f be a compactly supported function in $\mathbb{R}^3$. The 1D x-ray projection is an integral transform given by

$$\mathcal{R}f(\alpha, \beta) = \int_{\mathbb{R}} f(\alpha t + \beta) dt. \quad (2)$$

Denote $\varphi(\alpha, \beta) = \mathcal{R}f(\alpha, \beta)$ and define a differential $\varkappa\varphi$ on $(\mathbb{R}^3 - \{O\}) \times \mathbb{R}^3$ by

$$\varkappa\varphi = \sum_{j=1}^3 \frac{\partial \varphi(\alpha, \beta)}{\partial \beta_j} d\alpha_j. \quad (3)$$

By Eq. (1), we have

$$\begin{aligned} \mathcal{R}f(\alpha', \beta') &= \int_{\mathbb{R}} f(\alpha' t + \beta') dt = \int_{\mathbb{R}} f(\alpha \lambda t + \beta + \alpha t_0) dt \\ &= \int_{\mathbb{R}} f(\alpha(\lambda t + t_0) + \beta) dt = \frac{1}{\lambda} \int_{\mathbb{R}} f(\alpha t + \beta) dt \\ &= \frac{1}{\lambda} \mathcal{R}f(\alpha, \beta) \end{aligned} \quad (4)$$

FIG. 1. Representation of a line with two vectors.

and

$$\begin{aligned} \sum_{j=1}^3 \frac{\partial \varphi(\alpha', \beta')}{\partial \beta'_j} d\alpha'_j &= \frac{1}{\lambda} \sum_{j=1}^3 \frac{\partial \varphi(\alpha, \beta)}{\partial \beta'_j} d\alpha'_j \\ &= \sum_{j=1}^3 \frac{\partial \varphi(\alpha, \beta)}{\partial \beta_j} d\alpha_j, \end{aligned} \quad (5)$$

i.e., the differential $\kappa\varphi$ depends on h only, being independent of the choice of (α, β) .

Let H_0 be the set of oriented 1D lines in $\mathbb{R}^3$ through the origin. Since an oriented line has a direction, H_0 can be viewed as the unit sphere S^2 in $\mathbb{R}^3$, and $\alpha \in S^2$ defines the oriented line $\alpha t \in H_0, t \in \mathbb{R}$. Let us fix a 1D oriented curve γ . As shown in Fig. 2, γ can be represented as an oriented curve $C_\gamma \subset S^2$. For any vector $x \in \mathbb{R}^3$, denoted by $x + \gamma$, the set of oriented lines in γ shifted to x . Then, $x + \gamma$ can be expressed as $\alpha t + x, \alpha \in C_\gamma, t \in \mathbb{R}$. Define an operator $\mathcal{J}_\gamma$ on $\varphi = \mathcal{R}f$ by

$$\begin{aligned} (\mathcal{J}_\gamma \varphi)(x) &= \frac{1}{2\pi i} \int_{x+\gamma} \kappa \varphi \\ &= \frac{1}{2\pi i} \int_{C_\gamma} \sum_{j=1}^3 \frac{\partial \varphi(\alpha, \beta)}{\partial \beta_j} \Big|_{\beta=x} d\alpha_j. \end{aligned} \quad (6)$$

Take $\xi \in \mathbb{R}^3 - \{O\}$. Denoted by G_ξ the set of all lines $h \in H_0$ on the plane $x \cdot \xi = 0$. Then, G_ξ can be represented as a large circle C_ξ , which is the intersection between the unit sphere S^2 and the plane $x \cdot \xi = 0$, oriented by the right-hand rule with ξ pointing to the direction of the thumb.

Let us define the Crofton function $\text{Cr}_\gamma(\xi)$ as the index for the intersections between C_γ and C_ξ . An intersection between C_γ and C_ξ contributes 1 to $\text{Cr}_\gamma(\xi)$ if $v_\gamma \cdot \xi > 0$,

FIG. 2. Representation of oriented lines with a curve, which can be shifted to x .

where v_γ is the tangent vector of C_γ at the intersection (see Fig. 3). An intersection contributes -1 if $v_\gamma \cdot \xi < 0$, and 0 if $v_\gamma \cdot \xi = 0$. Note that $\text{Cr}_\gamma(-\xi) = -\text{Cr}_\gamma(\xi)$, and for a continuous or otherwise properly constructed γ , $\text{Cr}_\gamma(\xi) = 0, \pm 1$.

Based on the notations defined above, in Sec. III we will introduce the pseudo-differential operator defined by Gel'fand and Graev, and show that the composition operator $\mathcal{J}_\gamma \mathcal{R}$ is a pseudo-differential operator whose symbol is the Crofton function. In Sec. IV, we will present the Gel'fand and Graev inverse formula for the integral transform $\varphi = \mathcal{R}f$, which was derived from the properties of the pseudo-differential operator. In Sec. V, we will see that the composition operator $\mathcal{J}_\gamma \mathcal{R}$ in the 3D real space reveals a Hilbert transform relationship between the differentiated backprojection image and the original image, which was first suggested by Gel'fand and Graev. In Sec. VI, we will discuss some related issues.

III. PSEUDO-DIFFERENTIAL OPERATORS

Let $P(D) = \sum_n a_n D^n$ be a linear differential operator with constant coefficients a_n and on smooth functions on $\mathbb{R}^3$, and $P(\xi) = \sum_n a_n \xi^n$ be a polynomial of three variables, where

$$\begin{aligned} n &= (n_1, n_2, n_3), \quad \xi = (\xi_1, \xi_2, \xi_3), \quad \xi^n = \xi_1^{n_1} \xi_2^{n_2} \xi_3^{n_3}, \\ D &= (-i\partial_1, -i\partial_2, -i\partial_3), \quad D^n = (-i\partial_1)^{n_1} (-i\partial_2)^{n_2} (-i\partial_3)^{n_3}. \end{aligned}$$

Define the Fourier transform and its inversion as

$$\begin{aligned} \hat{f}(\xi) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx, \\ f(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{f}(\xi) d\xi. \end{aligned} \quad (7)$$

FIG. 3. Crofton function in different cases.

Then, by the Fourier inversion formula

$$\begin{aligned}
 P(D)u(x) &= P(D) \left(\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{u}(\xi) d\xi \right) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} P(D) e^{ix \cdot \xi} \hat{u}(\xi) d\xi \\
 &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\sum_n a_n (-i)^{n_1+n_2+n_3} \frac{\partial^{n_1+n_2+n_3} e^{ix \cdot \xi}}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} \right) \hat{u}(\xi) d\xi \\
 &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\sum_n a_n (-i)^{n_1+n_2+n_3} (i\xi_1)^{n_1} (i\xi_2)^{n_2} (i\xi_3)^{n_3} e^{ix \cdot \xi} \right) \hat{u}(\xi) d\xi \\
 &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\sum_n a_n \xi_1^{n_1} \xi_2^{n_2} \xi_3^{n_3} \right) e^{ix \cdot \xi} \hat{u}(\xi) d\xi = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} P(\xi) e^{ix \cdot \xi} \hat{u}(\xi) d\xi.
 \end{aligned}$$

This suggests the following definition. A pseudo-differential operator $C(D)$ on $\mathbb{R}^3$ is defined by

$$\begin{aligned}
 C(D)u(x) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(x-y) \cdot \xi} C(\xi) u(y) dy d\xi \\
 &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} C(\xi) \hat{u}(\xi) d\xi
 \end{aligned} \tag{8}$$

for a certain function $C(\xi)$, which is called the symbol of the pseudo differential operator $C(D)$.

Theorem 1. (Gel'fand and Graev¹) *The composition operator $\mathcal{J}_\gamma R$ defined by Eqs. (2) and (6) is a pseudo-differential operator on $\mathbb{R}^3$ with the symbol $\text{Cr}_\gamma(\xi)$.*

Proof. By Eqs. (2) and (6), we have

$$\begin{aligned}
 (\mathcal{J}_\gamma Rf)(x) &= \frac{1}{2\pi i} \int_{C_\gamma} \sum_{j=1}^3 \frac{\partial \varphi(\alpha, \beta)}{\partial \beta_j} \bigg|_{\beta=x} d\alpha_j = \frac{1}{2\pi i} \int_{C_\gamma} \sum_{j=1}^3 \frac{\partial}{\partial \beta_j} \left(\int_{\mathbb{R}} f(\alpha t + \beta) dt \right) \bigg|_{\beta=x} d\alpha_j \\
 &= \frac{1}{2\pi i} \int_{C_\gamma} \sum_{j=1}^3 \frac{\partial}{\partial \beta_j} \left(\int_{\mathbb{R}} \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(\alpha t + \beta) \cdot \xi} \hat{f}(\xi) d\xi dt \right) \bigg|_{\beta=x} d\alpha_j = \frac{1}{(2\pi)^{5/2}} \int_{C_\gamma} \sum_{j=1}^3 \left(\int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{i(\alpha t + x) \cdot \xi} \hat{f}(\xi) \xi_j d\xi dt \right) d\alpha_j \\
 &= \frac{1}{(2\pi)^{5/2}} \int_{C_\gamma} \sum_{j=1}^3 \left(\int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{f}(\xi) \xi_j \int_{\mathbb{R}} e^{itx \cdot \xi} dt d\xi \right) d\alpha_j = \frac{1}{(2\pi)^{3/2}} \int_{C_\gamma} \sum_{j=1}^3 \left(\int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{f}(\xi) \xi_j \delta(\alpha \cdot \xi) d\xi \right) d\alpha_j \\
 &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{f}(\xi) \left(\int_{C_\gamma} \sum_{j=1}^3 \delta(\alpha \cdot \xi) \xi_j d\alpha_j \right) d\xi,
 \end{aligned} \tag{9}$$

where we have used the Fourier inversion formula and the fact that

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{it\alpha \cdot \xi} dt = \delta(\alpha \cdot \xi),$$

with $\delta(\cdot)$ being the Dirac delta function. By Eq. (9), $\mathcal{J}_\gamma R$ is a pseudo-differential operator with the symbol

$$C(\xi) = \int_{C_\gamma} \sum_{j=1}^3 \delta(\alpha \cdot \xi) d\alpha_j = \sum_{j=1}^3 \left(\int_{C_\gamma} \delta(\alpha \cdot \xi) d\alpha_j \right) \xi_j.$$

We want to prove that $C(\xi) = \text{Cr}_\gamma(\xi)$ for $\xi \in \mathbb{R}^3 - \{0\}$. To that effect, we only need to prove $C(\xi) = \text{Cr}_\gamma(\xi)$ for a small local piece of γ , as illustrated in Fig. 3. Let $s, s_1 \leq s \leq s_2$, be a variable. For each oriented line $h(s) \in \gamma$ depending on s smoothly, we have its representation $\alpha(s)t, \alpha(s) \in C_\gamma, t \in \mathbb{R}$. Denote $u(s) = \alpha(s) \cdot \xi$, we have

$$C(\xi) = \int_{s_1}^{s_2} \delta(u(s)) \frac{du}{ds} ds = \int_a^b \delta(u) du,$$

where $a = u(s_1) = \alpha(s_1) \cdot \xi$ and $b = u(s_2) = \alpha(s_2) \cdot \xi$. If C_γ does not intersect C_ξ , $u(s)$ is never zero, and $\int_a^b \delta(u) du = 0$. If C_γ intersects C_ξ once, then $C(\xi) = 1$ if $a < 0 < b$ and $C(\xi) = -1$ if $a > 0 > b$, which is exactly $\text{Cr}_\gamma(\xi)$. If C_γ is tangent to C_ξ at one point, a and b are of the same sign, and $C(\xi) = 0$. Alternatively, we may compute

$$\begin{aligned}
 C(\xi) &= \frac{1}{2\pi} \int_a^b \int_{\mathbb{R}} e^{itu} dt du = \frac{1}{2\pi} \int_{\mathbb{R}} \int_a^b e^{itu} du dt \\
 &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{ibt} - e^{iat}}{t} dt,
 \end{aligned} \tag{10}$$

which is a Froullani integral. Equation (10) equals 0 if a and b are of the same sign, must be 1 if $a < 0 < b$, and be -1 if

FIG. 4. Various quasicycles.

$a > 0 > b$. Consequently, $C(\xi) = \text{Cr}_\gamma(\xi)$, and Theorem 1 follows. ■

IV. INVERSION FORMULA

First, we have a corollary.

Corollary. (Gel'fand and Graev¹) For two oriented curves γ_1 and γ_2 , the symbol of the pseudo-differential operator $(\mathcal{J}_{\gamma_1}\mathcal{R})(\mathcal{J}_{\gamma_2}\mathcal{R})$ is

$$C(\xi) = \text{Cr}_{\gamma_1}(\xi)\text{Cr}_{\gamma_2}(\xi).$$

Proof. This is the composition law of pseudo-differential operators. In fact, by Eq. (8), we have

$$\begin{aligned} C_1(D)C_2(D)u(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} C_1(\xi) (C_2(\widehat{D})u)(\xi) d\xi \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} C_1(\xi) C_2(\xi) \hat{u}(\xi) d\xi. \end{aligned}$$

A curve $\gamma \subset H_0$ is called a quasicycle if $|\text{Cr}_\gamma(\xi)|$ is a constant function for almost every ξ . In particular, γ is a quasicycle if $|\text{Cr}_\gamma(\xi)| = 1$ for almost all $\xi \in \mathbb{R}^3 - \{O\}$. This is the case when C_γ is a smooth curve on S^2 whose end points are diametrically opposite $C_\gamma(s_2) = -C_\gamma(s_1)$. Figure 4 shows

several quasicycles. For a quasicycle γ , we denote the constant $|\text{Cr}_\gamma(\xi)|$ by $c(\gamma)$. Note that in this case $\text{Cr}_\gamma(\xi)$ itself is not a constant in general.

By the Corollary, if γ is a quasicycle, then the symbol of $(\mathcal{J}_\gamma\mathcal{R})^2$ is

$$(\text{Cr}_\gamma(\xi))^2 = |\text{Cr}_\gamma(\xi)|^2 = (c(\gamma))^2,$$

which is a constant. A pseudo-differential operator with a constant symbol $(c(\gamma))^2$ is indeed $(c(\gamma)^2)E$, where E is the identity operator, by the Fourier inversion formula. Therefore, we have proved an inversion formula for the x-ray transform $\mathcal{R}$.

Theorem 2. (Gel'fand and Graev¹) If γ is a quasicycle in H_0 , then $(\mathcal{J}_\gamma\mathcal{R})^2 = (c(\gamma)^2)E$. Thus, for the integral transform $\varphi = \mathcal{R}f$, one has the inversion formula

$$\mathcal{J}_\gamma\mathcal{R}\mathcal{J}_\gamma\varphi = c(\gamma)^2 f. \quad (11)$$

In Sec. V, we will show a major implication of Eq. (11) for CT reconstruction for any curve γ that is a quasicycle.

V. HILBERT TRANSFORM

From Eqs. (9) and (10), we have

$$\begin{aligned} (\mathcal{J}_\gamma\mathcal{R}f)(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{f}(\xi) \frac{1}{2\pi i} \int_{\mathbb{R}} (e^{it(\alpha(s_2) \cdot \xi)} - e^{it(\alpha(s_1) \cdot \xi)}) \frac{dt}{t} d\xi \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (e^{it(\alpha(s_2) \cdot \xi)} - e^{it(\alpha(s_1) \cdot \xi)}) e^{ix \cdot \xi} \hat{f}(\xi) d\xi \frac{dt}{t} \\ &= \frac{1}{2\pi i} PV \int_{\mathbb{R}} (f(\alpha(s_2)t + x) - f(\alpha(s_1)t + x)) \frac{dt}{t}, \end{aligned} \quad (12)$$

where the Fourier inversion formula has been used, and the PV integral takes the Cauchy principal value. Equation (12) shows that $(\mathcal{J}_\gamma\mathcal{R}f)(x)$ is the difference between the Hilbert transforms at the point x along the oriented lines $\alpha(s_2)t + x$ and $\alpha(s_1)t + x$, $t \in \mathbb{R}$. The Hilbert transform relationship Eq. (12) is exactly the only example given in Sec. 8 in Ref. 1. When γ is a closed curve, i.e., when the corresponding C_γ is a closed curve on S^2 with $\alpha(s_2) = \alpha(s_1)$, from Eq. (12) we can see $(\mathcal{J}_\gamma\mathcal{R}f)(x) \equiv 0$. When γ is a quasicycle, i.e., when the corresponding C_γ is a smooth curve on S^2 whose end points are diametrically opposite, Eq. (12) can be simplified as

$$\begin{aligned} (\mathcal{J}_\gamma\mathcal{R}f)(x) &= \frac{1}{2\pi i} PV \int_{\mathbb{R}} (f(\alpha(s_2)t + x) - f(-\alpha(s_2)t + x)) \frac{dt}{t} \\ &= \frac{1}{\pi i} PV \int_{\mathbb{R}} (f(\alpha(s_2)t + x)) \frac{dt}{t}, \end{aligned} \quad (13)$$

which shows a Hilbert transform along the line parallel to $\alpha(s_2)$ and through the point x . In other words, the composition operator $\mathcal{J}_\gamma\mathcal{R}$ is a Hilbert transform, which is the DBP approach. It is well known that the Hilbert transform can be inverted using itself. Then, performing a Hilbert filtering operation on Eq. (13) along the line parallel to $\alpha(s_2)$ and through x , we can exactly reconstruct the image $f(x)$ on that

FIG. 5. Fan- or cone-beam geometry for (a) the Gel'fand and Graev formula and (b) the chord-based reconstruction.

line, which is a good example of the general inversion scheme Eq. (11).

In the CT field, projection data are usually measured in a parallel-, fan-, or cone-beam geometry. When a parallel-beam geometry is assumed, the scanning trajectory can be viewed as C_γ and represented by $\alpha(s)$ on the unit sphere, where s with $s_1 \leq s \leq s_2$ is a variable. For any fixed s , $\mathcal{R}f(\alpha(s), \beta)$ will be measured for all $\beta \in \mathbb{R}^3$. Particularly, $f(x)$ will be measured by $\mathcal{R}f(\alpha(s), x)$ for $s_1 \leq s \leq s_2$. When $\alpha(s_1) = -\alpha(s_2)$, we have Eq. (13). Because $\alpha(s)$ is independent of x , Eq. (13) holds for any x . Therefore, Eq. (11) becomes the BPF reconstruction formula in a parallel-beam geometry.

When a fan- or cone-beam geometry is considered,¹⁰ the scanning trajectory can be represented as a curve $\Lambda(s)$, where s with $s_1 \leq s \leq s_2$ is a variable. For any fixed s , $\mathcal{R}f(\alpha, \Lambda(s))$ will be measured for all $\alpha \in S^1$ or S^2 . Particularly, $f(x)$ will be measured by $\mathcal{R}f(\alpha(s, x), \Lambda(s)) = \mathcal{R}f(\alpha(s, x), x)$ with $\alpha(s, x) = (x - \Lambda(s))/\|x - \Lambda(s)\|$ for $s_1 \leq s \leq s_2$ [see Fig. 5(a)]. In this case, the curve C_γ depends on x and should be denoted as C_{γ_x} as pointed out in Sec. 9 in Ref. 1. Substituting $\alpha(s, x)$ into Eq. (6), we obtain

$$\begin{aligned} (\mathcal{J}_{\gamma_x} \mathcal{R}f)(x) &= \frac{1}{2\pi i} \int_{C_{\gamma_x}} \sum_{j=1}^3 \frac{\partial \varphi(\alpha(s, x), \beta)}{\partial \beta_j} \Big|_{\beta=x} d(\alpha(s, x))_j \\ &= \frac{1}{2\pi i} \int_{(x-\Lambda(s))} \sum_{j=1}^3 \frac{\partial \varphi(x - \Lambda(s), \beta)}{\partial \beta_j} \Big|_{\beta=x} \\ &\quad \times d(x - \Lambda(s))_j, \end{aligned} \quad (14)$$

where we have used the fact proved in Eq. (5) that the differential depends on the line only and is independent of the choice of (α, β) . Thus, by Eq. (4)

$$\begin{aligned} (\mathcal{J}_{\gamma_x} \mathcal{R}f)(x) &= \frac{1}{2\pi i} \int_{s_1}^{s_2} \sum_{j=1}^3 (\Lambda'(s))_j \frac{\partial \varphi(x - \Lambda(s), \beta)}{\partial \beta_j} \Big|_{\beta=x} ds \\ &= \frac{1}{2\pi i} \int_{s_1}^{s_2} \sum_{j=1}^3 \frac{(\Lambda'(s))_j}{\|x - \Lambda(s)\|} \frac{\partial \varphi(\alpha(s, x), \beta)}{\partial \beta_j} \Big|_{\beta=x} ds. \end{aligned} \quad (15)$$

Consequently, we have

$$\begin{aligned} (\mathcal{J}_{\gamma_x} \mathcal{R}f)(x) &= \frac{1}{2\pi i} \int_{s_1}^{s_2} \sum_{j=1}^3 \frac{(\Lambda'(s))_j}{\|x - \Lambda(s)\|} \left(\frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} (e^{i(\alpha(s, x)t+x) \cdot \xi} \hat{f}(\xi) \xi_j d\xi dt) \right) ds \\ &= \frac{1}{2\pi i} \int_{s_1}^{s_2} \frac{1}{\|x - \Lambda(s)\|} \left(\frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{i(\alpha(s, x)t+x) \cdot \xi} \hat{f}(\xi) (\Lambda'(s) \cdot \xi) d\xi dt \right) ds \\ &= \frac{1}{2\pi i} \int_{s_1}^{s_2} \frac{1}{\|x - \Lambda(s)\|} \frac{\partial \varphi(\alpha(s, x), \Lambda(q))}{\partial q} \Big|_{q=s} ds, \end{aligned} \quad (16)$$

where in the last step we have used the fact

$$\begin{aligned} \frac{\partial \varphi(\alpha(s, x), \Lambda(q))}{\partial q} \Big|_{q=s} &= \frac{\partial}{\partial q} \left(\frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} (e^{i(\alpha(s, x)t+\Lambda(q)) \cdot \xi} \hat{f}(\xi) d\xi dt) \right) \Big|_{q=s} \\ &= \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} (e^{i(\alpha(s, x)t+\Lambda(q)) \cdot \xi} \hat{f}(\xi) (\Lambda'(q) \cdot \xi) d\xi dt) \Big|_{q=s} \\ &= \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} (e^{i(\alpha(s, x)t+\Lambda(s)) \cdot \xi} \hat{f}(\xi) (\Lambda'(s) \cdot \xi) d\xi dt) \\ &= \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} (e^{i(\alpha(s, x)t+x) \cdot \xi} \hat{f}(\xi) (\Lambda'(s) \cdot \xi) d\xi dt). \end{aligned} \quad (17)$$

Combining Eqs. (12) and (14)-(16), we arrive at

$$\begin{aligned}
 (\mathcal{J}_{\gamma_x} \mathcal{R}f)(x) &= \frac{1}{2\pi i} \int_{C_{\gamma_x}} \sum_{j=1}^3 \frac{\partial \varphi(\alpha(s, x), \beta)}{\partial \beta_j} \bigg|_{\beta=x} d(\alpha(s, x))_j \\
 &= \frac{1}{2\pi i} \int_{s_1}^{s_2} \frac{1}{\|x - \Lambda(s)\|} \frac{\partial \varphi(\alpha(s, x), \Lambda(q))}{\partial q} \bigg|_{q=s} ds \\
 &= \frac{1}{2\pi i} PV \int_{\mathbb{R}} (f(\alpha(s_2, x)t + x) - f(\alpha(s_1, x)t + x)) \frac{dt}{t}.
 \end{aligned} \tag{18}$$

When x belongs to the chord connecting the two end points of the scanning trajectory $\Lambda(s)$, we have $\alpha(s_1, x) = -\alpha(s_2, x)$ and Eq. (18) is a DBP formula, which clearly suggests the BPF approach for image reconstruction. Combined with the finite Hilbert transform inversion technique,^{23,24} the BPF approach immediately lead to the BPF algorithms for standard and non-standard helical cone-beam scanning in the CT field.^{6-10,13,25-29}

VI. DISCUSSIONS AND CONCLUSION

For practical CT systems, the projection is a half-ray transform and can be defined as

$$\mathcal{R}^+f(\alpha, \Lambda) = \int_0^{+\infty} f(\alpha t + \Lambda) dt. \tag{19}$$

Because an object $f(x)$ to be reconstructed in biomedical applications is compactly supported and the trajectory $\Lambda(s)$ is always outside the convex hull of the object support in biomedical applications, we have $\mathcal{R}^+f(-\alpha, \Lambda) \equiv 0$ and $\mathcal{R}^+f(\alpha, \Lambda) \equiv \mathcal{R}f(\alpha, \Lambda)$. As a result, the object is compactly supported on the chord, as shown in Fig. 5(b). Thus, the image on the chord can be exactly reconstructed by performing the inverse Hilbert transform or inverting the truncated Hilbert transform.^{23,24} As a side note, when the BPF algorithms were developed in the CT field, two data extension schemes were used in the proofs. They are the even data extension $\mathcal{R}^+f(\alpha, \Lambda) + \mathcal{R}^+f(-\alpha, \Lambda)$ and the odd data extension $\mathcal{R}^+f(\alpha, \Lambda) - \mathcal{R}^+f(-\alpha, \Lambda)$. Clearly, Eq. (18) is based on the even extension. Normally, the two extensions are equivalent. However, in a rare case (the scanning trajectory intersects the convex hull of an object support), $\mathcal{R}^+f(-\alpha(s, x), \Lambda(s)) \not\equiv 0$, and only the odd extension allows theoretically exact reconstruction on the chord. Please see the relevant references for more details.^{6,8-10,25} We would like to point out that Gel'fand and Graev¹ provided no suggestion on how the non quasi-cycle formula could be beneficial for cone-beam reconstruction, whereas the results in Ref. 9 described a powerful methodology using the concepts of R-lines and M-lines.

After Ref. 1, the Hilbert transform relationship was extended by Rullgård in the case of the attenuated Radon transform for single photon emission computed tomography.⁵ Also, we would like to make a few comments on a Fourier-transform-based general CT scheme presented by Wei *et al.*¹⁰ First, Theorem 1 (Gel'fand and Graev) in the

parallel-beam scenario was re-generated in $\mathbb{R}^3$ when $As = 1$ in case II (Eq. 27 in Ref. 10) of that general CT scheme. The formula Eq. (13) is closely related to Eq. (28) in Ref. 10 and the existing FBP formulas with the even extension in the fan- and cone-beam geometry, which were re-derived in the same way by Wei *et al.*¹⁰ Second, while the theory in Ref. 1 provides a unified reconstruction formula in a higher dimensional space using the pseudo differential operator, the scheme in Ref. 10 works in 2D and 3D through a general weighting function. This weighting function can be specialized to give not only the BPF formulas but also other important CT algorithms.^{30,31}

A few more comments are in order on the relationship between the Gel'fand and Graev formula and some recent work in the CT field. First, the DBP approach (the same as the BPF approach) requires the inverse Hilbert transform after Gel'fand and Graev's formulation in Sec. 8 in Ref. 1. Evidently, Gel'fand and Graev revealed a fundamental link between the differential backprojection and the Hilbert transform. The overall purpose of their work is the recovery of an underlying function from its integral transform. Hence, the 2D or 3D image reconstruction from parallel-, fan-, or cone-beam projections is a special case of their general scheme. Therefore, once they presented the example in Sec. 8 in Ref. 1, it goes without saying that they had suggested the BPF in the parallel-beam geometry (it is unlikely that they did not know the inverse Hilbert transform). Furthermore, their Hilbert transform relationship assumes any path on a unit sphere, including but not restricted to a "quasicycle." That is, they are the first who proposed the BPF approach. On the other hand, we want to point out that the application of the inverse truncated Hilbert transform and the aforementioned odd data extension for DBP-based CT reconstruction of a compactly supported object is an original contribution by CT researchers.^{6,8,11-14,27-29}

Second, it is mathematically trivial to extend the parallel-beam BPF formula to divergent-beam BPF formulas. Indeed, Gel'fand and Graev pointed out this connection in a more general setting in Sec. 9 in Ref. 1. Note that they used the word "easily", and indeed we exactly followed their instruction in the case of the x-ray transform in a 3D real space, and re-derived the same fundamental link between the differential backprojection and the Hilbert transform, which is Eq. (18) in our manuscript.

In other words, both the DBP (or BPF) approach/formulas suggested by Gel'fand and Graev and the DBP (or BPF)

algorithms developed by CT researchers are valuable in different ways. Mathematically, the Gel'fand and Graev theory is very abstract and yet powerful, which gives a recipe dealing with image reconstruction in a general space from its integral transform of various forms, which are expressed in mathematical "jargons." Technically, it is not trivial to have working algorithms designed and coded directly from their theory. When an object to be reconstructed is compactly supported, the truncated Hilbert transform technique is crucial, which was not mentioned by Gel'fand and Graev. Thus, the mathematical conception of DBP/BPF was suggested by Gel'fand and Graev and, at the same time, its application for reconstruction of a compactly supported object and technical implementation should be credited to CT researchers.

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^{a)}Electronic mail: yey@math.uiowa.edu

^{b)}Electronic mail: hengyong-yu@ieee.org

^{c)}Electronic mail: ge-wang@ieee.org

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