Published in final edited form as:

J Theor Biol. 2011 October 7; 286(1): 85–91. doi:10.1016/j.jtbi.2011.05.005.

A Linearized and Incompressible Constitutive Model for Arteries

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Abstract

In many biomechanical studies, blood vessels can be modeled as pseudoelastic orthotropic materials that are incompressible (volume-preserving) under physiological loading. To use a minimum number of elastic constants to describe the constitutive behavior of arteries, we adopt a generalized Hooke's law for the co-rotational Cauchy stress and a recently proposed logarithmic-exponential strain. This strain tensor absorbs the material nonlinearity and its trace is zero for volume-preserving deformations. Thus, the relationships between model parameters due to the incompressibility constraint are easy to analyze and interpret. In particular, the number of independent elastic constants reduces from ten to seven in the orthotropic model. As an illustrative study, we fit this model to measured data of porcine coronary arteries in inflation-stretch tests. Four parameters, n (material nonlinearity), Young's moduli E_1 (circumferential), E_2 (axial), and E_3 (radial) are necessary to fit the data. The advantages and limitations of this model are discussed.

Keywords

Hooke's law; large deformation; strain measure; blood vessel

1 Introduction

Knowledge of mechanical property of biological tissues is important for understanding the function of an organism. Upon preconditioning, soft tissues can be viewed as pseudoelastic materials that undergo large deformation at physiological state. Due to the arrangement of microstructural constituents, some tissues possess orthotropic mechanical properties. For instance, blood vessels have circumferential, axial and radial symmetries.

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Research on the mechanical behavior of blood vessels has been active for several decades [1-10]. Numerous constitutive models have been proposed, among which the pseudoelastic stress-strain relations have been at center stage. Nevertheless, comprehensive knowledge of the admissible range of model parameters and particularly the elastic stability of a model is still incomplete [11, 12]. A major reason for this paucity of knowledge is due to the difficulties of anisotropic models involving finite deformations. There is a major advantage to simplifying the constitutive laws and thus clarifying the analysis of parameter admissibility and reliability of model elastic constants.

In this report, we present a generalized Hooke's law for arteries. It is assumed that the tissue volume is preserved in all deformation ranges of interest. Thus, the existence of three intrinsic relations between the elastic constants reduces the number of model parameters. To validate the model, it is shown that the linearized model can fit the experimental data of porcine coronary arteries very well with only four parameters.

2 Methods

2.1 Linearized Constitutive Law

The basic promise of this study is to employ a constitutive model that results in simple relations between model parameters when the incompressibility assumption is enforced. A cylindrically orthotropic elastic model that accounts for large deformation of arteries is necessary. For simplicity, we propose a linearized stress-strain relation (constitutive model) that makes the analysis easier and more interpretive.

2.1.1 Logarithmic-exponential strain—We make use of standard notations in continuum mechanics. The deformation gradient $\mathbf{F}$ can be decomposed as tensor product of a rigid rotation tensor $\mathbf{R}$ ($\mathbf{R}^T\mathbf{R} = \mathbf{I}$) and right stretch tensor $\mathbf{U}$, i.e., $\mathbf{F} = \mathbf{R}\mathbf{U}$. The right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$. Following the approach we adopted recently [13], a logarithmic-exponential (log-exp) strain tensor $\mathbf{D}$ that absorbs the material nonlinearity is defined as:

$$\mathbf{D} = \frac{1}{2} Q \ln \mathbf{C} = \exp[n(J_1 - 3)] \ln \mathbf{U} \quad (1)$$

with

$$Q = \exp[n(J_1 - 3)], \quad J_1 = \text{tr}(\mathbf{C}) = \text{tr}(\mathbf{U}^2), \quad (2)$$

where n is a nonlinearity parameter. That is, the tensor $\mathbf{D}$ is the product of the logarithmic (natural, Hencky) strain and a scalar Q which is proposed to account for the strong exponential strain-stress relation found in vascular tissues [4]. It can be shown, under the principal-axis representation, that $\mathbf{D}$ satisfies all the mathematical requirements for a strain tensor [24]. It can also be proven that $\text{tr}(\mathbf{D}) = \ln(J)Q$, where $J = \det(\mathbf{F}) = \det(\mathbf{U})$; i.e., $\text{tr}(\mathbf{D}) = 0$ for incompressible deformation where $J \equiv 1$. In fact, this property arises from the classical Hencky strain. Furthermore, $\mathbf{D}$ is a Lagrangian strain similar to the Green strain $\mathbf{E}$.

2.1.2 Co-rotational Cauchy stress—Deformation of blood vessels is considered incompressible under physiological loading. The Cauchy stress $\boldsymbol{\sigma}$ for incompressible materials is decomposed [e.g., 7] as $\boldsymbol{\sigma} = \boldsymbol{\sigma}^e - p\mathbf{I}$, where p is a Lagrange multiplier accounting for the hydrostatic pressure and $\boldsymbol{\sigma}^e$ is the “extra stress” due to deformation of the

material. $\boldsymbol{\sigma}^e$ is also interpreted as the deviatoric stress, i.e., $\text{tr}(\boldsymbol{\sigma}^e) = 0$ [7]. It is noted for incompressible materials that hydrostatic stress $\boldsymbol{\sigma} = -p\mathbf{I}$ does not lead to any deformation. In other words, the hydrostatic pressure p must be determined from the boundary condition rather than the stress-strain relation. The second Piola-Kirchhoff stress $\mathbf{S}$, however, cannot be physically decomposed into hydrostatic stress and “extra stress”, since $\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{F}^{-1}\boldsymbol{\sigma}^e\mathbf{F}^{-T} - p\mathbf{C}^{-1}$ (note $J = 1$). This brings some complexities for modeling anisotropic incompressible materials in Lagrangian frame; e.g., the linear relationship between $\mathbf{D}$ and $\mathbf{S}$ proposed in our previous study [13] becomes problematic since it is difficult to ensure $\text{tr}(\mathbf{D}) = 0$ under arbitrary stress $\mathbf{S}$.

Here, we present a modified formulation in order to directly handle the hydrostatic pressure and material incompressibility, with introduction of a co-rotational Cauchy stress $\mathbf{T}$ [13], as

$$\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U} = \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R} = \mathbf{R}^T\boldsymbol{\sigma}^e\mathbf{R} - p\mathbf{I} = \mathbf{T}^e - p\mathbf{I}. \quad (3)$$

Although $\mathbf{T}$ is in Lagrangian frame (because both $\mathbf{U}$ and $\mathbf{S}$ are in Lagrangian frame), it preserves the decomposition for $\boldsymbol{\sigma}$; i.e., $\mathbf{T}^e$ is a deviatoric “extra stress” generated by material deformation:

$$\text{tr}(\mathbf{T}^e) = \text{tr}(\mathbf{R}^T\boldsymbol{\sigma}^e\mathbf{R}) = \text{tr}(\boldsymbol{\sigma}^e) = 0, \quad (4)$$

and the trace $\text{tr}(\mathbf{T})$ is exclusively contributed by the hydrostatic stress $-p\mathbf{I}$ which is independent of deformation.

Specifically for coronary arteries, it is found that the triaxial testing results [13] can be accurately described using a linear stress ($\mathbf{T}^e$)-strain ($\mathbf{D}$) relation, with appropriate choice of the scaling parameter n in Eqs. (1) & (2).

2.1.3 Linearized and Incompressible Strain-stress relation—It is proposed that the stress $\mathbf{T}$ and the log-exp strain $\mathbf{D}$ are connected by a constant fourth-order compliance tensor $\mathbf{M}$, as:

$$\mathbf{D} = \mathbf{M}:\mathbf{T} = \mathbf{M}:(\mathbf{T}^e - p\mathbf{I}). \quad (5)$$

The tensor $\mathbf{M}$ has major and minor symmetries, and must satisfy certain conditions such as $\mathbf{M}:\mathbf{I} = \mathbf{0}$ [25]; i.e., to ensure zero deformation ($\mathbf{D} = \mathbf{0}$) when the stress is hydrostatic ($\mathbf{T} = -p\mathbf{I}$). If we consider the orthotropic symmetry of vessel tissue, Eq. (5) can be expressed in a matrix form (generalized Hooke’s law):

$$\begin{pmatrix} D_{\theta\theta} \\ D_{zz} \\ D_{rr} \\ \sqrt{2}D_{zr} \\ \sqrt{2}D_{\theta r} \\ \sqrt{2}D_{\theta z} \end{pmatrix} = \begin{bmatrix} 1/E_\theta & -\nu_{\theta z}/E_\theta & -\nu_{\theta r}/E_\theta & 0 & 0 & 0 \\ -\nu_{\theta z}/E_\theta & 1/E_z & -\nu_{zr}/E_z & 0 & 0 & 0 \\ -\nu_{\theta r}/E_\theta & -\nu_{zr}/E_z & 1/E_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2G_{zr} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2G_{\theta r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2G_{\theta z} \end{bmatrix} \begin{pmatrix} T_{\theta\theta}^e \\ T_{zz}^e \\ T_{rr}^e \\ \sqrt{2}T_{zr}^e \\ \sqrt{2}T_{\theta r}^e \\ \sqrt{2}T_{\theta z}^e \end{pmatrix}, \quad (6a)$$

or

$$\{D\}=[M]\{T^e\} \quad (6b)$$

where θ , z , and r denote circumferential, axial, and radial directions. The parameters E_θ , E_z and E_r can be interpreted as Young's moduli in reference to the log-exp strains, $G_{\theta z}$, $G_{\theta r}$ and G_{zr} are shear moduli, and $\nu_{\theta z}$, $\nu_{\theta r}$ and ν_{zr} are Poisson's ratios. Except for the definition of strain and stress, this constitutive relation is identical to the classical generalized Hooke's law for orthotropic elastic materials. Note that the use of factor $\sqrt{2}$ in Eq. (6) is to preserve the tensorial properties in matrix presentation [25].

Unlike the general orthotropic materials where the six parameters E 's and ν 's in Eq. (6) are independent, the incompressibility condition $\mathbf{M}:\mathbf{I}=\mathbf{0}$ imposes restrictions [15, 16]

$$\nu_{ij}=\frac{E_i}{2}\left(\frac{1}{E_i}+\frac{1}{E_j}-\frac{1}{E_k}\right) \quad (ij=\theta z, \theta r, zr; k \neq i, j) \quad (7)$$

such that Eq. (6) is rewritten for vessel tissue as

$$\begin{pmatrix} \frac{2D_{rr}-D_{\theta\theta}-D_{zz}}{\sqrt{6}} \\ \frac{D_{zz}-D_{\theta\theta}}{\sqrt{2}} \\ \sqrt{2}D_{zr} \\ \sqrt{2}D_{\theta r} \\ \sqrt{2}D_{\theta z} \end{pmatrix} = \begin{bmatrix} \frac{3}{2E_r} & \frac{\sqrt{3}}{2}\left(\frac{1}{E_\theta}-\frac{1}{E_z}\right) & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2}\left(\frac{1}{E_\theta}-\frac{1}{E_z}\right) & \frac{1}{E_\theta}+\frac{1}{E_z}-\frac{1}{2E_r} & 0 & 0 & 0 \\ 0 & 0 & 1/2G_{zr} & 0 & 0 \\ 0 & 0 & 0 & 1/2G_{\theta r} & 0 \\ 0 & 0 & 0 & 0 & 1/2G_{\theta z} \end{bmatrix} \begin{pmatrix} \frac{2T_{rr}^e-T_{\theta\theta}^e-T_{zz}^e}{\sqrt{6}} \\ \frac{T_{zz}^e-T_{\theta\theta}^e}{\sqrt{2}} \\ \sqrt{2}T_{zr}^e \\ \sqrt{2}T_{\theta r}^e \\ \sqrt{2}T_{\theta z}^e \end{pmatrix} \quad (8a)$$

or

$$\{\widehat{D}\}=[\widehat{M}]\{\widehat{T}^e\} \quad (8b)$$

which involves six, instead of nine, material parameters: $\{E_\theta, E_z, E_r, G_{\theta z}, G_{\theta r}, G_{zr}\}$.

Together with $\text{tr}(\mathbf{D})=D_{\theta\theta}+D_{zz}+D_{rr}=0$ and $\text{tr}(\mathbf{T}^e)=T_{\theta\theta}^e+T_{zz}^e+T_{rr}^e=0$, Eqs. (8) are identical to Eqs. (6) with the advantage that the relation is invertible (note: matrix $[M]$ in Eqs. (6) is singular), i.e.,

$$\{\widehat{T}^e\}=[\widehat{M}]^{-1}\{\widehat{D}\} \quad (9)$$

which is convenient for strain-based computations.

Equilibrium equations and boundary conditions for finite-strain deformations are conventionally formulated in terms of second Piola-Kirchhoff stress $\mathbf{S}$ or Cauchy stress $\boldsymbol{\sigma}$. Given a deformation gradient $\mathbf{F}$, the log-exp strain $\mathbf{D}$ is calculated by Eqs. (1, 2), then the deviatoric co-rotational Cauchy stress $\mathbf{T}^e$ is calculated by Eq. (9) and condition $\text{tr}(\mathbf{T}^e)=0$, and finally $\mathbf{S}$ or $\boldsymbol{\sigma}$ is obtained by Eq. (3). For some deformation modes such as inflation-stretch experiment of vessels, the shear stress vanish, and the stress-strain relation Eqs. (8–9) can be further simplified as in Eq. (11).

2.2 Experiments and Parametric Identification

2.2.1 Triaxial torsion experiment—The experimental data are taken from a previously published work [19]. For completeness, the experimental measurements are briefly summarized here. Five pairs of vessel segments (3–4 cm long) were used for study of the left anterior descending (LAD) artery. The arterial bifurcations were identified and ligated to prevent leakage during pressurization. Each specimen was mounted to the triaxial torsion machine and tested in an organ bath at room temperature. The loading combines axial stretch, inflation and axial torsion. After the vessels were preconditioned (cyclically loaded to minimize the viscous response), short rings (1–2 mm) were cut from the proximal and distal LAD artery for measurement of the inner and outer radius r_i and r_o for no-load state (Fig. 1(a), with no axial stretch and internal pressure), radius R_i and R_o , inner and outer circumference C_i and C_o , and opening angle Φ for zero-stress (Fig. 1(b)) state [17].

During the experiment, the axial stretch λ_z was varied from 1.2 to 1.4 in increments of 0.1, the axial twist angle α was performed between -25° and 25° , and the internal pressure P^e (superscript e denotes *experiment*) was continuously varied from 0 to 200 mmHg at each fixed λ_z . Outer radius r_o^e of the deformed vessel, the axial force F_z^e and the torque M^e were recorded. Since some loading (e.g., large pressure and small stretch) can lead to a curved vessel, only data of straight segments were used for analysis. After the experiment for the intact artery, the adventitia layer was mechanically dissected and the media layer was tested using the same protocol. All the geometric and mechanical measurements were recorded as well.

As formulated in [20], the right Cauchy-Green deformation tensor $\mathbf{C}$ has the following nonzero components:

$$C_{rr}=\lambda_r^2, \quad C_{\theta\theta}=\lambda_\theta^2, \quad C_{zz}=\lambda_z^2+\xi_\theta^2, \quad C_{z\theta}=C_{\theta z}=\lambda_\theta\xi_\theta, \quad (10)$$

where λ_θ , λ_z , and λ_r are the local material stretches in the circumferential, axial, and radial directions, respectively, computed from the applied axial stretch λ_z , the deformed radius r_o^e and the zero-stress data R_i , R_o and Φ [17]. The torsion term ξ_θ is calculated by $\xi_\theta(r)=\alpha r/L_0$, where r is the current radius and L_0 is the initial (non-load) length of the specimen. In the log-exp strain (Eqs. 1, 2), $J_1=\text{tr}(\mathbf{C})=C_{rr}+C_{\theta\theta}+C_{zz}=\lambda_\theta^2+\lambda_z^2+\lambda_r^2+\xi_\theta^2$ for axial torsion deformation according to Eq. (10). The stress-strain relation is given by:

$$\begin{pmatrix} \frac{2D_{rr}-D_{\theta\theta}-D_{zz}}{\sqrt{6}} \\ \frac{D_{zz}-D_{\theta\theta}}{\sqrt{2}} \\ \sqrt{2}D_{\theta z} \end{pmatrix} = \begin{bmatrix} \frac{3}{2E_r} & \frac{\sqrt{3}}{2}\left(\frac{1}{E_\theta}-\frac{1}{E_z}\right) & 0 \\ \frac{\sqrt{3}}{2}\left(\frac{1}{E_\theta}-\frac{1}{E_z}\right) & \frac{1}{E_\theta}+\frac{1}{E_z}-\frac{1}{2E_r} & 0 \\ 0 & 0 & 1/2G_{\theta z} \end{bmatrix} \begin{pmatrix} \frac{2T_{rr}^e-T_{\theta\theta}^e-T_{zz}^e}{\sqrt{6}} \\ \frac{T_{zz}^e-T_{\theta\theta}^e}{\sqrt{2}} \\ \sqrt{2}T_{\theta z}^e \end{pmatrix}. \quad (11)$$

2.2.2 Identification of normal parameters—The material parameters involved in the stress-strain relation Eq. (11), $\{n, E_\theta, E_z, E_r\}$, were first fitted from stretch-inflation data where the axial twist angle is $\alpha = 0$. As formulated in [2, 6, 13, 21], the complete axisymmetric Cauchy stress field $\boldsymbol{\sigma}$ ($=\mathbf{T}^e - p\mathbf{I}$ noting that $\mathbf{R} = \mathbf{I}$ here) was solved by integrating the Eulerian equilibrium equation in radial direction and taking into account the stress-free condition on the outer surface of the vessel. The *calculated* (denoted by superscript c) internal pressure $P^c(r_o^e, \lambda_z)$ and axial force $F_z^c(r_o^e, \lambda_z)$ were obtained by integration across the vessel wall as described in [17]:

$$P^c = \int_{r_i^e}^{r_o^e} \frac{\sigma_{rr}^e - \sigma_{\theta\theta}^e}{e} dr, \text{ and } F_z^c = \int_{r_i^e}^{r_o^e} 2\pi\sigma_{zz}^e r dr + \pi P^c (r_i^e)^2, \quad (12)$$

where σ_{rr}^e and $\sigma_{\theta\theta}^e$ are the components of the “extra stress” $\boldsymbol{\sigma}^e$ calculated by $\boldsymbol{\sigma}^e = \mathbf{R}\mathbf{T}^e\mathbf{R}^T$, and are detailed in the Appendix. The second relation is due to the fact that the axial force measured by force transducer includes contribution from both the tissue and the pressure [17]. Similar to the previous works [17, 22], we searched for $\{n, E_\theta, E_z, E_r\}$ that minimizes an error function given by:

$$err1(n, E_\theta, E_z, E_r) = \frac{\sum [(P^c - P^e)^2 + w_F (F_z^c - F_z^e)^2 / A_l^2]}{\sum (P^{e2} + w_F F_z^{e2} / A_l^2)}, \quad (13)$$

where summation is conducted over all stretch-inflation data points, A_l is the luminal area (calculated from r_o^e and the zero-stress geometry using incompressible condition). We found that $w_F = 0.4$ gave overall good fitting results. In all measurements and calculations, the unit was kPa for pressure, mN for the axial force, and mm^2 for the luminal area. It should be noted that we used a combined fitting error here, while in Wang et al [22] the fitting errors for pressure and force were treated separately. A limited-memory Broyden-Fletcher-Goldfarb-Shanno (L-BFGS) quasi-Newton minimization method [23] was used for the data fitting.

2.2.3 Identification of shear parameters—The shear parameter $G_{\theta z}$ was identified from torsion data with twist angle $\alpha \neq 0$. Given a trial $G_{\theta z}$, the deviatoric stress $\mathbf{T}^e$ was calculated via Eq. (11), and the deviatoric Cauchy stress $\boldsymbol{\sigma}^e$ was obtained as $\boldsymbol{\sigma}^e = \mathbf{R}\mathbf{T}^e\mathbf{R}^T$. The total torque M^c was then calculated as

$$M^c = 2\pi \int_{r_i^e}^{r_o^e} \sigma_{\theta z}^e r^2 dr \quad (14)$$

where r_o^e and r_i^e are the outer and inner radius of the deformed vessel. It is noted that the shear stress $\sigma_{\theta z}$ is coupled with the normal stresses, as indicated by the definition of $\mathbf{D}$ and that $\mathbf{R} \neq \mathbf{I}$ in torsion. Therefore, $\boldsymbol{\sigma}^e$ depends on normal parameters $\{n, E_\theta, E_z, E_r\}$. Finally, we identified $G_{\theta z}$ by minimizing an error function:

$$err2(G_{\theta z}) = \frac{\sum (M^c - M^e)^2}{\sum M^{e2}}. \quad (15)$$

3 Results

The fitted material parameters are listed in Table 1 of five pairs of experimentally measured LAD specimens: intact vessels LAD 1–LAD 5 and the media layers LAD-M 1–LAD-M 5. Figure 2 shows the plots of outer radius r_o and internal pressure of intact vessel LAD 5 and media LAD-M 5, respectively, from experimental measurements P^e (symbols) and theoretical predictions P^c (curves) that were obtained by solving boundary value problems with the fitted material constants. It can be seen that the agreement between theory and experiment is excellent. Figure 3 shows a comparison of the measured axial force F_z^e and prediction F_z^c under various axial stretch and pressure loading combinations. The general

trend is satisfactory considering that the measured force (small magnitude, more sensitive to loading change) is not as accurate as the pressure data.

The media layer has a larger nonlinearity parameter n than the corresponding intact vessel, i.e., 1.93 ± 0.38 vs. 1.44 ± 0.18 . This implies that the load in the media will increase faster than that in the overall vessel in the tested range. The Young's moduli E_θ , E_z and E_r are all found to be positive. The fitting errors defined in Eq. (13) are generally less than 5% which are excellent when considering the variance of biological data. For both intact vessel and media, the Poisson's ratio $\nu_{\theta r}$ and ν_{zr} are positive, while $\nu_{\theta z}$ is negative for all specimens (Table. 1) in the range of -0.1 to -0.25 . This negative Poisson's ratio is consistent with experimental observation that the vessel tends to stretch axially with increase of pressure ($T_{\theta\theta} > 0$ and increases), and may buckle at high pressure.

As given in Table 1, the shear parameter $G_{\theta z}$ is fitted with slightly higher error and the value shows higher variation among the vessels in comparison to the Young's moduli. Figure 4 shows a comparison of the measured torque M^e and prediction M^c as function of $J\alpha/L$ [19], where J is the polar moment of inertia of the deformed vessel, α is the torsion angle (in rad), and L is the deformed length. Under various axial stretch (shown with $\lambda_z = 1.3$ in the figure) and pressure loading combinations, the general trend of the experimental data is satisfactorily predicted, considering that the measurement is not as accurate as the pressure-radius data.

Discussion

For a linear orthotropic material with small deformation, the number of independent parameters due to incompressibility and the corresponding admissibility of elastic constants have been studied [15, 16]. We extended these results to large deformation of arteries using a linearized constitutive model. The new formulation is a modified version of the generalized Hooke's law in [13]. The major advantage of the present model lies in the fact that it requires fewer independent parameters (only four without shear moduli). The volumetric incompressibility constraint is used to eliminate three redundant elastic constants, which typically cannot be done in a nonlinear constitutive law due to mathematical complexity. For example, Fung's model of blood vessels [1, 2, 4, 5] requires eight material parameters to describe the triaxial torsion stress-strain relation, compared to four in the present model. One advantage is the narrower range of admissible parameter values and hence more deterministic material constants. Written in a linearized form, the arterial stress-strain relation is easier to understand and many well developed concepts for the generalized Hooke's law can be inherited.

As compared with our previous generalized Hooke's law [13], the present model rigorously takes into consideration the material incompressibility, and fits the experimental data as well as former models. We expect that the nonlinearity parameter n in Eq. (1) is slightly larger than that in the model of [13], since the co-rotational Cauchy stress $\mathbf{T}$ is slightly larger than the second Piola-Kirchhoff stress $\mathbf{S}$ under physiological loading (stretch ratios in circumferential and axial directions are larger than one).

The arterial wall is known to be heterogeneous and may be treated mechanically as a two-layer composite made of intima-media and adventitia [4, 6, 17–19]. We fitted our model to the media data and found that the nonlinearity of porcine LAD media might be larger than that of the intact wall (Tables 1). Further experiments and measurements are needed to confirm this observation.

The shear and normal strain components are coupled in this model, as in Eqs. (1, 10), such that the radial and axial Cauchy stresses σ_{rr} and σ_{zz} change with the torsion angle. This

explains experimental observation that the vessel radius and axial force slightly change in torsion test under constant pressure. Unlike our previous data analysis that assumed constant radius in torsion [20], the present data fitting takes into account the change, and the resulting shear parameter better describes the vessel behavior. There have been histological structure-based models to explain the coupling of distension and torsion by introducing two intercrossed helical fiber families in the vessel tissue [26, 27]. Some microscopy images of porcine LAD, however, do not show directly intercrossed helical fiber families [28]. The explanation of the coupling effect requires further consideration of the cells and ground substance that connect the fibers.

The physical meaning of the negative Poisson's ratio $\nu_{\theta z}$ found for the vessel specimens is that the circumferential extension of the vessel tissue ($D_{\theta\theta} > 0$, or equivalently the Green strain $E_{\theta\theta} > 0$) is accompanied by axial elongation ($D_{zz} > 0$, or equivalently $E_{zz} > 0$). When the axial displacement is confined, such as in the triaxial experiments where the vessels were ligated onto a cannula [19], large circumferential extension results with an axial compression stress that causes buckling of the vessel. The buckling criterion is found to depend on the axial stretch ratio. In general, vessels subjected to larger axial stretch buckle at higher pressure. For example, for porcine LAD vessels axially stretched to or beyond the physiological stretch ratio ($\lambda = 1.3$ – 1.4), buckling is observed at high pressure, typically >150 mmHg. When the vessels are slightly stretched, a much lower pressure, ~ 40 mmHg for LAD with $\lambda < 1.1$, may cause buckling. The exact pressure to initiate buckling, however, is difficult to determine as it is affected by many factors such as small non-symmetry of the vessel. Therefore, the negative $\nu_{\theta z}$ is a result of the inherent properties of the vascular tissue determined by the internal microstructures. In fact, negative Poisson's ratio, while is uncommon for conventional materials, exists for composite materials with special micro-architectures [30–33], such as the re-entrant foam that has a Poisson's ratio of -0.7 [30]. Because of the complex and anisotropic fiber-cell microstructures in the vascular tissue, the negative Poisson's ratio is not physically excluded.

A limitation of the model is that unconventional stress and strain measures are used. It may require some extra computation to convert these measures to conventional ones, or vice versa. Another limitation is that a strain energy (or hyperelastic potential) may not exist for this kind of stress-strain relationship; i.e., we discarded the hyperelasticity assumption. Regardless, a nominal strain potential could be utilized to conveniently write the constitutive law in a scalar form [13]. The lack of explicit form for the strain energy may be a limitation for application in computational methods that are developed based on energy principles. On the other hand, major finite element computational software are derived based on the principle of virtual work that only requires user-supplied stress-strain relation and the tangent stiffness tensor, which are all provided for the proposed model. For example, we have embedded the proposed model into ANSYS and FEAP and conducted simulation of angioplasty in LAD [29]. The lack of energy form also makes it difficult to investigate the material stability of the model, such as polyconvexity or rank-one convexity. On the other hand, the model has been found to be stable in finite-element simulations [29] where the arteries deform within the typical in vivo range. Therefore, the proposed model is a simpler alternative to existing models of coronary arteries.

5 Conclusions

In summary, we proposed a generalized Hooke's law for blood vessels. The number of independent model parameters is reduced by three when the incompressibility constraint is enforced. This linearized constitutive relation for large deformations resembles the classic Hooke's law for small strains. Therefore, it is convenient to use developed methods to analyze the interconnections between material constants introduced by incompressibility

constraint and to examine the admissible range of model parameters. Although only experimental data of porcine coronary arteries were used to extract elastic constants, the model can be extended to other materials that have similar mechanical properties.

Acknowledgments

This research was supported in part by the National Institute of Health-National Heart, Lung, and Blood Institute Grant 1 R01 HL087235-01 and 2 R01 HL055554-12.

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APPENDIX

A. Deformation

As detailed in our previous work [20], the triaxial deformation gradient $\mathbf{F}$ and the right Cauchy-Green deformation tensor $\mathbf{C}$ of the vessel with respect to the zero-stress state are

$$\mathbf{F} = \begin{pmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & \xi_\theta \\ 0 & 0 & \lambda_z \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 & \lambda_\theta \xi_\theta \\ 0 & \lambda_\theta \xi_\theta & \lambda_\theta^2 + \xi_\theta^2 \end{pmatrix} \quad \text{in coordinate } \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\} \quad (\text{A1.1})$$

where $\xi_\theta = \lambda_z a r \cdot L^{-1}$, with r being the current radius of the material point, and L the current length of the vessel. The eigenvalues $\{\Lambda_i\}$ and of eigenvectors $\{\mathbf{u}_i\}$ ($i=1,2,3$) of $\mathbf{C}$ are calculated numerically, as $\mathbf{C} = \sum_{i=1,3} \Lambda_i \mathbf{u}_i \otimes \mathbf{u}_i$. Then, spectral decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ is conducted, as:

$$\mathbf{U} = \sum_{i=1,3} \sqrt{\Lambda_i} \mathbf{u}_i \otimes \mathbf{u}_i, \text{ and } \mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{F} \cdot \left(\sum_{i=1,3} (\sqrt{\Lambda_i})^{-1} \mathbf{u}_i \otimes \mathbf{u}_i \right) \quad (\text{A1.2})$$

Finally, the logarithmic-exponential strain tensor $\mathbf{D}$ defined in Eqs. (1,2) is:

$$\mathbf{D} = \frac{1}{2} Q \ln \mathbf{C} = \exp[n(\Lambda_1 + \Lambda_2 + \Lambda_3 - 3)] \left(\sum_{i=1,3} \ln \Lambda_i \mathbf{u}_i \otimes \mathbf{u}_i \right) \quad (\text{A1.3})$$

B. Stress

It is immediately solve from Eq. (11) that together with condition $\text{tr}(\boldsymbol{\sigma}^e) = 0$ that

$$\begin{pmatrix} T_{\theta\theta}^e \\ T_{zz}^e \\ T_{rr}^e \\ \sqrt{2}T_{\theta z}^e \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \frac{3}{2E_r} & \frac{\sqrt{3}}{2} \left(\frac{1}{E_\theta} - \frac{1}{E_z} \right) & 0 \\ \frac{\sqrt{3}}{2} \left(\frac{1}{E_\theta} - \frac{1}{E_z} \right) & \frac{1}{E_\theta} + \frac{1}{E_z} - \frac{1}{2E_r} & 0 \\ 0 & 0 & 1/2G_{\theta z} \end{bmatrix}^{-1} \begin{pmatrix} \frac{2D_{rr} - D_{\theta\theta} - D_{zz}}{\sqrt{6}} \\ \frac{D_{zz} - D_{\theta\theta}}{\sqrt{2}} \\ \sqrt{2}D_{\theta z} \end{pmatrix} \quad (\text{A2.1})$$

B.1. Stress components in axisymmetric deformation (Eq. 12)

In axisymmetric deformation, $\mathbf{R} = \mathbf{I}$ such that $\boldsymbol{\sigma}^e = \mathbf{T}^e$. Therefore,

$$\begin{pmatrix} \sigma_{rr}^e \\ \sigma_{\theta\theta}^e \\ \sigma_{zz}^e \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{bmatrix} \frac{3}{2E_r} & \frac{\sqrt{3}}{2} \left(\frac{1}{E_\theta} - \frac{1}{E_z} \right) \\ \frac{\sqrt{3}}{2} \left(\frac{1}{E_\theta} - \frac{1}{E_z} \right) & \frac{1}{E_\theta} + \frac{1}{E_z} - \frac{1}{2E_r} \end{bmatrix}^{-1} \begin{pmatrix} \frac{2D_{rr} - D_{\theta\theta} - D_{zz}}{\sqrt{6}} \\ \frac{D_{zz} - D_{\theta\theta}}{\sqrt{2}} \end{pmatrix} \quad (\text{A2.2})$$

The integration of the radial equilibrium $d\sigma_{rr}/dr + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0$, subjected to boundary conditions $\sigma_{rr} = 0$ on the external surface of the vessel ($r = r_{\text{out}}$), gives

$$\sigma_{rr}(r) = \int_{r_{\text{out}}}^r \frac{1}{s} (\sigma_{rr}^e(s) - \sigma_{\theta\theta}^e(s)) ds \quad (\text{A2.3})$$

noting that $\sigma_{rr} - \sigma_{\theta\theta} = \sigma_{rr}^e - \sigma_{\theta\theta}^e$. In addition, the axial stress σ_{zz} can be derived, as:

$$\sigma_{zz} = \sigma_{rr} + \sigma_{zz}^e - \sigma_{rr}^e \quad (\text{A2.4})$$

B.2. Stress components in axial torsion (Eq. 14)

In axial torsion, the rigid rotation tensor $\mathbf{R} \neq \mathbf{I}$, therefore, the shear stress $\sigma_{\theta z}$ involves all components in $\mathbf{T}^e$ through relation $\boldsymbol{\sigma}^e = \mathbf{R} \mathbf{T}^e \mathbf{R}^T$. Specifically, the shear stress in Eq. (14) is:

$$\sigma_{\theta z} = \sigma_{\theta z}^e = R_{\theta\theta} R_{z\theta} T_{\theta\theta}^e + (R_{\theta z} R_{z\theta} + R_{\theta\theta} R_{zz}) T_{\theta z}^e + R_{\theta r} R_{zr} T_{rr}^e + R_{\theta z} R_{zz} T_{zz}^e \quad (\text{A.2.5})$$

Highlights

- A generalized Hooke's stress-strain model has been developed for vessel wall.
- The co-rotational Cauchy stress and log-exponential strain shows linear relationship.
- The model requires fewer material parameters than Fung's model for vessels.
- The model fits triaxial test data of porcine coronary arteries with four parameters.

Fig. 1.

A vessel at (a) no-load and deformed state; and (b) zero-stress state. r_i and r_o , R_i and R_o , are the inner and outer radii in the (r, θ, z) and (R, Θ, Z) coordinate systems, respectively. Φ denotes the opening angle.

Fig. 2. Outer radius r_o and internal pressure P at axial stretches $\lambda_z = 1.2$ and 1.4 for intact vessel LAD-5 (a) and media LAD-M 5 (b). Symbols are experimental data points [19], solid curves are computed using the fitted material parameters.

Fig. 3. Axial force F_z and internal pressure P at axial stretches $\lambda_z = 1.2$ and 1.4 for intact vessel LAD-5 (a) and media LAD-M 5 (b). Symbols are experimental data points [19], solid curves are computed using the fitted material parameters.

Fig. 4.

Torque M and torsion angle α at axial stretches $\lambda_z = 1.3$ for intact vessel LAD-5 (a) and media LAD-M 5 (b). J is the polar moment of inertia of the deformed vessel, L is the deformed length. Symbols are experimental data points [19], solid curves are computed using the fitted material parameters.

Table 1

Material parameters of porcine left anterior descending arteries (LAD). M refers to media layer of the vessel. n is nonlinearity parameter, E_θ , E_z and E_r (in kPa) are Young's moduli, $G_{\theta z}$ is shear modulus, $\nu_{\theta z}$, $\nu_{\theta r}$ and ν_{zr} are Poisson's ratios, $err1$ and $err2$ (in %) are fitting error as defined in Eqs. (13) and (16), $R^2_{F_z}$, R^2_p and R^2_M are the R^2 values of axial force, pressure and torque.

Intact Vessel	n	E_θ	E_z	E_r	$G_{\theta z}$	$\nu_{\theta z}$	$\nu_{\theta r}$	ν_{zr}	$err1$	$err2$	$R^2_{F_z}$	R^2_p	R^2_M
LAD 1	1.26	30.50	43.07	15.49	6.45	-0.13	1.13	1.26	3.32	4.66	0.95	0.95	0.91
LAD 2	1.55	18.97	32.29	9.06	5.72	-0.25	1.25	1.55	2.25	4.13	0.97	0.96	0.92
LAD 3	1.26	28.43	35.94	12.43	8.56	-0.25	1.25	1.26	4.22	2.50	0.93	0.95	0.95
LAD 4	1.68	32.06	40.75	15.57	4.16	-0.14	1.14	1.68	2.94	1.87	0.97	0.98	0.97
LAD 5	1.43	21.88	35.35	10.27	5.27	-0.26	1.26	1.43	2.08	3.48	0.98	0.98	0.94
mean	1.44	26.37	37.48	12.56	6.04	-0.20	1.20	1.44	-	-	-	-	-
SD	0.18	5.67	4.35	2.97	1.65	0.07	0.07	0.18	-	-	-	-	-
Media													
LAD-M 1	1.96	10.86	13.76	4.95	4.62	-0.20	1.20	1.25	1.93	2.33	0.99	0.99	0.96
LAD-M 2	1.52	20.34	22.82	9.65	5.66	-0.11	1.11	1.12	4.31	3.08	0.94	0.93	0.95
LAD-M 3	1.88	17.61	20.25	7.82	3.03	-0.19	1.19	1.22	4.22	3.87	0.95	0.93	0.95
LAD-M 4	1.75	39.64	32.26	16.88	6.74	-0.06	1.06	1.05	5.26	4.35	0.93	0.91	0.93
LAD-M 5	2.54	10.42	13.86	5.17	3.28	-0.13	1.13	1.18	4.02	4.66	0.94	0.92	0.92
mean	1.93	19.77	20.59	8.90	4.67	-0.14	1.14	1.16	-	-	-	-	-
SD	0.38	11.90	7.64	4.87	1.57	0.06	0.06	0.08	-	-	-	-	-