An empirical model of diagnostic x-ray attenuation under narrow-beam geometry

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Purpose: The purpose of this study was to develop and validate a mathematical model to describe narrow-beam attenuation of kilovoltage x-ray beams for the intended applications of half-value layer (HVL) and quarter-value layer (QVL) estimations, patient organ shielding, and computer modeling.

Methods: An empirical model, which uses the Lambert W function and represents a generalized Lambert-Beer law, was developed. To validate this model, transmission of diagnostic energy x-ray beams was measured over a wide range of attenuator thicknesses [0.49–33.03 mm Al on a computed tomography (CT) scanner, 0.09–1.93 mm Al on two mammography systems, and 0.1–0.45 mm Cu and 0.49–14.87 mm Al using general radiography]. Exposure measurements were acquired under narrow-beam geometry using standard methods, including the appropriate ionization chamber, for each radiographic system. Nonlinear regression was used to find the best-fit curve of the proposed Lambert W model to each measured transmission versus attenuator thickness data set. In addition to validating the Lambert W model, we also assessed the performance of two-point Lambert W interpolation compared to traditional methods for estimating the HVL and QVL [i.e., semi-logarithmic (exponential) and linear interpolation].

Results: The Lambert W model was validated for modeling attenuation versus attenuator thickness with respect to the data collected in this study ($R^2 > 0.99$). Furthermore, Lambert W interpolation was more accurate and less sensitive to the choice of interpolation points used to estimate the HVL and/or QVL than the traditional methods of semilogarithmic and linear interpolation.

Conclusions: The proposed Lambert W model accurately describes attenuation of both monoenergetic radiation and (kilovoltage) polyenergetic beams (under narrow-beam geometry). © 2011 American Association of Physicists in Medicine. [DOI: 10.1118/1.3592933]

Key words: CT, mammography, radiography, polyenergetic, half-value layer

I. INTRODUCTION

The Lambert-Beer law describes attenuation of a monoenergetic beam (composed of photons of a single energy) and is represented as

$$I = I_0 e^{-\mu x}, \quad (1)$$

where I is the intensity of photons transmitted through an attenuating material of thickness x , μ is the linear attenuation coefficient, and I_0 is the unattenuated intensity. In a polyenergetic beam, the energy spectra of the photons changes as the beam passes through an attenuating material; as a result, the linear attenuation coefficient is not a constant value but

rather varies with beam attenuation.¹ Because x-rays are polyenergetic, attenuation of a clinical x-ray beam cannot be accurately modeled by Eq. (1).

Development of an equation to describe x-ray beam attenuation may benefit several applications, one of which is half-value layer (HVL) and quarter-value layer (QVL) estimations. The HVL is defined as the attenuator thickness that reduces the measured exposure to exactly one-half of the unattenuated exposure; the HVL is the most common metric used to describe the quality of an x-ray beam and is therefore routinely estimated during quality assurance (QA) testing.^{1,2} Similarly, the QVL is defined as the attenuator thickness that reduces measured exposure to exactly one-quarter of the

unattenuated exposure. Although estimation of the QVL is not as routine as that of the HVL, it is necessary for determining the homogeneity coefficient, which is used to characterize the polyenergetic nature of an x-ray beam.² Additionally, some Monte Carlo computer models use both the HVL and QVL to construct virtual computed tomography (CT) scanners.³ Typically, two-point interpolation is performed to estimate the HVL or QVL from two measured data points $[(x_1, I_1), (x_2, I_2)]$ bracketing the HVL or QVL; that is, with one thickness (x_1) less than and one thickness (x_2) greater than the HVL or QVL.⁴ Alternatively, two-point extrapolation uses two thicknesses (x_1 and x_2), which are both less than or both greater than the HVL or QVL. Traditionally, either two-point semilogarithmic (i.e., exponential) or linear interpolation has been used for HVL and QVL estimations, despite the fact that neither function accurately describe attenuation in radiographic imaging. The measurements used to estimate the HVL or QVL are gathered under narrow-beam geometry, which refers to detection of photons from the primary beam and does not take into account the contribution of scatter; conversely, broad-beam geometry refers to radiation that is outside the primary beam (i.e., scatter).²

Four mathematical models for describing attenuation of megavoltage x-ray beams (two of which were evaluated in this study) were discussed by Kleinschmidt,⁵ who suggested that a function ($\bar{\mu}$) can be written to describe the mean attenuation coefficient of a polyenergetic beam. The model that Kleinschmidt determined to be optimal was excluded from our analysis because it contained three unknown variables and thus conflicted with the objective of this study, which was to develop an attenuation model that could be used to estimate the HVL and QVL by two-point interpolation. The first of the two mean attenuation coefficient functions ($\bar{\mu}$) discussed by Kleinschmidt and considered in our analysis was developed by Bjärngard and Shackford⁶ as:

$$\bar{\mu} = \mu_0 - \lambda x, \quad (2)$$

where μ_0 and λ are unknown parameters. The second function we considered was proposed by Yu *et al.*⁷ as an alternative to Eq. (2),

$$\bar{\mu} = \frac{\mu_0}{1 + \lambda x}, \quad (3)$$

with all variables defined as above.

Because Eqs. (2)–(3) were intended for use in describing high-energy (megavoltage) photon beams, an empirical model for $\bar{\mu}$ was proposed for diagnostic-energy (kilovoltage) x-ray beams with the intent that the model be utilized to describe narrow-beam attenuation through homogeneous materials typically used for HVL and QVL estimations:

$$\bar{\mu} = \mu_0 + \lambda \left(\frac{I}{I_0} \right). \quad (4)$$

Replacing constant μ by function $\bar{\mu}$ [i.e., Eq. (4)], Eq. (1) can be rewritten as (see Appendix A for derivation):

$$I = I_0 \frac{w(\lambda x e^{-\mu_0 x})}{\lambda x}, \quad (5)$$

where I , x , and I_0 are defined as in Eq. (1) and W refers to the Lambert W function. The Lambert W function has been shown to have several applications in physics⁸ and is defined as the multivalued inverse of the function

$$W(z)e^{W(z)} = z, \quad (6)$$

where z is a complex number.⁹ By L'Hopital's rule, Eq. (5) is reduced to Eq. (1) when used to model attenuation of a monoenergetic beam ($\mu_0 = \mu$ and $\lambda = 0$). Therefore, Eq. (5) represents a more general form of the Lambert–Beer law [Eq. (1)].

Although Eqs. (2) and (3) were not originally developed for or validated under the conditions of this study (i.e., kilovoltage x-ray beam attenuation), there were no constraints included in the publications describing their derivation that would specifically exclude them from being applied to this type of data. Therefore, the following attenuation models [Eqs. (7) and (8)] were derived [in the same manner as Eq. (5)] by substituting Eqs. (2) and (3), respectively, into Eq. (1):

$$I = I_0 e^{-\mu_0 x + \lambda x^2}, \quad (7)$$

$$I = I_0 e^{-\mu_0 x / (1 + \lambda x)}. \quad (8)$$

The purpose of this study was to validate the proposed Lambert W model [Eq. (5)] using transmitted exposure versus attenuator thickness data obtained (under narrow-beam geometry) on different radiographic equipment, including CT, general radiography, and two different mammography systems. Validating the model across several radiographic imaging modalities, which have different inherent filtration and thus different x-ray beam spectra, ensures that its clinical usefulness is not limited to a specific modality. Versatility of the proposed model was evaluated by collecting data over a range of conditions, including two peak kilovoltages (kVps) per modality and two attenuating materials (aluminum and copper) for general radiography. After validating the Lambert W model over a broad range of conditions, we evaluated the accuracy of using two-point Lambert W interpolation to estimate the HVL and QVL compared to accepted methods (i.e., semilogarithmic and linear interpolation) and also to interpolations based on the models described by Yu *et al.* and Bjärngard and Shackford.

II. MATERIALS AND METHODS

II.A. Data collection

Exposure measurements were collected through varying thicknesses of attenuating materials on different radiographic imaging equipment (general radiography, mammography, and CT). Two unattenuated exposure measurements (I_0) were initially collected and averaged for every data set. Data points were recorded as measured exposure (I) versus attenuator thickness (x). On all machines tested, exposure was detected (under narrow-beam geometry) using an exposure meter (RadCal Corporation, Monrovia, CA) connected to a RadCal ionization chamber (6-cm³ general purpose chamber for general radiography, 6-cm³ dedicated mammography chamber for mammography, and 3-cm³ pencil chamber for

TABLE I. Details of the experimental conditions under which exposure versus attenuator thickness data were collected for general radiography, on two models of mammography systems (Hologic Selenia and GE Senographe 2000D), and using a computed tomography scanner.

Radiographic imaging modality	Equipment model	Measurement details ^a	kVp ^b	Range of attenuator thicknesses ^c (mm)	Number of data points collected
General radiography	GE Definium	Cu filters	80	0.1–0.45	7
			120	0.1–0.45	7
		Al 1100 filters	80	0.49–14.87	31
			120	0.49–14.87	31
Mammography	Hologic Selenia	Standard paddle	26	0.1–0.79	8
			35	0.1–1.08	11
	GE Senographe 2000D	Standard paddle	26	0.09–1.75	19
			35	0.09–1.93	21
		Fenestrated paddle	26	0.09–1.84	20
Computed tomography	GE LightSpeed QX/i	N/A	80	0.49–20.28	41
			120	0.49–33.03	68

^aA 6-cm³ ionization chamber was used to collect general radiography exposure data; data were collected using both type 1100 aluminum alloy (Al 1100) and pure copper (Cu) attenuators. Exposure measurements for all of the mammography data sets were made with a dedicated mammography ionization chamber; data were collected on the mammography GE Senographe 2000D system using two different breast compression paddles. The experimental setup used to collect CT data, which uses a 10-cm long pencil ionization chamber to detect exposure, is shown in Fig. 1.

^bBoth mammography systems use Mo/Mo at 26 kVp and Mo/Rh (Hologic Selenia) or Rh/Rh (GE Senographe 2000D) at 35 kVp.

^cFor all data sets, average unattenuated exposure (I_0) was first determined and then either Al 1100 or Cu (for general radiography only) filters were added. The Al 1100 attenuators for the general radiography, mammography, and CT data sets were added in approximately 0.5, 0.1, and 0.5-mm increments, respectively, within the specified attenuator thickness range; the number of attenuator thicknesses tested within this range is equal to the number of exposure versus thickness data points collected.

CT). Attenuator thicknesses were measured using a digital caliper (with 0.01-mm accuracy) for all data sets.

II.A.1. General radiography

To detect exposure from general radiography [Definium; General Electric (GE) Healthcare, Waukesha, WI], a 6-cm³ ionization chamber was connected to a RadCal exposure meter. A tabletop x-ray configuration was used and the chamber was held in place by a RadCal stand. The x-ray field size was collimated to the size of the chamber and the Definium's built-in laser lights were employed to center the chamber within the field. These standard procedures were used to measure exposure at 80 and 120 kVp using type 1100 aluminum alloy (Al 1100) and pure copper filters.¹⁰ Table I summarizes the methods used, including details regarding the attenuators.

II.A.2. Mammography

A RadCal exposure meter was connected to a mammography chamber to measure exposure on Selenia (Hologic, Inc., Danbury, CT) and Senographe 2000D (GE Healthcare) digital mammography units. Standard methods were used to collect exposure readings at 26 kVp [molybdenum (Mo)/Mo target/filter combination for both systems] and at 35 kVp [Mo/rhodium (Rh) for the Hologic Selenia system and Rh/Rh for the GE Senographe 2000D system].⁴ Additionally, although QA testing standards typically specify presence of the compression paddle in attenuation-related measurements, one data set was collected in this study without the attenuation contribution from the paddle. To accomplish this, we used a fenestrated paddle, which featured a square opening or “window” (designed for needle localization), to hold the attenuators and then focused the

beam to an area within this window. Table I lists the Al 1100 thickness ranges tested on each mammography unit at each kVp and also summarizes other relevant experimental information.

II.A.3. Computed tomography

The experimental setup shown in Fig. 1 was used to gather exposure versus attenuator thickness data on a LightSpeed QX/i CT scanner (GE Healthcare). Following

FIG. 1. The experimental setup used to collect data on a GE LightSpeed QX/i computed tomography scanner. With the x-ray tube held stationary at the bottom of the gantry, varying thicknesses of Al 1100 filters (indicated by the arrow) were placed over the x-ray tube output port and transmitted exposure was detected with a 10-cm pencil ionization chamber (circled).

standard methods, the x-ray tube was positioned at the bottom of the scanner gantry and held stationary at that location throughout testing; sheets of Al 1100 were placed at the bottom of the gantry over the Mylar window so that they directly covered the x-ray tube output port.¹¹ Exposure readings were measured at 80 and 120 kVp (20-mm nominal beam width, large focal spot size, and “body” scan field of view) using a 10-cm pencil ionization chamber positioned free-in-air at gantry isocenter. The thicknesses of added aluminum, along with other information related to the methods, appear in Table I.

II.B. Data analysis

The proposed Lambert W model [Eq. (5)] was validated (within the context of this study) as an acceptable means of modeling attenuation of diagnostic x-ray beams. Additionally, the model’s fit to the measured transmission data was compared to that of the monoenergetic Lambert-Beer law [Eq. (1)] and also to that of the polyenergetic attenuation models described by Eqs. (7) and (8). After validating and comparing Eq. (5) to other attenuation models, the accuracy of using two-point interpolation based on the Lambert W function to estimate the HVL and QVL was assessed. The Lambert W estimates were compared to estimates from the traditional methods of semilogarithmic and linear interpolation and also to estimates from interpolations based on Eqs. (7) and (8).

II.B.1. Validation of Lambert W model

Prior to validating the Lambert W model [Eq. (5)], we will justify the reasoning behind its development. Since Eq. (4) was used to derive Eq. (5) and thus is the basis of the Lambert W model, evidence in support of a linear relationship between $\bar{\mu}$ and (I/I_0) (with slope λ and y-intercept μ_0) could provide such justification. However, because $\bar{\mu}$ was not quantified in this study, we could not directly test for the presence of such a linear relationship. Instead, we rearranged Eq. (B2) [derived from the substitution of Eq. (4) into Eq. (1), as shown in the appendix] such that the right side of the equation matched that of Eq. (4) and $\bar{\mu}$ was replaced by a term which used the data at hand:

$$\frac{-\ln\left(\frac{I}{I_0}\right)}{x} = \mu_0 + \lambda \left(\frac{I}{I_0}\right). \quad (9)$$

For every data set, $[-\ln(I/I_0)]/x$ versus (I/I_0) was plotted and goodness of fit was judged through least squares linear regression; Fig. 2 shows one such plot along with the corresponding regression analysis. However, because these plots were generated by transforming the data to create a linear graph, some assumptions of linear regression were violated.¹² Additionally, transmission (I/I_0) , which was the dependent variable of the measured data, appears in both the independent and dependent variable terms in Eq. (9). Therefore, while it was inappropriate to interpret results of the linear regression analysis, the plots were used to visually assess

FIG. 2. Scatter plot of $[-\ln(I/I_0)]/x$ versus (I/I_0) for general radiography data collected on a GE Definium at 120 kVp using Al 1100 filters. The calculated R^2 value of 0.9969 indicates a close fit of the linear trendline (which has the following equation: $[-\ln(I/I_0)]/x = 0.09 + 0.12 \cdot [(I/I_0)]$) to the data, thus providing support in favor of Eq. (9) and indirectly justifying the validity of Eq. (4). Note that the R^2 value and the estimates of μ_0 and λ listed here differ from those determined through nonlinear regression even though the same data were analyzed [for the reasons explained in Sec. II B 1]. Table II in the supplemental data¹⁷ lists the R^2 value for this data set as 0.9999, as well as estimates of μ_0 and λ determined through nonlinear regression based on the Lambert W model [Eq. (5)].

the linear relationship described by Eq. (9) as a means of indirectly validating Eq. (4).

Since the linear scatter plots provided justification for Eq. (4), proper analysis of the Lambert W model [Eq. (5)] was warranted. Hence, Eq. (5) was fit to the detected transmission (I/I_0) versus attenuator thickness (x) data in MATLAB R2008b (MathWorks Inc., Natick, MA) by nonlinear least squares regression, which used an iterative search algorithm¹³ to determine the best-fit values of unknown parameters μ_0 and λ . For each data set, 95% confidence intervals for the best-fit parameters were calculated, as well as, the coefficient of determination (R^2) and the sum of squares error (SSE) of the best-fit curves. Additionally, the HVL and QVL were estimated by substituting the best-fit values of μ_0 and λ into Eqs. (B4)–(B5) (shown in Appendix B); the HVL and QVL were estimated for data set comparison purposes only and were not used in the analysis of two-point Lambert W interpolation for HVL and QVL estimations. The same nonlinear regression analysis was also performed for Eqs. (1), (7), and (8).

Since the R^2 values for all four models were (for the most part) very similar to one another, Akaike’s information criterion (AIC) was used to provide another source of comparison by which to gauge the models’ relative goodness of fit.¹⁴ For every data set, an AIC value was calculated for each of the four attenuation models [Eqs. (1), (5), (7), and (8)]. AIC values by themselves are meaningless, and so evidence ratios were used to compare the AIC values calculated for the models described by Eqs. (1), (7), and (8) to the AIC values calculated for the Lambert W model [Eq. (5)]. The evidence ratios described the likelihood that the Lambert W model was correct (e.g., an evidence ratio of 10 indicated that the Lambert W model was 10 times more likely to be correct

than the model it was being compared to) by computing the probability that it was correct divided by the probability that another model [i.e., Eqs. (1), (7) or (8)] was correct.¹² Rough rules of thumb used when interpreting the AIC values were that a difference between two AIC values of less than 2, which corresponds to an evidence ratio of less than 2.7, indicates substantial evidence for the other model, while a difference between 4 and 7 indicates considerably less support for the other model, and a difference greater than 10 (or an evidence ratio greater than 148) indicates essentially no support in favor of the other model.¹⁵

II.B.2. HVL and QVL estimations

Lambert W interpolation was compared to traditional methods (semilogarithmic and linear interpolation) for HVL and QVL estimations, as well as interpolations based on Eqs. (7) and (8). Each interpolation method was judged by the accuracy of its predictions, which was evaluated by comparing measured transmission to predicted transmission through the HVL or QVL nearest neighbor. Additionally, the methods were compared in terms of the sensitivity of their predictions to the choice of interpolated data points. Finally, although interpolation is typically used for HVL and QVL estimations, we also considered whether extrapolation could yield accurate estimates.

The first step in obtaining our predictions was to divide the measured data points (except for unattenuated exposure, I_0) into pairs and to calculate μ_0 and λ for each pair using Eqs. (B6)–(B7) (see Appendix B). Although μ_0 and λ can be calculated using more than two data points, such as in Sec. II B 1 where all of the collected data points were used to determine the best-fit values of μ_0 and λ , typically only two data points (i.e., one pair) are experimentally collected for the purpose of HVL or QVL estimation. The number of pairs analyzed and thus the number of μ_0 and λ values calculated depended on the number of data points collected for each data set (shown in Table I); for example, seven data points were recorded for the general radiography data sets using Cu attenuators, which translated to 15 pairs being analyzed. All analyzed pairs were categorized according to whether they bracketed (interpolation) or lay on one side of (extrapolation) the HVL or QVL. Furthermore, a subcategory of interpolation was considered by analyzing those pairs that bracketed both the HVL and QVL.

The calculated values of μ_0 and λ for each data pair and the HVL or QVL nearest neighbor (x_{nearest}) for the data set that the pair came from were substituted into the Lambert W model [Eq. (5)] to predict transmission ($I_{\text{predicted}}/I_0$) through thickness x_{nearest} ; transmission through x_{nearest} was also predicted by semilogarithmic and linear interpolation and by interpolations based on the Bjärngård and Shackford [Eq. (7)] and Yu *et al.* [Eq. (8)] models. The measured data points nearest the HVL or QVL (i.e., x_{nearest}) were used to approximately represent these values since the HVL and QVL were not directly measured and therefore their true values were unknown.

Predicted and measured transmissions were compared through their relative percent difference (RPD) as

FIG. 3. Sample calculation using data collected on a Hologic Selenia mammography system at 26 kVp. The following steps were performed to calculate the relative percent differences (RPDs) and related statistical analysis for this data set: (1) identify the measured data point closest to the HVL ($[x_{\text{nearest}}, (I_{\text{nearest}}/I_0)]$); (2) group all other data points, except for (0, 1), into pairs and separate the pairs according to whether they bracket (interpolation) or lay on one side of (extrapolation) the HVL; (3) calculate μ_0 and λ for each pair [from Eqs. (B6)–(B3) in Appendix B]; (4) substitute x_{nearest} , μ_0 and λ into Eq. (5) to calculate ($I_{\text{predicted}}/I_0$) for each pair; (5) use Eq. (10) to calculate the RPD between each (I_{nearest}/I_0) and ($I_{\text{predicted}}/I_0$); (6) calculate the mean and range of RPDs and also the fraction of interpolation and extrapolation pairs that yielded RPDs $\leq 5\%$. Similar steps can be followed to evaluate the accuracy of QVL estimation, the other measured data sets, and the other four interpolation methods.

$$\text{RPD} = \left| \frac{(\text{Predicted} - \text{Measured})}{\text{Measured}} \right| \cdot 100\%. \quad (10)$$

RPDs were not calculated for pairs containing the nearest neighbor data point(s) of interest because, in these cases, the predicted transmission is by definition equal to the measured transmission (i.e., RPD equals 0%); therefore these RPDs do not provide any information on accuracy of the predictions.

FIG. 4. Residual plots for general radiography (GE Definium) data collected at 80 and 120 kVp using two different attenuating materials (Al 1100 and Cu).

A pair was considered to be acceptable for estimating the HVL or QVL if its RPD was less than or equal to 5%. This criterion was used because the exposure measurement precision of all systems was within 5% (as indicated by the percent coefficient of variation of 10 repeated exposure measurements; data not shown); additionally, this criterion is consistent with the reported measurement precision for CT scanners of within $\pm 5\%$.¹⁶ To characterize and compare the accuracy of the HVL and QVL estimation methods, the mean and range of RPDs were calculated for each data set and each method. Sensitivity of each estimation method to the choice of interpolation or extrapolation points was also

gauged by determining the percentage of acceptable pairs (i.e., pairs with RPDs $\leq 5\%$).

Figure 3 provides an illustrative example showing the steps used to assess the success of HVL estimation by Lambert W interpolation for the Hologic Selenia mammography data set at 26 kVp.

III. RESULTS

III.A. Validation of Lambert W equation

Plots of the residuals [i.e., differences between the measured transmissions and the transmission values predicted by

FIG. 5. Residual plots for data collected on a Hologic Selenia and a GE Senographe 2000D mammography unit at 26 and 35 kVp; data were collected on the GE Senographe 2000D system at 26 kVp using both the standard compression paddle and a fenestrated compression paddle.

FIG. 6. Residual plots for data collected on a GE QX/i LightSpeed CT scanner for 80 and 120 kVp; the experimental setup used to collect exposure measurements is shown in Fig. 1.

the best-fit lines for the models described by Eqs. (1), (5), (7), and (8)] for the general radiography, mammography, and CT data sets appear in Figs. 4, 5, and 6, respectively; additional details of the nonlinear regression analysis can be found in Table II of the supplemental data.¹⁷

The R^2 values for the Lambert W model [Eq. (5)] ranged from 0.9995 to 1.000 across all data sets, indicating a very close fit of the model to the data. When compared to other attenuation models, the AIC evidence ratios indicated that the Lambert W model ranged (across all 11 data sets) from 4.35×10^7 to 4.27×10^{81} , 1.03×10^1 – 2.78×10^{46} , and 0.165×10^1 – 2.29×10^{38} times more likely to be the correct model compared to the Lambert-Beer [Eq. (1)], Bjärngard and Shackford [Eq. (7)], and Yu *et al.* [Eq. (8)] models, respectively. Furthermore, the Lambert W model featured higher R^2 values, across all data sets, than the models described by Eqs. (1) and (7). Equation (8), however, had a slightly higher R^2 value and was also around 6 times more likely to be correct than the Lambert W model for the Hologic Selenia mammography data set at 35 kVp. An evidence ratio of 6 is not large enough to provide persuasive evidence in favor of the Yu *et al.* model over the Lambert W model; additionally, the Lambert W model featured higher AIC evidence

ratios for the other 10 data sets and was well over 1×10^6 times more likely to be the correct model for the CT data sets.

III.B. HVL and QVL estimations

Figure 7 shows the percentage of acceptable pairs (i.e., pairs with RPDs $\leq 5\%$) for predicting transmission through either the HVL or QVL nearest neighbor by each interpolation method. Although Fig. 7 does not specifically quantify accuracy of the predictions (beyond the criteria of 5%), it does indicate that (in general) Lambert W interpolation was the most versatile estimation method in terms of flexibility in the choice of interpolation data points. Table III in the supplemental data¹⁷ also lists the fraction and corresponding percentage of acceptable interpolation pairs, as well as the mean and range of RPDs for each interpolation method.

Lambert W interpolation was shown to be successful in all of the HVL cases analyzed (1563 out of 1563 pairs), regardless of the choice of data points interpolated, and in over 98% of the QVL cases analyzed (2151 out of 2176 pairs). The mean and range of RPDs between the measured and predicted (by Lambert W interpolation) transmissions through the HVL nearest neighbor across the general

FIG. 7. This bar graph provides a visual representation of the Lambert W, Yu *et al.*, Bjärngard and Shackford, semilogarithmic and linear interpolation methods' respective sensitivities to the choice of interpolation points used to predict transmission through the attenuator thickness nearest the HVL or QVL (x_{nearest}) for each data set. The relative percent differences (RPDs) between the predicted and measured transmissions through x_{nearest} were calculated and the percentage of acceptable interpolation pairs (i.e., pairs with RPDs $\leq 5\%$) for estimating the (a) HVL and (b) QVL is shown. Not all of the data sets appear in both graphs either because a data point less than the HVL [other than (0, 1), which cannot be interpolated] or greater than the QVL was not measured, and therefore, interpolation could not be performed.

radiography, mammography, and CT data sets, respectively, were 0.44% (0.00–2.00%), 0.59% (0.00–2.37%), and 0.60% (0.00–3.37%); similarly, the mean and range of RPDs for the QVL nearest neighbor across the general radiography, mammography, and CT data sets were 1.55% (0.00–6.90%), 1.36% (0.01–5.87%), and 0.56% (0.00–3.75%), respectively. Across all data sets and all interpolation pairs (HVL and QVL), the Lambert W interpolated predictions yielded RPDs ranging from 0.00 to 6.90% (99.3% of the 3739 pairs analyzed had RPDs $\leq 5\%$) with a mean RPD of 0.73%. These results can be compared to the mean and range of RPDs (calculated across all 3739 pairs) of 1.74% (0.00–11.0%) for interpolation based on the Yu *et al.* model (94.1% of pairs had RPDs $\leq 5\%$), 2.78% (0.00–20.8%) for interpolation based on the Bjärngard and Shackford model (83.3% of pairs had RPDs $\leq 5\%$), 3.48% (0.00–26.7%) for semilogarithmic interpolation (75.0% of pairs had RPDs $\leq 5\%$), and 18.4% (0.04–112%) for linear interpolation (24.3% of pairs had RPDs $\leq 5\%$).

The special case of using Lambert W interpolation to estimate both the HVL and QVL from a single pair of two measured data points, as opposed to using two pairs (i.e., four data points) to interpolate the HVL and QVL separately, had a 97.2% acceptance rate (across all data sets). This rate can be compared to 76.8, 47.8, 40.9, and 1.13% acceptance when using Yu *et al.*, Bjärngard and Shackford, semilogarithmic, and linear interpolations, respectively, to jointly estimate the HVL and QVL.

Lambert W extrapolation was evaluated and RPDs of less than or equal to 5% were achieved by approximately 89.8% of pairs (2545 out of 2908 pairs and 2112 out of 2280 pairs used to predict transmission through the HVL and QVL nearest neighbors, respectively). Lambert W extrapolation was more versatile than Yu *et al.*, Bjärngard and Shackford, semilogarithmic, and linear extrapolations, which had acceptance rates of 87.8, 77.6, 57.4, and 18.4%, respectively. However, the diminished success rate of Lambert W extrapolation compared to Lambert W interpolation (89.8 versus 99.3% acceptance) indicates that, based on our results, interpolation rather than extrapolation should be used to estimate the HVL or QVL.

IV. DISCUSSION

In this study, we proposed an empirical model, which uses the Lambert W function, to describe narrow-beam attenuation of diagnostic x-ray beams through homogenous materials, and our analyses showed that the proposed model accurately describes such attenuation. Furthermore, accuracy of the Lambert W attenuation model [Eq. (5)] over the broad range of testing conditions considered (three radiographic imaging modalities, two kVps per modality, and two attenuating materials) demonstrates the versatility of this model.

Although there is an established method for assessing attenuation under broad-beam geometry,¹⁸ this method does not necessarily hold true when describing attenuation of the primary beam (and vice versa) because the x-ray spectra are very different under these two geometries. Therefore, the focus of this study was on developing a narrow-beam

attenuation model. While several such models have been developed for high-energy beams,^{6,7} to the best of the authors' knowledge, no such model has been verified for determining narrow-beam attenuation of diagnostic energy beams.

The proposed Lambert W model appears to closely model narrow-beam attenuation of kilovoltage x-ray beams in light of the high R^2 values computed in this study, which ranged from 0.9995 to 1.000. These promising results may motivate future development of the empirical Lambert W model, which presently relies on experimental data and could be enhanced by generation of look-up tables for constants μ_0 and λ , similar to the National Institute of Standards and Technology (NIST) mass attenuation coefficient tables. If look-up tables were developed, parameters μ_0 and λ would be specified according to the kVp, attenuating material, inherent filtration, and possibly other factors that affect the polyenergetic beam spectra, such as the bowtie filter for CT data. Our data sets demonstrate the need for specification according to these factors. For example, the effect of filtration is evident when comparing the μ_0 and λ values for data collected on a GE Senographe 2000D mammography system using both a standard and a fenestrated paddle (thickness of the standard paddle contributed to the total beam filtration, while the fenestrated paddle did not). Furthermore, the effect of the peak kilovoltage and attenuating material is apparent when comparing the general radiography data sets at both 80 and 120 kVp using Al 1100 and Cu attenuators. Besides the expected differences in μ_0 and λ that we observed across data sets, unexpected (but possibility justified) inconsistencies in the best-fit parameters were also observed. Although similar HVLs and QVLs were estimated (at the same KVps) across the two mammography systems included in this study, their best-fit constants were significantly different at the $\alpha = 0.05$ level (as indicated by the lack of overlap in the 95% confidence intervals for μ_0 and λ). Differences in these parameters could be attributed to physical differences in the machines such as the Hologic Selenia system using a Mo/Rh target/filter combination at 35 kVp versus the GE Senographe 2000D using a Rh/Rh target/filter combination. In order to develop generic (nonmanufacturer specific) look-up tables, inconsistencies in μ_0 and λ for data sets with the same attenuating material, kVp, and inherent filtration would need to be resolved and the stability of μ_0 and λ across these data sets would need to be established.

Two-point Lambert W interpolation was considered to be a successful means of predicting transmission through the HVL nearest neighbor, yielding RPDs $\leq 5\%$ across all data sets regardless of the choice of interpolation data points. In addition to showing less sensitivity (in general) to the choice of interpolation points compared to the other interpolation methods analyzed, Lambert W interpolation displayed a higher degree of accuracy in its predictions. Of the four other interpolation methods evaluated, linear interpolation had the weakest performance, while interpolations based on the Yu *et al.* model most closely matched the performance of the Lambert W model. Despite its diminished success compared to Lambert W based interpolation, linear interpolation was observed to yield accurate predictions (within 5%) as long as

the interpolated data points were within $\pm 25\%$ of the HVL or $\pm 10\%$ of the QVL; similarly, semilogarithmic interpolation was observed to yield accurate predictions when the interpolated data points were within $\pm 50\%$ of the HVL or $\pm 25\%$ of the QVL. The Bjärngård and Shackford model was generally less versatile and yielded less accurate estimates of transmission through the HVL and QVL nearest neighbors compared to the other therapy-based model evaluated in this study (i.e., the Yu *et al.* model); furthermore, semilogarithmic interpolation was superior to interpolations based on the Bjärngård and Shackford model for some data sets. Although the Yu *et al.* model had a higher average RPD across all data sets (1.74 versus 0.77%), higher average and maximum RPDs (across most data sets), and was (in general) less versatile in its percentage of acceptable pairs than the Lambert W model, the Lambert W model's principal advantage was its usefulness for estimating both the HVL and QVL simultaneously. While accuracy of the HVL and QVL estimations was not directly evaluated in this study, we expect it to be roughly equivalent to the level of accuracy observed in the predicted transmissions through the HVL and/or QVL nearest neighbors. Therefore, we can assume that Lambert W interpolation allows for accurate estimation of both the HVL and QVL from the unattenuated exposure (I_0) and two measured data points (e.g., exposure measured through 0.5 and 33 mm of Al 1100 on a CT scanner at 120 kVp). Thus, it is feasible that two-point Lambert W interpolation could be implemented into quality assurance practices. To estimate the HVL and/or QVL, the unattenuated exposure and two measured exposure versus attenuator thickness data points could be substituted into Eqs. (B4)–(B7) (in Appendix B), which could be incorporated in a spreadsheet as part of QA practices; Eqs. (B4)–(B7) are the appropriate forms to use when estimating HVL and/or QVL because they describe attenuator thickness in terms of intensity (transmission) rather than Eq. (5), which expresses intensity as a function of attenuator thickness.

For all HVL or QVL estimation practices, the authors recommend that the attenuator thicknesses be measured with a digital caliper. Our findings indicated that the nominal thicknesses of Al 1100 differed from the measured thicknesses by up to 10%. Therefore, by not measuring the attenuator thicknesses, the HVL and QVL estimates could be misrepresented, for instance, by as much ± 0.8 and ± 1.79 mm of Al 1100, respectively, for a CT scanner at 120 kVp. Additionally, constants μ_0 and λ would change if the nominal rather than the measured attenuator thicknesses were used to determine these values, potentially leading to further misrepresentation of the HVL and QVL estimates. Since the nominal attenuator thicknesses were only precise to $\pm 10\%$, it is reasonable that the original acceptance criterion of 5% be adjusted accordingly. We can apply this adjustment to our data, despite the fact that we measured the attenuator thicknesses, because we assessed performance of the Lambert W model by comparing measured and predicted transmissions rather than by comparing the HVL and QVL estimates to their true values. Since all RPDs observed in this study (0.00–6.90%) met the adjusted acceptance criterion, we can

conclude that Lambert W interpolation accurately estimated transmission through both the HVL and QVL regardless of the choice of interpolation data points.

Besides using the proposed Lambert W model to estimate the HVL and QVL, Eq. (5) could also be applied for other purposes, such as empirically determining attenuation through a given thickness of radiation protective clothing (e.g., in-plane bismuth breast shields, lead aprons, etc.); the international standard IEC 61331–3 specifies that the lead-equivalent thickness of a material used for radiation-shielding garments be measured under narrow-beam geometry.¹⁹ Additionally, Eq. (5) can be used in Monte Carlo modeling, as well as in other computer modeling of x-ray beam attenuation through inherent, bowtie, or other filtration. However, in order to apply Eq. (5) in either of these circumstances, additional analysis would need to be performed to validate the Lambert W model under the specific conditions being tested or modeled.

V. CONCLUSION

The empirical model proposed in this study represents a generalized Lambert-Beer law that describes attenuation of both monoenergetic and polyenergetic radiation. In the case of monoenergetic radiation, the proposed Lambert W model mathematically reduces to the Lambert-Beer law. Applications of the Lambert W model relevant to polyenergetic radiation include HVL estimation and narrow-beam patient shielding (e.g., lead apron) calculations. HVL estimation is simplified by the Lambert W model because only two attenuator thicknesses need to be supplied for each radiographic modality (e.g., 2 and 10 mm of Al 1100 for general radiography). Additionally, the Lambert W model can be utilized to jointly interpolate both the HVL and QVL from the same two measured data points. Besides yielding more accurate estimates of the HVL and/or QVL compared to traditional methods (i.e., semilogarithmic and linear interpolation), other advantages of using Lambert W interpolation are that it is less time-consuming (as it requires less guess-work and fewer measurements) and less vulnerable to experimenter error (since it is relatively insensitive to the choice of interpolation points).

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APPENDIX A: DERIVATION OF EQ. (5)

Initially, Eq. (4) was substituted into Eq. (1) for μ :

$$I = I_0 e^{-\left(\mu_0 + \lambda \left(\frac{t}{t_0}\right)\right)x} \quad (\text{A1})$$

Eq. (A1) was rearranged to achieve separation of variables:

$$\left(\frac{I}{I_0}\right) e^{\lambda x \left(\frac{t}{t_0}\right)} = e^{-\mu_0 x} \quad (\text{A2})$$

Both sides of Eq. (A2) were multiplied by λx :

$$\lambda x \left(\frac{I}{I_0} \right) e^{\lambda x \left(\frac{I}{I_0} \right)} = \lambda x e^{-\mu_0 x} \quad (\text{A3})$$

Term u was created such that:

$$u = \lambda x \left(\frac{I}{I_0} \right) \quad (\text{A4})$$

Substituting u into Eq. (A3):

$$ue^u = \lambda x e^{-\mu_0 x} \quad (\text{A5})$$

By definition of W , Eq. (A5) becomes:

$$u = W(\lambda x e^{-\mu_0 x}) \quad (\text{A6})$$

Setting Eq. (A4) equal to Eq. (A6) and rearranging:

$$I = I_0 \frac{W(\lambda x e^{-\mu_0 x})}{\lambda x} \quad (5)$$

APPENDIX B: EQUATIONS FOR CALCULATING HVL AND QVL

Rearranging Eq. (A1):

$$\left(\frac{I}{I_0} \right) = e^{-\left(\mu_0 + \lambda \left(\frac{I}{I_0} \right) \right) x} \quad (\text{B1})$$

Taking the natural logarithm of both sides of Eq. (B1):

$$\ln \left(\frac{I}{I_0} \right) = -\left(\mu_0 + \lambda \frac{I}{I_0} \right) x \quad (\text{B2})$$

Rearranging Eq. (B2):

$$x = \frac{-\ln \left(\frac{I}{I_0} \right)}{\mu_0 + \lambda \frac{I}{I_0}} \quad (\text{B3})$$

Eq. (B3) can be written as the HVL or QVL by substituting $1/2$ or $1/4$, respectively, for (I/I_0) :

$$\text{HVL} = \frac{\ln(2)}{\mu_0 + \frac{\lambda}{2}} \quad (\text{B4})$$

$$\text{QVL} = \frac{\ln(4)}{\mu_0 + \frac{\lambda}{4}} \quad (\text{B5})$$

where μ_0 and λ are calculated as follows for two-point Lambert W interpolation of measured data points $(x_1, I_1/I_0)$ and $(x_2, I_2/I_0)$:

$$\mu_0 = \frac{x_1 \left(\frac{I_1}{I_0} \right) \ln \left(\frac{I_2}{I_0} \right) - x_2 \left(\frac{I_2}{I_0} \right) \ln \left(\frac{I_1}{I_0} \right)}{x_1 x_2 \left(\left(\frac{I_2}{I_0} \right) - \left(\frac{I_1}{I_0} \right) \right)} \quad (\text{B6})$$

$$\lambda = \frac{x_2 \ln \left(\frac{I_1}{I_0} \right) - x_1 \ln \left(\frac{I_2}{I_0} \right)}{x_1 x_2 \left(\left(\frac{I_2}{I_0} \right) - \left(\frac{I_1}{I_0} \right) \right)} \quad (\text{B7})$$

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