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Nonlinear Growth Curves in Developmental Research

Kevin J. Grimm, Ph.D.,

Department of Psychology, University of California, Davis

Nilam Ram, Ph.D., and

Department of Human Development and Family Studies, The Pennsylvania State University and the Max Plank Institute of Human Development

Fumiaki Hamagami, Ph.D.

Center for the Advanced Study of Teaching and Learning, University of Virginia

Abstract

Developmentalists are often interested in understanding change processes and growth models are the most common analytic tool for examining such processes. Nonlinear growth curves are especially valuable to developmentalists because the defining characteristics of the growth process such as initial levels, rates of change during growth spurts, and asymptotic levels can be estimated. A variety of growth models are described beginning with the linear growth model and moving to nonlinear models of varying complexity. A detailed discussion of nonlinear models is provided, highlighting the added insights into complex developmental processes associated with their use. A collection of growth models are fit to repeated measures of height from participants of the Berkeley Growth and Guidance Studies from early childhood through adulthood.

Understanding, estimating, and capturing the defining characteristics of growth processes are key components of developmental research. Defining characteristics may include initial levels, rates of change, periods of acceleration and deceleration, when the process enters and leaves different developmental phases, and final or asymptotic levels. Growth curves are often estimated to understand these aspects of developmental processes and nonlinear growth curves are essential for capturing these various change components. Growth curve models have been used in diverse disciplines, especially in biological sciences to study crop growth, population processes, and bacterial growth. Among early applications was Rao's (1958) use of anthropometric data collected by Rao and Bhattacharyya (1952, 1953) to illustrate a systematic approach to modeling changes in human height. The contemporary growth models that emerged from this approach provide descriptions of systematic growth and change of living organisms across time.

In psychological science, developmentalists are often interested in examining and understanding change processes related to psychological dimensions, such as fluid intelligence, personality, and behavior problems. Researchers use growth models to obtain descriptions of change processes, including information regarding within-person change, average change, between-person differences in change, and determinants of change (Baltes & Nesselroade, 1979; Bryk & Raudenbush, 1992; McArdle, 1986, 1988; Meredith & Tisak, 1990). Many applications consider and model linear change patterns because of their simplicity and interpretability. However, many developmental processes are more complex, several core theoretical notions of development do not posit simple linear change (e.g., Bronfenbrenner,

1979), and many empirical data are not best characterized by linear change patterns. In such situations, researchers must consider models capable of representing nonlinear developmental patterns and be specific about where between-person differences appear in those patterns (see Burchinal & Appelbaum, 1991, Grimm & Ram, 2009, Ram & Grimm, 2007).

Acknowledging the nonlinearity present in many human growth processes, polynomial (e.g., quadratic & cubic) representations of change are often examined, with or without between-person differences in the additional change components. Occasionally, researchers consider the latent basis growth model (McArdle & Epstein, 1987; Meredith & Tisak, 1990), a flexible growth model that allows for complex nonlinear patterns of change, but also limited in that all between-person differences in change are reduced to differences in a single component. Rarely do researchers consider nonlinear latent growth curves that follow specific nonlinear functions, such as exponential or logistic, that allow for multiple between-person differences components. Our purpose in this paper is to provide exposure to a collection of nonlinear latent growth curves and the procedures by which they may be applied to developmental data.

Acknowledging that different terms are used in different fields and throughout the literature, we use the term *nonlinear* to describe patterns of change not adequately represented by linear (i.e., straight line) trajectories. Additional nuances within the larger collection of nonlinear growth curve models are subsequently introduced and discussed.

Example of Nonlinear Change

Consider individual changes in height from early childhood through adulthood. To help visualize these change patterns, repeated measurements of height obtained from five females and five males at regular intervals between age 3 to 19 years are plotted in Figure 1A and 1B, respectively. As seen in these figures, changes in height are not adequately characterized by a straight line - individual growth is not linear. Instead, changes in height proceed through a number of developmental phases. A period of stable growth is followed by accelerated growth, and then decelerated growth towards a final asymptote. Further, it can be noted that these developmental phases appear to begin and end at different ages for different individuals (e.g., accelerations in growth manifest at different ages for different individuals). To model changes in height properly, a statistical model must accommodate the nonlinear developmental pattern and important between-person differences in aspects of the developmental process (e.g., timing of growth spurt, final height, & rates of change).

Models of Nonlinear Change: Theory and Method

Models of change should, ideally, provide adequate representations of developmental theory (Ram & Gerstorf, 2009; Ram & Grimm, 2007; Wohlwill, 1973). A model is of little use, when its parameters do not map onto theoretical notions of change or what is known about the developmental system under investigation. For example, when modeling changes in height from early childhood to adulthood, it is important consider what is known regarding the developmental system. Biological processes coinciding with the onset of puberty, changes occurring during puberty, the asymptotic nature of adult height, as well as how people differ with respect to these aspects of development should inform the statistical model of change (Collins, 2006). Models that do not include parameters that map onto specific and theoretically important aspects of development may not be helpful in understanding the process of development and its determinants. It must be acknowledged, though, that in some areas, there may not yet be strong theories regarding the nature of the developmental process. In such circumstances, it is useful to fit a set of models that may be able to adequately represent the observed change patterns and have parameters that help researchers understand important features of the developmental process. Given the nature of knowledge discovery, there is, and should be, an ongoing interplay between the analytical models and theories. Each should inform

the other as we iterate between hypothesis/model construction and hypothesis/model testing. Cross-validation and replication play important roles in determining the appropriateness of developmental theory and accompanying growth model, as well as the refinement of those theories and models.

In this paper, we describe several types of growth models that may be useful for representing developmental theory. We start with the simple linear growth curve and move towards nonlinear latent growth curves with additive and multiplicative between-person differences. Our general thesis is that nonlinear latent growth curves with multiplicative between-person differences are among the tools necessary to adequately model many developmental processes. Although complex, these models provide opportunities for, and enable separation of, multiple aspects of change, such as the amount of change, rate of change, and timing of changes in developmental processes in ways that allow researchers to distinguish among them and obtain a clearer picture of the underlying developmental process and determinants.

On our path from simple to complex models, we first review linear, quadratic, and latent basis growth models. Building on these, we then discuss nonlinear latent growth curves with additive and multiplicative random coefficients. Finally, we illustrate their utility by fitting a collection of growth models to the longitudinal height data introduced above, and demonstrate how the models may be expanded to include between-person predictors of the different change components. In doing so, we hope to make nonlinear growth models more accessible to the developmental research community.

Growth Curve Analysis

Growth curve analysis (Browne, 1993; Browne & du Toit, 1991; McArdle, 1986; 1988; Meredith & Tisak, 1990; Rao, 1958; Tucker, 1958), often referred to as latent curve analysis as the parameters of the model are latent, is a method for describing within-person change and between-person differences in within-person change. In the following sections, we describe specific types of growth curves and highlight important similarities and differences among them.

Linear growth

The linear growth model is probably the most commonly fit growth curve and can be written as

$$y_{nt} = i_n + s_n \cdot \left(\frac{t - k_1}{k_2} \right) + e_{nt} \quad (1)$$

where y_{nt} are the repeated measures of attribute y for individual n ($n = 1, 2, \dots, N$) at time t ($t = 1, 2, \dots, T$), i_n is the intercept or predicted score when $t = k_1$ for individual n , k_1 is a constant used to center, or adjust, the measure of time (i.e., t) so the intercept has a desired interpretation, s_n is the linear slope or the predicted amount of change in y for a one unit change in t/k_2 for individual n , k_2 is a scaling coefficient used to adjust the metric of t and the interpretation of s_n , and e_{nt} is a time-specific residual score for individual n at time t . Often k_1 is set equal to 1, making the intercept reflect an individual's level at the first time of measurement, $t = 1$; if $k_1 = T$, the intercept represents an individual's level at the last time of measurement. Regardless of choice of k_1 in a linear growth model, the slope coefficient, s_n , is unaltered. Similarly, k_2 is often set equal to 1, scaling the slope in terms of the original time metric between measurement occasions. Regardless of the choice of k_2 , the intercept coefficient, i_n , is unaltered. Individual intercepts, i_n , and slopes, s_n , are assumed to follow multivariate normal distributions that can be described with means (μ_i & μ_s), variances (σ_i^2 & σ_s^2), and covariance ($\sigma_{i,s}$). The assumption

of multivariate normality can be seen as a limitation of current implementations of growth curve models; however, fixed effect parameters, which describe the average growth function, are quite robust to violations of normality.

The time-specific residual score is assumed to have a normal distribution characterized by a mean of zero and an estimated variance, σ_e^2 . However, it is important to note that growth curves can also be fit to non-normal data, including variables with dichotomous, ordinal, and Poisson distributions (e.g., Muthén, 1996), by changing the assumptions about how residuals are distributed. Often, residual variances are constrained to be equal at all time points. This assumption of a single residual variance, as opposed to a separately estimated residual variance at each measurement occasion, is common in longitudinal analysis because the same entity is repeatedly assessed. However, this assumption is not necessary in many applications and is testable. Recent work on the topic (e.g., Ferron, Dailey, & Yi, 2002; Grimm & Widaman, 2010; Kwok, West, & Green, 2007; Sivo, Fan, & Witta, 2005) has suggested that the choice of residual structure can have a large impact on the estimation of latent variable covariance parameters and model fit. In ideal situations, the assumption of a single residual variance is reasonable. However, there are times to consider alternative structures, such as when variables show evidence of artificial ceiling or floor effects. For consistency and simplicity in the presentation of growth models here, we treat σ_e^2 as a single residual variance throughout our model descriptions.

The linear growth curve (as with many growth curves) can be fit in both structural equation modeling (SEM) and multilevel modeling frameworks (see Ferrer, Hamagami, & McArdle, 2004), and each framework has its advantages and disadvantages (see Ghisletta & Lindenberger, 2004). Terminology also differs between these frameworks and we highlight these differences. In the SEM framework, the linear growth curve is fit as a restricted common factor model with a mean structure. Specifically, i_n and s_n are *latent variables*, factor loadings

for i_n are fixed to 1, and factor loadings for s_n are fixed to $\left(\frac{t - k_1}{k_2}\right)$. The latent variables have means (μ_i & μ_s), variances (σ_i^2 & σ_s^2), and covariance ($\sigma_{i,s}$) describing the average trajectory and individual variation surrounding the average trajectory. In the multilevel (mixed effects, random coefficient) modeling framework, i_n and s_n are often referred to as *random coefficients* as their values vary over persons, which can be described by fixed effects (μ_i & μ_s) and random effects (σ_i^2 , σ_s^2 , & $\sigma_{i,s}$) parameters. The growth model is fit as a multilevel

regression, with occasions nested within persons and a variable coded $\left(\frac{t - k_1}{k_2}\right)$ used as a predictor of the outcome variable, y_{nt} .

Estimated parameters of the linear growth curve can also be presented with mean and variance/covariance expectations. For example, the expected value of the intercept is the sample mean for the intercept or $E(i_n) = \mu_i$. Similarly, mean expectations for the slope and residuals can be written as $E(s_n) = \mu_s$ and $E(e_{nt}) = 0$. Variance and covariance expectations can be written as $\text{var}(i_n) = \sigma_i^2$, $\text{var}(s_n) = \sigma_s^2$, $\text{cov}(i_n, s_n) = \sigma_{i,s}$. Additional expectations include $\text{var}(e_{nt}) = \sigma_e^2$, $\text{cov}(i_n, e_{nt}) = 0$, and $\text{cov}(s_n, e_{nt}) = 0$ to indicate the variance of residuals and the assumption that residuals are uncorrelated with the intercept and slope.

The linear growth model is simple to estimate because the latent variables or random coefficients (i_n & s_n) are additive. That is, the predicted score at time t for individual n is the

sum of the individual's intercept score and his/her slope score multiplied by $\left(\frac{t - k_1}{k_2}\right)$, which varies with time but does not vary across individuals. The parameters of the linear growth

model are easy to interpret. The intercept is the predicted score when $t = k_1$. The linear slope represents the average rate of change, which is constant within individual, but allowed to vary across individuals. In the linear growth model the slope is perfectly related to the total amount of expected change during the observation period. By design, knowledge of the slope implies knowledge of the total amount of growth, and how that growth is (equally) distributed over time.

Note that for generality, we use t to represent *time*, but recognize and use different time metrics that theoretically guide the developmental process, such as chronological age, pubertal age (e.g., Tanner stage), grade, time since marriage, time until death, etc. (see McArdle & Bell, 2000; Ram et al., 2010). The choice of time metrics is of the utmost importance when attempting to understand and model developmental processes and, therefore, should be thoughtfully considered. Within the linear model, additional care should be taken on generalizations to times outside the specific observation period under study. A limitation of the model is that, if projected onwards to future times, individual trajectories continue towards positive or negative infinity. Given the physical reality of human development, the lack of asymptotic constraints suggest that the linear model may only be appropriate for modeling change over very specific developmental periods (e.g., from ages 3 to 6 years).

Growth Curves with Nonlinear Change Patterns

In this section we describe several different growth models that can describe nonlinear change patterns. These include models of quadratic growth, latent basis growth, and a collection of additive and multiplicative nonlinear latent curve models.

Quadratic growth

Many researchers turn to the quadratic growth model when a linear change model does not fit well or when a nonlinear trend is seen in the longitudinal plot. The quadratic model allows for a specific type of nonlinearity in the change pattern and can be written as

$$y_{nt} = i_n + sl_n \cdot \left(\frac{t - k_1}{k_2} \right) + sq_n \cdot \left(\frac{t - k_1}{k_2} \right)^2 + e_{nt} \quad (2)$$

where i_n is the intercept for individual n , sl_n is the linear slope for individual n , sq_n is the quadratic slope for individual n , and e_{nt} is the time-specific residual score for individual n at time t . Similar to the linear growth curve, the interpretation of the intercept remains the predicted score when $t = k_1$; but, the interpretations of the linear and quadratic slopes are more complicated as they both affect the rate of change. The linear slope represents the instantaneous rate of change at $t = k_1$, and the quadratic slope represents change in the rate of change at $t = k_1$ (see Bollen & Curran, 2006). Thus, larger absolute values of the linear slope represent a faster rate of change at $t = k_1$, and larger absolute values of the quadratic slope indicate that the rate of change is changing more rapidly, producing greater curvature in represented trajectories. The intercept, linear slope, and quadratic slope have means (μ_i , μ_{sl} , & μ_{sq}), variances (σ_i^2 , σ_{sl}^2 , & σ_{sq}^2), and covariances ($\sigma_{i,sl}$, $\sigma_{i,sq}$, & $\sigma_{sl,sq}$). The residual scores have a mean of zero and an estimated variance (σ_e^2).

As a structural model, sq_n is an additional latent variable with factor loadings equal to

$\left(\frac{t - k_1}{k_2} \right)^2$. As with the linear model, the quadratic growth model is simple to estimate because the random coefficients remain additive. A drawback, though, is that the parameters of the quadratic model can be difficult to interpret (see Cudeck & du Toit, 2002) and map onto

theoretical notions of developmental processes. Interpretation of parameters is difficult because the linear and quadratic slopes both affect the rate of change, possibly in different ways at different values of t . Cudeck and du Toit (2002) re-parameterized the quadratic model to highlight important features of curve that are more easily interpretable. The parameters include an initial level, the time at which the curve reaches its maximum (or minimum), and the maximum (or minimum) score.

As noted with the linear model, expected trajectories of the quadratic model can be algebraically tractable, but empirically unrealistic outside of the observation period. In most cases, the quadratic model is unlikely to provide a good representation of developmental theory because of the limited interpretability of its parameters and its parabolic change pattern. In sum, quadratic growth models only provide reasonable representations of change within the range of the data and are not good at representing the final asymptotic level of a developmental process (Grimm & Ram, 2009; Widaman, 2007).

Latent basis growth

The latent basis growth model (McArdle & Epstein, 1987; Meredith & Tisak, 1990) is a flexible and often overlooked growth model for modeling nonlinear change patterns. The latent basis growth model can be written as

$$y_{nt} = i_n + s_n \cdot (\alpha_t) + e_{nt} \quad (3)$$

where y_{nt} are repeated measures of attribute y for individual n at time t , α_t are basis coefficients that represent how the within-person change process unfolds, i_n is the intercept or predicted score for individual n when $\alpha_t = 0$, s_n is the shape factor or the predicted amount of change in y for a one unit change in α_t for individual n , and e_{nt} is a time-specific residual score for individual n at time t . The intercept and shape factor have means (μ_i & μ_s), variances (σ_i^2 & σ_s^2), and a covariance ($\sigma_{i,s}$), and residual scores have a mean of zero and an estimated variance (σ_e^2).

Identification constraints must be placed on α_t to define s_n . Often $\alpha_1 = 0$ and $\alpha_T = 1$, making i_n interpreted as an individual's predicted score at $t = 1$ and s_n as the total amount of change that occurred from $t = 1$ to $t = T$. Estimated loadings (α_2 to α_{T-1}) represent the proportion of the total amount of change that has occurred up to that point in time. Similar to the linear model, the total amount of change is perfectly related to the rate of change because there is only one change component, s_n . However, the rate of change is not constrained to be constant across time as in the linear model, but changes proportionally across time in accordance with the basis coefficients, α_1 to α_T .

The latent basis model is able to capture a variety of nonlinear change patterns because it does not have a specific functional form. That is, the model is atheoretical regarding the structure of change. The latent basis model will reproduce the mean structure of the data and uses a single factor to capture interindividual-differences in the change process. In this sense, the latent basis model rescales time for optimal fit. Thus, the latent basis model will be the best fitting model, in terms of the chi-square, for any growth model with one such interindividual-differences variable (e.g., will fit better than a linear model). Drawbacks of the latent basis model are that it is parameter heavy, especially with many measurement occasions, because all but two shape factor loadings are estimated, the estimated parameters can be difficult to map onto theoretical notions of developmental processes, and the model does not make predictions outside the observation period.

Nonlinear latent curve models

We use the term *nonlinear latent curve models* to describe a series of growth models that follow specific nonlinear patterns characterized by known mathematical functions like the Gompertz, logistic, exponential, and Richards curves (Browne, 1993; Browne & du Toit, 1991; Grimm & Ram, 2009). Having been simultaneously developed in multiple fields, there is a variety of terminology that exists for describing these types of models and their subsets (see e.g., Cudeck & Harring, 2007; Browne & du Toit, 1991). Terminological distinctions among models center particularly on how latent variables (random coefficients) are combined within the model (whether they enter the model linearly or nonlinearly). We further classify nonlinear latent growth curves into two types: models with additive random coefficients and models with multiplicative random coefficients. This distinction is fleshed out below; however, for clarity we note some of the alternate terminologies used to describe these models. In discussions of multilevel or random coefficient models (RCM), additive models have been referred to as partially nonlinear RCMs (Blozis & Cudeck, 1999; Cudeck & Harring, 2007; Harring, Cudeck, & du Toit, 2006) and multiplicative models as fully nonlinear RCMs (Davidian & Giltinan, 1995). In the structural equation modeling tradition, the former have been discussed as nonlinear structured latent curve models because the parameters enter the model nonlinearly, but the random coefficients enter the model linearly. The latter cannot be directly estimated in structural equation modeling and, therefore, have not been referred to with a specific term. However, Browne and du Toit (1991; Browne, 1993) presented a technique for approximating these models using available SEM software. Details of this work are described in Appendix A. In this paper, we use the terms additive nonlinear latent growth curves and multiplicative nonlinear latent growth curves to highlight differences in how random coefficients are combined in the model.

A benefit of nonlinear latent curve models is that their functional forms are numerous, and therefore many different patterns of change can be modeled (Ram & Grimm, 2007). These nonlinear models describe specific patterns of change over time, and, as such, offer developmentalists a collection of models useful for testing specific theory-based hypotheses about change. Additionally, several models have parameters that are easy to interpret in terms of how the process unfolds that can be mapped onto theories regarding psychological or physical growth. A series of nonlinear models is described next. We focus on the exponential model to highlight similarities and differences across types of nonlinear latent curves. We use an exponential model in our initial presentation because of its relative simplicity and its utility for studying development processes has been demonstrated in several domains, including learning processes, language development, and life-span cognitive development (e.g., Burchinal & Appelbaum, 1991; Browne & du Toit, 1991; Ghisletta et al., 2010; McArdle, Grimm, Hamagami, Bowles, & Meredith, 2009; Ram, Rabbitt, Stollery, & Nesselrode, 2005).

Additive nonlinear latent curves—Additive nonlinear latent curves are models that follow specific nonlinear functions and have additive random coefficients. So, similar to the models discussed thus far, predicted scores are simply the *sum* of the participant's scores on the random coefficients (e.g., intercept & slope), each multiplied by a function of time that does not vary across participants. For example, an exponential model with additive random coefficients can be written as

$$y_{nt} = i_n + s_n \cdot (1 - \exp(-\alpha \cdot t)) + e_{nt} \quad (4)$$

where i_n is the predicted score when $t = 0$ for individual n , s_n is the total amount of change from i_n to the upper asymptote for individual n , α is the rate of approach to the upper asymptote, and e_{nt} is the time-dependent residual. In this model, α is an estimated (fixed effect) parameter

and does not vary across participants whereas i_n and s_n are random coefficients that do vary across participants. The random coefficients have means (μ_i & μ_s), variances (σ_i^2 & σ_s^2), and covariance ($\sigma_{i,s}$), and residual scores have a mean of zero and an estimated variance (σ_e^2). If α was a chosen parameter, as opposed to an estimated parameter, then time (t) could be rescaled and the model could be fit using linear mixed-effects modeling and linear structural equation modeling programs. This type of model is common in Randomized Control Trials as time is often transformed using log (e.g., $\ln(t+1)$) or square root (e.g., $\sqrt{t}$) functions (e.g., Hedeker & Gibbons, 1997).

Additive nonlinear latent curves are relatively easy to estimate using most SEM and multilevel modeling software (see Blozis & Cudeck, 1999; Grimm & Ram, 2009; Ram & Grimm, 2007; Harring, Cudeck, & du Toit, 2006) because the random coefficients are additive. For example, in this exponential model, the value of $(1 - \exp(-\alpha \cdot t))$ is only dependent on time and the estimated value of the rate parameter, α . As a structural equation model, s_n is a latent variable with factor loadings equal to $(1 - \exp(-\alpha \cdot t))$. Estimation of such models in a structural equation modeling framework requires the ability to impose nonlinear constraints. Programs such as Lisrel, *Mplus*, Mx, OpenMx, and PROC CALIS in SAS allow for such constraints. Similarly, for multilevel modeling programs to estimate this model, nonlinear constraints are required (e.g., PROC NL MIXED in SAS, nlme in R, & HLM).

These additive nonlinear latent curves are elegant, often have interpretable parameters, and are able to model complex developmental patterns with few parameters (in contrast to the latent basis model described above). Furthermore, they are better able to represent the entirety of the developmental process. However, in the additive exponential model, participants only differ with respect to two aspects of the developmental process, the intercept (i_n) and total amount of change (s_n). Because s_n is the only random coefficients controlling the change function, both the rate of change and the total amount of change for each person are contained within s_n , making s_n a complex, potentially confounded random coefficient, just as in the linear and latent basis models discussed previously.

Multiplicative nonlinear latent curves—Multiplicative nonlinear latent curves allow for additional sets of between-person differences in the nonlinear function. An exponential model with multiplicative random coefficients can be written as

$$y_{nt} = i_n + s_n \cdot (1 - \exp(-\alpha_n \cdot t)) + e_{nt} \quad (5)$$

where i_n is the predicted score when $t = 0$ for individual n , s_n is the total amount of change from i_n to the upper asymptote for individual n , α_n is the rate of approach to the upper asymptote for individual n , and e_{nt} is the time-dependent residual. Now, α_n is a *random coefficient*, as opposed to a (fixed effect) parameter, and therefore allowed to vary across participants. The individual rate of approach (α_n) has a mean (μ_α), variance (σ_α^2), and covariances with the intercept ($\sigma_{i,\alpha}$) and total change ($\sigma_{s,\alpha}$). In contrast to the additive model, the rate of change (α_n) is now distinct from the total amount of change (s_n) at the individual level and these aspects of change may be theoretically and practically important and have different determinants and consequents. The model is considered to be multiplicative because $(1 - \exp(-\alpha_n \cdot t))$ varies across both time and persons and is multiplied by another random coefficient, s_n . Thus, individual predicted scores are not simply the sum of random coefficients multiplied by parameters; rather, s_n is a multiplier of α_n . Note that, for this model to be estimated directly in a structural modeling framework, the factor loadings for s_n would have to vary across participants and time.

Because of how the random coefficients enter the model, multiplicative nonlinear latent curves can be difficult to estimate (see du Toit & Cudeck, 2009; Vonesh & Carter, 1992). However, these models allow for between-person differences in various aspects of a change process. If separating variation in rates of change, timing, and total amounts of change is of theoretical importance, as in the height example, a multiplicative model is needed.

There are several ways to fit multiplicative nonlinear latent curves. The model can be estimated directly using nonlinear mixed effects modeling programs (e.g., NLMIXED in SAS & nlme in R) using various estimation techniques or using nonlinear mixed effects or structural equation modeling programs via a first-order Taylor series approximation method (see Browne, 1993; Browne & du Toit, 1991). Generally, this approach is preferred because of difficulties often encountered with the direct estimation approach. Appendix B contains an *Mplus* script for the exponential growth curve of Eq. 5 using the first-order Taylor series approximation method (described in Appendix A). Next, we illustrate how the use and estimation of such models can proceed in practice.

Illustrative Example

Data

Longitudinal height data come from the Berkeley Growth Study (Bayley, 1940) and Berkeley Guidance Study (MacFarlane, 1939). Details regarding participant recruitment and sample characteristics are contained in previously published work (e.g., Bayley, 1940; MacFarlane, 1939). Height measurements made between three and 19 years of age were extracted from both studies resulting in longitudinal data from 155 males and 167 females. The individual height trajectories contained within Figure 1 are characterized by a long period of relatively stable growth, followed by accelerated growth during puberty, and a leveling off at the participant's adult height.

Models

In modeling changes in height we considered the models described above including the linear, quadratic, and latent basis model. Additionally, we selected a nonlinear latent growth curve (in additive & multiplicative forms) that could be tethered to theoretical notions of the developmental processes. Several nonlinear curves have been developed to capture the complex change patterns exhibited in human growth (see Karkach, 2006). However, many of these models are atheoretical and only attempt to fit the data, as opposed to capture known features of growth. In contrast, the Preece-Baines Model (Preece & Baines, 1978; Zemel & Johnston, 1994) was developed specifically to model human growth patterns and has interpretable parameters that map onto developmental growth theory. The Preece-Baines Model can be written as

$$y_{nt} = h_{1n} - \frac{2(h_{1n} - h_{2n})}{\exp[s_{0n}(t - a_n)] + \exp[s_{1n}(t - a_n)]} + e_{nt}, \quad (6)$$

where h_{1n} is individual n 's adult height, h_{2n} is individual n 's height at $t = a_n$, the age when individual n 's height is changing fastest (middle of puberty), s_{0n} is the rate of change during childhood for individual n , s_{1n} is the rate of change during puberty for individual n , and e_{nt} is the time dependent residual. In its additive form, h_{1n} and h_{2n} were the only random coefficients. In its multiplicative form, h_{1n} , h_{2n} , a_n , s_{1n} , and s_{2n} were allowed to vary across persons. In our application of these models, chronological age was used as the time metric and an invariant residual variance was assumed. After determining the best fitting model, we proceeded to illustrate how the modeling framework could be expanded to include predictor variables.

Specifically, we introduced gender into the model as a predictor of the random coefficients (e.g., h_{1n} , h_{2n} , a_n , s_{1n} , & s_{2n}).

Model Fit

Statistical model fit was examined using the χ^2 statistic, Comparative Fit Index (CFI; Bentler, 1990), Tucker-Lewis Index (TLI), Root Mean Square Error of Approximation (RMSEA; Steiger & Lind, 1980), Bayesian Information Criterion (BIC), and Akaike Information Criterion (AIC). CFI and TLI values greater than .95 (Hu & Bentler, 1999) indicate the model adequately represents the data and an RMSEA less than .10 (MacCallum et al., 1996) indicates adequate fit. Information criteria are useful for comparing models, but do not provide an absolute measure of model fit. Models with lower BIC and AIC values are considered better fitting. Note that statistical model fit is based on the difference between observed mean and covariance structure and the model implied mean and covariance structure. Models showing better fit indicate a smaller difference between observed and model implied statistics and do not necessarily imply the model is a closer representation of the truth, simply that the model remains viable.

All models were fit using *Mplus* (Muthén & Muthén, 1998–2007). Input and output scripts as well as descriptions of several nonlinear growth models are available at <http://psychology.ucdavis.edu/labs/Grimm/personal/downloads.html>. Additionally, we refer readers to several books on nonlinear regression models (Bates & Watts, 1998; Huet, Bouvier, Poursat, & Jolivet, 2004; Ratkowsky, 1989; Seber & Wild, 1989) and nonlinear random coefficient models (Davidian & Giltinan, 1995; Pinheiro & Bates, 2000), which provide equations and descriptions of a variety of nonlinear models.

Results

Fit statistics for the linear, quadratic, latent basis, as well as the additive and multiplicative forms of the Preece-Baines models are contained in Table 1. The linear, quadratic, latent basis, and additive Preece-Baines model showed very poor fit based on the fit indices (CFIs & TLIs < .52, RMSEA > .24). The multiplicative Preece-Baines model had better fit indices; however, still showed greater misfit than generally desired with CFI = .853, TLI = .866, and RMSEA = .13. However, residual variances were very small, and the model accounted for at least 95% of the variability at each occasion. Browne et al. (2002) argued that many measures of model fit, even the likelihood ratio chi-square, will often appear to reflect relatively poor fit – when a model fits data very closely – that is, if residual variances are quite small. This appeared to be the case in the present application. Thus, following Browne et al.'s demonstration, we accepted the fit of the Preece-Baines model to the height data as adequate.

The multiplicative Preece-Baines model had five random coefficients and 21 estimated parameters, making it a somewhat heavily parameterized model. However, given the large number of measurement occasions to which the model was fit ($T = 30$), there were plenty of degrees of freedom ($df = 452$) for hypothesis testing. Previous empirical research has shown that as many as seven random coefficients may be needed to adequately fit human growth data (Jolicoeur, Pontier, Pernin, & Sempé, 1988). The mean function from the Preece-Baines model describes a person who has an adult height of 173.08 cm, a height in the middle of puberty, located at 13.23 years of age, of 160.43 cm, a rate of change before puberty of .11, and a rate of change during puberty of .88. Significant variation was found in all random coefficients, indicating that people differed in their rate of change during childhood, timing of puberty, rate of change during puberty, height during puberty, and final adult height. Additionally, children's adult height was almost perfectly correlated ($r = .99$) with their height during puberty, negatively correlated ($r = -.51$) with rate of change during childhood, negatively correlated ($r = -.20$) with rate of change during puberty, and positively correlated ($r = .55$) with when

they experienced puberty. Children's height during puberty correlated negatively ($r = -.40$) with their rate of change during childhood, did not correlate significantly with rate of change during puberty, and correlated positively ($r = .55$) with their age during puberty. Children's rate of growth during childhood correlated positively ($r = .58$) with their rate of change during puberty and negatively ($r = -.78$) with the age when they experienced puberty. Finally, children's rate of change during puberty correlated negatively ($r = -.44$) with the age when they experienced puberty. Interpreted together, the pattern of relationships suggested that the tallest adults tended to experience puberty later in life and tended to show slower, more gradual changes in height during childhood. Furthermore, children who grew more rapidly during childhood tended to grow more rapidly during puberty and tended to experience puberty earlier in life. Finally, children who grew more rapidly during puberty tended to experience puberty earlier in life. Figure 2 is a plot of the predicted average growth pattern and a 95% confidence bound on the between-person differences in the growth process described by the multiplicative Preece-Baines model. From this figure, different phases of development become apparent as the rate of growth can be seen to increase between 11 and 12 years of age and then slow between 15 and 16 years of age.

Theoretically, gender, as one of many possible interindividual differences characteristics, is a likely determinant of the variation in the aspects of intraindividual change and can help account for the correlations between the various aspects of development (e.g., boys, compared to girls, tend to be taller in adulthood, & experience puberty later in life). Gender (mean centered) was included as a predictor of the five random coefficients in the Preece-Baines model. Girls had a final adult height that was 13.53 ($b = -13.53, p < .001$) centimeters less than boys, a height during puberty that was 11.47 ($b = -11.47, p < .001$) centimeters less than boys, grew at a slightly faster rate during childhood ($b = .02, p < .001$), grew at a faster rate during puberty relative to their growth spurt ($b = 0.13, p < .05$), and experienced puberty 2.11 years ($b = -2.11, p < .001$) earlier than boys. These relations map onto to what is known about gender differences in human growth, and demonstrate the specificity with which interindividual differences in change can be examined.

Discussion

Developmental processes are complex entities, the representation and understanding of which often require complex models. In psychology, we often limit ourselves to linear change models, failing to consider models that allow for description and further understanding of some of the complexities of developmental processes and their determinants. In this paper, we provided an overview of an array of growth models useful for describing developmental processes, ranging from simple linear models to highly complex nonlinear growth models.

Effectively modeling, describing, and understanding change processes is difficult. An optimal growth model should allow the form of changes to be readily interpretable so that the meaning of change processes is clear and can be interpreted in relation to theory. Our presentation began with the linear growth model, a simple functional form that is easily interpretable, but is likely to only provide an adequate representation of developmental change for very restricted time spans. We then described a few of the many change models able to represent patterns of developmental change that are not linear in nature.

If a linear model does not adequately capture within-person change processes, many researchers move to models with quadratic, cubic, and higher-order polynomial components. These models are able to better capture many developmental patterns, but difficulty arises in interpretation, especially when time-invariant covariates are added as predictors. Furthermore, parameters of these models are unlikely to map onto theories of the developmental process. Additionally, researchers often add higher-order terms (e.g., a quadratic term), but constrain

the variation in the higher-order terms to zero. This approach allows for nonlinearity in the mean change pattern, but does not allow for or capture additional between-person differences in the patterns of change – further complicating interpretation. In these cases, a latent basis model may more accurately capture the nature of the nonlinear mean change pattern and provide interpretable parameter estimates. However, like the polynomial models, parameters of the latent basis model are unlikely to map onto developmental theories of change processes.

The next series of models discussed were characterized by specific nonlinear functions. These models were broken down into models with additive and multiplicative random coefficients. Compared to the latent basis model, additive nonlinear latent curves tend to be more parsimonious, especially with many measurement occasions, and often have interpretable parameters that can be mapped onto theoretical notions about the underlying change processes. These models, however, may not allow for between-person differences in key aspects of the growth process. Multiplicative nonlinear curves have the benefits of additive nonlinear models, but also allow for between-person differences in various (possibly all) components underlying the developmental process. Thus, these models resolve the limitation that all subjects follow the same de-individualized curve. By allowing for additional sources of heterogeneity in the modeling of a growth process, researchers can more completely decompose the underlying growth process and understand how covariates relate to different aspects of the developmental process. In these models, rates of change, phases of change, and total amounts of change are separable. Further, each of these aspects of the developmental process can be examined with respect to determinants and causes. In sum, multiplicative nonlinear growth curves offer many possibilities. They not only delineate interindividual differences among these parameters, but also permit the possibility of examining relations of covariates with very specific aspects of change. Although these models are admittedly more complex to use, fitting simpler additive nonlinear latent curves increases the risks of misrepresenting data by averaging over observed heterogeneity in growth parameters and disallowing the possibility of examining determinants of potentially important individual differences in the developmental process under investigation.

As seen in our examples, the complexity of the growth processes under study required nonlinear change patterns and incorporation of multiple between-person differences to adequately fit the observed data. Height changes have multiple phases (e.g., childhood, puberty, & adulthood) with different rates of change. Children differed in both the timing of these phases and the rates of change during each phase. The multiplicative version of the Preece-Baines Model was able to capture these aspects of change more accurately by incorporating between-person differences in each change component. Thus, the multiplicative nonlinear latent curve approach to modeling change allowed us to gain insights into the change processes that would not be possible using more simplified change analysis.

The analysis of longitudinal data with nonlinear latent curves has been recently extended in a variety of ways. Blozis (2004) and Harring (2009) describe models for examining nonlinear change in latent variables measured by common factor models. Harring (2005) and Kelley (2005, 2008) discuss the estimation of nonlinear models in heterogeneous populations with unknown class membership – a combination of nonlinear latent curve models with the finite mixture model (McLachlan & Peel, 2000). Choi, Harring, and Hancock (2009) describe an innovative way to fit logistic response models using conventional structural equation modeling software. Additionally, nonlinear latent curve models can and have been used to study multiple processes that have nonlinear change patterns to examine correlations between aspects of each developmental process (e.g., Blozis, 2007; Blozis, Conger, & Harring, 2007).

Additional Change Models

Additional change models not discussed here include spline or multiphase models (e.g., Cudeck & Klebe, 2002; Ram & Grimm, 2007; Wang & McArdle, 2008) and latent difference score models (Hamagami & McArdle, 2001; McArdle, 2001; McArdle & Hamagami, 2001). Multiphase models allow for different phases of development – similar to what was needed to model changes in height. Multiphase models have one or more transition (e.g., knot) points, where one phase of development is ending and another phase is beginning. These models are often fit with simple components within each phase (e.g., linear), but a variety of nonlinear change patterns and between-person differences in these patterns can be included (e.g., Ram & Grimm, 2007). In the structural modeling framework, transition points are fixed and do not vary over persons; however, in the multilevel framework, transition points can be random coefficients and vary over persons (see Wang & McArdle, 2008). Allowing for variability in these transition points would be necessary to account for between-person differences in many growth processes.

Latent difference score models are a collection of models that provide for systematic change processes and time-dependency among sets of repeated measures, combining features of traditional growth curves and time-series models (McArdle, 2001; McArdle & Hamagami, 2001). By allowing for systematic and time-dependent change, latent difference score models can capture a wide variety of linear and nonlinear change patterns. For example, the dual change score model is equivalent, in terms of fit and model implied trajectory, to the additive form of the exponential model (Eq. 4).

Drawbacks

The benefits of multiplicative nonlinear latent curves are obvious. They allow us to describe and better understand smooth, non-disjointed change processes, between-person differences in coefficients that govern the change patterns, and determinants of the various different aspects of change. However, several drawbacks accompany the use of these models. First, due to their mathematical complexity, one often encounters convergence problems during estimation of parameters, which can be unstable with few measurement occasions. However, the approximation method described by Browne and du Toit (Browne, 1993; Browne & du Toit, 1991) makes model fitting simpler and these nonlinear models can now be fit with many currently available programs. Following Browne and duToit's approach, even the most complex models fit in our empirical example converged within a few minutes. A second limitation lies in the condition that nonlinear growth models require more data than fitting simple linear growth curves. For example, consider a growth process that includes an initial phase, followed by a growth phase, and an asymptotic phase. If our choice of measurement occasions misses the asymptotic phase, we might mistakenly choose a *j*-shaped growth model, such as the exponential model, rather than an *s*-shaped growth model (e.g., logistic growth process). Moreover, because multiplicative nonlinear latent curves tend to have more than two random coefficients, a larger number of measurement occasions are needed to distinguish between different change components. Finally, if it is necessary to capture transitions periods and how they vary across individuals, measurement during these transitions and when changes are accelerating and decelerating is important. Fitting complex change models to limited data can lead to highly collinear random coefficients and unstable estimates, resulting in considerable overfitting relative to the amount of data. Decisions must be made regarding the extent of between-person differences in each component to achieve convergence and limit these problems.

Concluding Remarks

Modeling and understanding change processes is complex. Researchers must consider many issues before attempting to decipher any change process – issues including the selection of

appropriate measures capable of capturing changes in individuals, selection of appropriate samples, use of an adequate number of measurement occasions, and appropriate timing of measurement occasions (Collins, 2006). During the analysis phase, researchers must consider the potential presence of practice effects (e.g., Ferrer, Salthouse, Stewart, & Schwartz, 2004; McArdle & Woodcock, 1997), influence of time-varying covariates (e.g., schooling; McCoach, O'Connell, Reis, & Levitt, 2006), and ways in which attrition may affect results and conclusions. All of these issues impact the types of growth models that can be fit and thus constrain the richness of the theoretical conjectures we can investigate.

Our view is that researchers should not automatically limit themselves to simple change models, especially when their data and theories are inconsistent with such models. We believe that analyses of growth and development necessitate active conceptualization of which growth components are required to capture a given underlying developmental process. Conceptualization of a growth process involves decisions concerning how the process starts, evolves, is influenced by external forces, as well as how the rate of change is determined, how the acceleration of growth is determined, how the maturation phase emerges, and how all of these growth components vary across individuals. Fitting more complex change models have the potential to enable researchers to understand developmental processes and their determinants more fully. In turn, more sensitive or accurate modeling of data should provide empirical results that can be used to test and modify developmental theories. In attempting to account for more complex patterns in data, theoretical accounts will provide more and more accurate portrayals of developmental change and the processes that moderate or mediate change.

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Appendix A – First-Order Approximation Method

To aid in the estimation of multiplicative nonlinear latent curves Browne and du Toit (1991; Browne, 1993) showed how these models can be approximated by and fit as additive nonlinear growth curves. Several papers have followed describing how these models can be fit with available structural equation modeling programs (e.g., Blozis, Harring, & Mels, 2008). We summarize the approach.

Multiplicative nonlinear latent curves can be estimated as additive models because many multiplicative nonlinear functions can be rewritten in terms of additive random coefficients multiplied by the partial derivatives of the target function with respect to each random coefficient (Browne & du Toit, 1991). The target function is the mean growth function and therefore has only fixed parameters. When rewritten this way, factor loadings of the random coefficients are complex nonlinear functions, but only vary with time, as is the case with additive nonlinear growth curves making estimable within the structural equation modeling framework.

For example, the target function for the exponential curve of Eq. 5 is

$$\mu_t = \mu_i + \mu_s \cdot (1 - \exp(-\mu_a \cdot t)) \quad (\text{A1})$$

which contains only fixed-effect parameters – the means of the random coefficients. Taking the partial derivative of Eq. A1 with respect to each random coefficient, we can then rewrite the exponential model at the individual level (e.g., Eq. 5) as

$$y_{nt} = x_{in} \cdot \frac{\partial \mu_t}{\partial \mu_i} + x_{sn} \cdot \frac{\partial \mu_t}{\partial \mu_s} + x_{an} \cdot \frac{\partial \mu_t}{\partial \mu_\alpha} + e_{nt}, \quad (\text{A2})$$

where y_{nt} are repeated measures of y for individual n at time t , x_{in} is the predicted score at $t =$

0 for individual n , $\frac{\partial \mu_t}{\partial \mu_i}$ is the partial derivative of the target function with respect to μ_i , x_{sn} is

the amount of change from x_{in} to the upper asymptote for individual n , $\frac{\partial \mu_t}{\partial \mu_s}$ is the partial derivative of the target function with respect to μ_s , x_{an} is the rate of change for individual n ,

and $\frac{\partial \mu_t}{\partial \mu_\alpha}$ is the partial derivative of the target function with respect to μ_α . The partial derivatives of the target function are

$$\begin{aligned} \frac{\partial \mu_t}{\partial \mu_i} &= 1 \\ \frac{\partial \mu_t}{\partial \mu_s} &= 1 - \exp(-\mu_\alpha \cdot t) \\ \frac{\partial \mu_t}{\partial \mu_\alpha} &= \mu_s \cdot t \cdot \exp(-\mu_\alpha \cdot t) \end{aligned} \quad (\text{A3})$$

The model of Eq. A2 is additive because the random coefficients (x_{in} , x_{sn} , & x_{an}) are multiplied by functions (i.e., partial derivatives) that are *constant* across individuals (but vary with time) and can be *summed* to calculate predicted score for each individual. The interpretations for x_{in} , x_{sn} , and x_{an} are the same as the random effects for the model of Eq. 5. However, the expectations for the means are slightly different from what might be expected. The mean expectations are $E(x_{in}) = \mu_i$, $E(x_{sn}) = \mu_s$, and $E(x_{an}) = 0$. The mean expectation for x_{an} is 0 and not μ_α , because α_n is part of the nonlinear function (enters the model nonlinearly). However, μ_α is a parameter in the derivative equations (i.e., factor loadings for x_{in} , x_{sn} , & x_{an} ; Eq. A3) and is therefore estimated as part of the model. More importantly, the expectations for the variances are $\text{var}(x_{in}) = \sigma_i^2$, $\text{var}(x_{sn}) = \sigma_s^2$, and $\text{var}(x_{an}) = \sigma_\alpha^2$ and for the covariances are $\text{cov}(x_{in}, x_{sn}) = \sigma_{i,s}$, $\text{cov}(x_{in}, x_{an}) = \sigma_{i,\alpha}$, and $\text{cov}(x_{sn}, x_{an}) = \sigma_{s,\alpha}$, thus having the same as expectations as in Eq. 5.

Appendix B – Mplus Example Script

```
TITLE: Exponential Growth Model (y_nt = i_n + s_n * (1 - exp(-a_n*t));

DATA: FILE = temp.dat;

VARIABLE:
  NAMES = id x1-x10; USEVARIABLES = x1-x5;

ANALYSIS: TYPE = MEANSTRUCTURE; ESTIMATOR = ML; ITERATIONS = 5000;

MODEL:

    i BY      x1* (L11)
              x2  (L12)
```

```

x3 (Li3)
x4 (Li4)
x5 (Li5);

s BY x1* (Ls1)
x2 (Ls2)
x3 (Ls3)
x4 (Ls4)
x5 (Ls5);

a BY x1* (La1)
x2 (La2)
x3 (La3)
x4 (La4)
x5 (La5);

!Means
[x1-x5@0]; [i*50] (Mu_i); [s*10] (Mu_s); [a@0];

!Variances & Covariances

x1-x5 (Ve); i s a; i WITH s a; s WITH a;

MODEL CONSTRAINT:

new(Mu_a*.5);

Li1 = 1;
Li2 = 1;
Li3 = 1;
Li4 = 1;
Li5 = 1;

Ls1 = (1 - exp (- Mu_a * 0));
Ls2 = (1 - exp (- Mu_a * 1));
Ls3 = (1 - exp (- Mu_a * 2));
Ls4 = (1 - exp (- Mu_a * 3));
Ls5 = (1 - exp (- Mu_a * 4));

La1 = Mu_s * (0) * (exp(- Mu_a * 0));
La2 = Mu_s * (1) * (exp(- Mu_a * 1));
La3 = Mu_s * (2) * (exp(- Mu_a * 2));
La4 = Mu_s * (3) * (exp(- Mu_a * 3));
La5 = Mu_s * (4) * (exp(- Mu_a * 4));

OUTPUT: SAMPSTAT STANDARDIZED;

```

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Figure 1.
Longitudinal plots of height for (A) five girls and (B) five boys from the Berkeley Growth and Guidance Studies.

Figure 2.
Predicted mean curve with 95% confidence bounds for height based on the multiplicative Preece-Baines Model.

Table 1

Fit statistics for growth models fit to the longitudinal height data

	Linear	Quadratic	Latent Basis	Preece-Baines ^a	Preece-Baines ^b
$\chi^2(df)$	13829 (467)	8814 (463)	9795 (439)	9878 (464)	2724 (452)
Parameters	6	10	34	9	21
CFI	.138	.462	.397	.393	.853
TLI	.238	.520	.432	.460	.866
RMSEA (C.I.)	.307 (.302–.311)	.244 (.239–.248)	.265 (.260–.269)	.258 (.254–.263)	.129 (.124–.133)
AIC	27575	22568	23597	23631	16500
BIC	27597	22605	23724	23664	16578

Notes:

df = degrees of freedom, CFI = Comparative Fit Index, TLI = Tucker-Lewis Index, RMSEA = Root Mean Square Error of Approximation, AIC = Akaike Information Criteria, BIC = Bayesian Information Criteria,

^a random coefficients were additive,

^b random coefficients were multiplicative.