Symplectic homology product via Legendrian surgery

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This research announcement continues the study of the symplectic homology of Weinstein manifolds undertaken by the authors [Bourgeois F, Ekholm T, Eliashberg Y (2009) arXiv:0911.0026] where the symplectic homology, as a vector space, was expressed in terms of the Legendrian homology algebra of the attaching spheres of critical handles. Here, we express the product and Batalin–Vilkovisky operator of symplectic homology in that context.

symplectic field theory | Lagrangian manifold | Hochschild homology

1 Introduction

Weinstein manifolds are symplectic counterparts of affine (Stein) complex manifolds. They are exact symplectic manifolds that admit symplectic handle decompositions with isotropic core disks. In particular, the index of any handle is at most half of the dimension of the manifold. We refer the reader to refs. 1–3 for precise definitions. Middle-dimensional handles are called *critical*, and handles below middle dimension are called *subcritical*. Given a Weinstein manifold X , its symplectic homology $\mathrm{SH}(X)$ is a symplectic invariant that was first defined by Floer and Hofer in ref. 4. It vanishes if X is subcritical (see ref. 5). A general Weinstein manifold X is obtained by attaching critical handles to a subcritical manifold X_0 along a collection Λ of Legendrian spheres in the contact manifold Y_0 , which is the ideal boundary of X_0 . In ref. 1, $\mathrm{SH}(X)$ was expressed in terms of the Legendrian homology algebra of $\Lambda \subset Y_0$.

Symplectic homology comes with topological quantum field theory–like operations, which on the homology level are all expressed in terms of a (pair-of-pants) product and a Batalin–Vilkovisky (BV) operator (see refs. 6 and 7). In this paper, we express the product and the BV operator in terms of the Legendrian homology algebra of Λ and some additional operations on it. The BV operator is determined by the algebra itself, whereas the expression of the product involves a duality operation that can be identified with the first term in the extension of Legendrian homology algebra to Legendrian rational symplectic field theory (SFT).

The paper is organized as follows. In Section 2 we introduce algebraic constructions that we later apply to Legendrian homology algebras. In Section 3 we express the Legendrian homology algebra results from ref. 1 in the terminology of Section 2 and also include a discussion on the construction of a version of rational SFT in the Legendrian setting (completion of this construction is work in progress). In Section 4 we present the symplectic homology product and the BV operator in terms of Legendrian homology algebra and a duality operator, and in Section 5 we use the result to compute the symplectic homology, with product and BV operator, of cotangent bundles of spheres. Although proofs are mostly omitted, we note that the proofs of the algebraic results from Section 2 are fairly straightforward.

2 Algebraic Preliminaries

In this section, we describe a number of constructions in a purely algebraic setting. The constructions will be applied in geometrically relevant situations in Sections 3 and 4.

2.1 A Tensor Algebra and Associated Objects. We associate to a differential graded algebra A over a field K a pair of dual A -modules

and a corresponding tensor algebra. We also discuss cyclic versions of these objects, which are vector spaces over K .

2.1.A The algebra A . Let R be an algebra over a field K of characteristic 0 generated by idempotents $1_1, \dots, 1_m$, where $1_k 1_l = \delta_{kl} 1_k$, where δ_{kl} is the Kronecker delta. We define a graded module V over R , which is generated by a countable set $\mathcal{C}$ decomposed as a disjoint union $\mathcal{C} = \bigcup_{1 \leq i, j \leq m} \mathcal{C}_{ij}$. The degree of an element $a \in V$ will be denoted by $|a|$. The decomposition of $\mathcal{C}$ induces a decomposition $V = \bigoplus_{1 \leq i, j \leq m} V^{(ij)}$, where $V^{(ij)}$ is generated by $\mathcal{C}_{ij}$. When applied on the left, the element 1_j , $j = 1, \dots, m$, acts as the identity on $V^{(ij)}$ and as the 0-map on $V^{(il)}$, $l \neq j$. Similarly, when applied on the right, the element 1_j acts as the identity on $V^{(ji)}$ and as 0 on $V^{(li)}$, $l \neq j$.

We define an extension $\widehat{V}$ of the module V as follows. The module $\widehat{V}$ is generated by $\widehat{\mathcal{C}} \cup \mathcal{X}$, where $\widehat{\mathcal{C}} = \{\widehat{c} : c \in \mathcal{C}\}$ and $\mathcal{X} = \{x_1, \dots, x_m\}$, where $|\widehat{c}| = |c| + 1$, for $c \in \mathcal{C}$ and $|x_j| = 0$, $j = 1, \dots, m$. We define

$$\widehat{V} := K\langle \mathcal{X} \cup \widehat{\mathcal{C}} \rangle = K\langle \mathcal{X} \rangle \oplus V[1],$$

where $K\langle \mathcal{X} \rangle$ denotes the m -dimensional module freely generated by the set $\mathcal{X}$ and where $V[1]$ denotes V with grading shifted up by 1. The module $\widehat{V}$ inherits a decomposition $\widehat{V} = \bigoplus_{1 \leq i, j \leq m} \widehat{V}^{(ij)}$ from V , where

$$\widehat{V}^{(ij)} = \begin{cases} V^{(ij)}[1] & \text{if } i \neq j, \\ Kx_j \oplus V^{(ij)}[1] & \text{if } i = j. \end{cases}$$

Here, the idempotents act as before on $V \approx V[1]$ as a subset of $\widehat{V}$, and as follows on the new generators: $1_j x_i = \delta_{ij} x_j$ and $x_i 1_j = \delta_{ij} x_j$, $1 \leq i, j \leq m$.

We will use the following notation. If $a \in V$, then $\widehat{a}$ denotes the element in $V[1] \subset \widehat{V}$ that corresponds to a . Furthermore, if F and G are either V or $\widehat{V}$ and if (u, v) is an ordered pair of elements in $F^{(ij)} \oplus G^{(kl)}$, then (u, v) is *composable* if $j = k$.

Let

$$A := R \oplus V \oplus V^{\otimes_2} \oplus \dots \quad [2.1]$$

be the tensor algebra over R generated by V with its natural grading. Then, A is generated additively by composable monomials $w = c_1 \dots c_k$, $c_j \in V$, $j = 1, \dots, k$, where we say that w is composable provided (c_j, c_{j+1}) is composable for $1 \leq j \leq k-1$. (Here, as often in the following, we will suppress the tensor sign in the notation for tensor products). The decomposition of V induces a corresponding decomposition of A , $A = \bigoplus_{1 \leq i, j \leq m} A^{(ij)}$, where $A^{(ij)}$ is generated by composable monomials $c_1 \dots c_k$ with $c_1 \in V^{(is)}$ for some $s \in \{1, \dots, m\}$ and $c_k \in V^{(jt)}$ for some $t \in \{1, \dots, m\}$.

We will assume that the algebra A has a differential $d: A \rightarrow A$ of degree -1 , which leaves the subspaces $A^{(ij)}$, $i, j \in \{1, \dots, m\}$ invariant [i.e., $d^2 = 0$, d satisfies the Leibniz rule, and

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$d(A^{(ij)}) \subset A^{(ij)}$. We will usually write $H(A)$ for the homology algebra $H_*(A, d)$.

2.1.B The A -module M and the K -complex M^{cyc} . Define

$$M = M(A) := A \otimes_R \widehat{V} \otimes_R A. \quad [2.2]$$

Then, M is a graded left-right module over the differential graded algebra A , which is additively generated by composable monomials $w = c_1 \dots c_k u f_1 \dots f_l$, where $c_i \in \mathcal{C}$, $i = 1, \dots, k$, $u \in \widehat{\mathcal{C}} \cup \mathcal{X}$, and $f_j \in \mathcal{C}$, $j = 1, \dots, l$. We say that a composable monomial w is *cyclically composable* if the pair (f_l, c_1) is composable. Let $M^{\text{diag}} \subset M$ be the submodule generated (over K) by cyclically composable monomials, and define M^{cyc} as the quotient space of M^{diag} in which monomials that differ by graded cyclic permutation are identified. If $w \in M^{\text{diag}}$, then we write $[w]$ for its equivalence class in M^{cyc} ; i.e., if $\pi: M^{\text{diag}} \rightarrow M^{\text{cyc}}$ denotes the natural projection, then $[w] := \pi(w) \in M^{\text{cyc}}$.

The differential $d: A \rightarrow A$ induces a differential $d_M: M \rightarrow M$ as follows. If $w_1 u w_2 \in M$, $u \in \widehat{V}$, and $w_1, w_2 \in A$, then

$$\begin{aligned} d_M(w_1 u w_2) &:= (dw_1) u w_2 + (-1)^{|w_1|} w_1 (d_M u) w_2 \\ &\quad + (-1)^{|w_1 u|} w_1 u (dw_2). \end{aligned} \quad [2.3]$$

Here,

$$d_M u = \begin{cases} ax_i - x_j a + S(da) & \text{if } u = \hat{a}, a \in V^{(ij)}, \\ 0 & \text{if } u = x_j, j \in \{1, \dots, m\}, \end{cases} \quad [2.4]$$

where the R -module homomorphism $S: A \rightarrow M$ is defined by

$$\begin{aligned} S(a_1 \dots a_k) &= \hat{a}_1 a_2 a_3 \dots a_k + (-1)^{|a_1|} a_1 \hat{a}_2 a_3 \dots a_k + \dots \\ &\quad + (-1)^{|a_1 \dots a_{k-1}|} a_1 a_2 \dots a_{k-1} \hat{a}_k, \end{aligned} \quad [2.5]$$

and $S(1_j) = 0$, for $j = 1, \dots, m$.

Lemma 2.1. *The map $d_M: M \rightarrow M$ is a differential (i.e., $d_M^2 = 0$) that leaves M^{diag} invariant and descends to a differential $M^{\text{cyc}} \rightarrow M^{\text{cyc}}$, still denoted d_M , in such a way that the natural projection $\pi: (M^{\text{diag}}, d_M) \rightarrow (M^{\text{cyc}}, d_M)$ is a chain map.*

We write $H(M)$ and $H(M^{\text{cyc}})$, respectively, for the homologies $H_*(M, d_M)$ and $H_*(M^{\text{cyc}}, d_M)$. Note that $H(M)$ is a left-right module over $H(A)$, whereas $H(M^{\text{cyc}})$ is just a K -vector space.

Remark 2.2: We can write

$$M^{\text{cyc}} = \left(\bigoplus_{j=1}^m A x_j \right) \oplus A \otimes_R V[1] := {}'M^{\text{cyc}} \oplus {}''M^{\text{cyc}}$$

(i.e., $'M^{\text{cyc}}$ is the submodule of M^{cyc} generated by $\mathcal{X}$, and $''M^{\text{cyc}}$ is its complement). Let $p: M^{\text{cyc}} \rightarrow {}'M^{\text{cyc}}$ be the projection, and let $''d_M := p \circ (d_M)|_{{}''M^{\text{cyc}}}$. The second summand $''M^{\text{cyc}}$ can be viewed as a noncommutative version of the “Lie algebra of vector fields” on the space V . Indeed, we can identify the cyclic monomial $[w\hat{u}] \in M^{\text{cyc}}$, $w \in A$, $u \in V$, with the vector field $w \frac{\partial}{\partial u}$. The differential $''d_M$ can be naturally viewed as a vector field on V , and hence an element of $''M^{\text{cyc}}$, given by $\sum_{c \in \mathcal{C}} dc \frac{\partial}{\partial c}$. For any $X \in {}''M^{\text{cyc}}$, its differential $''d_M X$ is just the Lie bracket $[{}''d_M, X]$ of vector fields, or equivalently the Lie derivative $L_{{}''d_M} X$.

This interpretation corresponds to the “satellite philosophy” from ref. 8. Another interpretation was pointed out to the authors by M. Abouzaid: $H(M^{\text{cyc}})$ is isomorphic to the Hochschild homology $HH_*(A, A)$.

2.1.C The extended tensor algebra U and the space $\mathcal{U}^{\text{bal}}$. Recall that the module V is generated over R by a set $\mathcal{C}$ and that the module

M is generated over A as a left-right module by $\widehat{\mathcal{C}} \cup \mathcal{X}$, where $\widehat{\mathcal{C}} = \{\hat{c}: c \in \mathcal{C}\}$ and $\mathcal{X} = \{x_1, \dots, x_m\}$. Let $\widehat{V}^*$ denote the K -vector space dual to $\widehat{V}$. Then, $\widehat{V}^*$ is the direct product of the one-dimensional K -vector spaces that are generated by elements of $\widehat{\mathcal{C}}^* \cup \mathcal{X}^*$, where $\widehat{\mathcal{C}}^* = \{\hat{c}^*: c \in \mathcal{C}\}$ and $\mathcal{X}^* = \{x_j^*: x_j \in \mathcal{X}\}$. Furthermore, we have

$$\widehat{V}^* = \bigoplus_{1 \leq i, j \leq m} \widehat{V}^{*(ij)}, \quad \text{where } \widehat{V}^{*(ij)} := (\widehat{V}^{(ij)})^*.$$

In $\widehat{V}^*$, the idempotent 1_j , $j = 1, \dots, m$, acts from the left as the identity on $\widehat{V}^{*(ij)}$ and as 0 on $\widehat{V}^{*(il)}$, $l \neq j$, and acts from the right as the identity on $\widehat{V}^{*(ij)}$ and as 0 on $\widehat{V}^{*(il)}$, $l \neq j$. Fix an integer $n \geq 2$ (it will correspond to half the dimension of the Liouville domain X_0 in Section 3). We endow $\widehat{V}^*$ with a grading defined as follows:

$$|\hat{c}^*| = n - 3 - |\hat{c}|, \quad \text{for } \hat{c} \in \widehat{\mathcal{C}}, \quad [2.6]$$

$$|x_j^*| = n - 3, \quad \text{for } j = 1, \dots, m. \quad [2.7]$$

Define $M^* := A \otimes_R \widehat{V}^* \otimes_R A$ and define the tensor algebra

$$U(0, 0) := \bigoplus_{k \geq 0} (M \oplus M^*)^{\otimes_A k} \quad [2.8]$$

as a certain completion of the free associative algebra generated by $\widehat{\mathcal{C}} \cup \widehat{\mathcal{C}}^* \cup \mathcal{X} \cup \mathcal{X}^*$. Its elements are infinite series in the $(\widehat{\mathcal{C}}^* \cup \mathcal{X}^*)$ -variables with polynomial coefficients in the $(\widehat{\mathcal{C}} \cup \mathcal{X})$ -variables (compare to the above definitions of M as an infinite sum and M^* as an infinite product). For simplicity of notation, we will suppress completions from notation and use the direct sum symbol in decompositions. Consider the extension U of $U(0, 0)$ obtained by adding multiplicative generators $\hbar$, $\hbar^{-1}$ and σ , σ^{-1} of degrees

$$|\sigma| = 1, \quad |\sigma^{-1}| = -1, \quad |\hbar| = n - 3, \quad |\hbar^{-1}| = -(n - 3),$$

which satisfy the following relations:

$$\sigma \sigma^{-1} = \sigma^{-1} \sigma = 1, \quad \hbar \hbar^{-1} = \hbar^{-1} \hbar = 1, \quad \sigma \hbar = (-1)^{n-3} \hbar \sigma,$$

and

$$\begin{aligned} \sigma u &= \begin{cases} (-1)^{|u|} u \sigma, & u \in \mathcal{C}; \\ (-1)^{|u|+1} u \sigma, & u \in \widehat{\mathcal{C}} \cup \mathcal{X} \cup \widehat{\mathcal{C}}^* \cup \mathcal{X}^*, \end{cases} \\ \hbar u &= \begin{cases} (-1)^{|u|(n-3)} u \hbar, & u \in \widehat{\mathcal{C}} \cup \mathcal{X} \cup \widehat{\mathcal{C}}^*; \\ (-1)^{(|u|+n-3)(n-3)} u \hbar, & u \in \widehat{\mathcal{C}}^* \cup \mathcal{X}^*. \end{cases} \end{aligned}$$

The algebra U can be decomposed according to tensor type: $U = \bigoplus_{p, q \geq 0} U_{p, q}^p$, where $U_{p, q}^p \subset U$ is the K -subspace generated by monomials with q factors from M and p from M^* . We decompose further: $U_{p, q}^p = \bigoplus_{s, h \in \mathbb{Z}} U_{p, q}^p(s, h)$, where $U_{p, q}^p(s, h) \subset U_{p, q}^p$ is the subspace spanned by all monomials of total σ -degree s and total $\hbar$ -degree h . In particular, $U_0^0(0, 0) = A$, $U_1^0(0, 0) = M$, and $U_0^1(0, 0) = M^*$. We also write $U^p := \bigoplus_{q \geq 0} U_{p, q}^p$, $U^+ := \bigoplus_{p > 0} U^p$, $U_q := \bigoplus_{p \geq 0} U_{p, q}^p$, $U_q(s, h) := \bigoplus_{p \geq 0} U_{p, q}^p(s, h)$, and $U^p(s, h) := \bigoplus_{q \geq 0} U_{p, q}^p(s, h)$.

For $X \in U_{p, q}^p(s, h)$, let

$$st(X) := p + h \quad \text{and} \quad jp(X) := p + q + s.$$

In analogy with the definition of M^{diag} in Section 2.1.B, we define U^{diag} . Further, define U^{cyc} as the quotient of U^{diag} obtained by dividing by the following *graded cyclic permutation* rule: If $a_1, \dots, a_k \in \widehat{\mathcal{C}} \cup \widehat{\mathcal{C}}^* \cup \mathcal{X} \cup \mathcal{X}^*$ and if $a_1 \dots a_k \in U^{\text{diag}}$, then $a_2 \dots a_k a_1 \in U^{\text{diag}}$, and we identify $a_1 \dots a_k$ with $\varepsilon_1 \varepsilon_2 \varepsilon_3 a_2 \dots a_k a_1$, where

$$\begin{aligned} \varepsilon_1 &= (-1)^{|a_1||a_2 \dots a_k|}, \\ \varepsilon_2 &= \begin{cases} 1 & \text{if } a_1 \in \mathcal{E} \cup \{\hbar, \hbar^{-1}\}, \\ (-1)^{jp(a_2 \dots a_k)} & \text{if } a_1 \in \widehat{\mathcal{E}} \cup \mathcal{U} \widehat{\mathcal{E}}^* \cup \{\sigma, \sigma^{-1}\}, \end{cases} \\ \varepsilon_3 &= \begin{cases} 1 & \text{if } a_1 \in \mathcal{E} \widehat{\mathcal{E}} \mathcal{U} \cup \{\sigma, \sigma^{-1}\}, \\ (-1)^{st(a_2 \dots a_k)} & \text{if } a_1 \in \widehat{\mathcal{E}}^* \cup \mathcal{U}^* \cup \{\hbar, \hbar^{-1}\}. \end{cases} \end{aligned}$$

We will often think of cyclic monomials as written on the unit circle S^1 with the first letter at $1 \in S^1$. This allows us to speak about generators in a cyclic monomial as ordered in the counterclockwise direction starting from some fixed generator in the monomial. In this language, the above permutations correspond to rotations of the circle.

The vector space U^{cyc} inherits the decomposition by tensor type:

$$U^{\text{cyc}} = \bigoplus_{p,q \geq 0} (U_q^p)^{\text{cyc}} = \bigoplus_{p,q \geq 0} \bigoplus_{s,h \in \mathbb{Z}} (U_q^p(s,h))^{\text{cyc}}.$$

If the monomials X and X' differ by a graded cyclic permutation (i.e., if $\llbracket X \rrbracket = \llbracket X' \rrbracket$), then, for $z \in \{\sigma, \sigma^{-1}, \hbar, \hbar^{-1}\}$, $\llbracket Xz \rrbracket = \llbracket X'z \rrbracket$ and $\llbracket zX \rrbracket = \llbracket zX' \rrbracket$. We thus define $z\llbracket X \rrbracket := \llbracket zX \rrbracket$ and $\llbracket X \rrbracket z := \llbracket Xz \rrbracket$, and note that this definition does not depend on choice of representative X .

A monomial $X \in (U_q^p(s,h))^{\text{cyc}}$ is *balanced* if $h = -1$ and $p - q - s = 0$. The subspace of U^{cyc} generated by balanced monomials will be denoted by U^{bal} . It contains balanced versions M^{bal} and $(M^*)^{\text{bal}}$ of M^{cyc} and $(M^*)^{\text{cyc}}$, respectively. More precisely,

$$\begin{aligned} M^{\text{bal}} &:= (U_1^0(-1, -1))^{\text{cyc}} = U_1^0 \cap U^{\text{bal}}, \\ (M^*)^{\text{bal}} &:= (U_0^1(1, -1))^{\text{cyc}} = U_0^1 \cap U^{\text{bal}}. \end{aligned}$$

2.1.D The operator $\mathcal{S}$: $U^{\text{cyc}} \rightarrow U^{\text{cyc}}$. Consider a monomial $X \in (U_q^p(s,h))^{\text{cyc}}$ and fix a factor $c \in \mathcal{E}$ in X . Then, $X = \llbracket X'_c c X''_c \rrbracket$ for monomials $X'_c, X''_c \in U$. Define

$$\mathcal{S}(X) = \sum_c \llbracket X'_c \sigma^{-1} \hat{c} X''_c \rrbracket \in (U_{q+1}^p(s-1,h))^{\text{cyc}},$$

where the sum is taken over all factors $c \in \mathcal{E}$ in X . It is straightforward to check that this gives a well defined operation and by definition $\mathcal{S}(U^{\text{bal}}) \subset U^{\text{bal}}$. In particular, if $X = \llbracket c_1 \dots c_m \hat{a}^* \rrbracket \in (U_0^1(0,0))^{\text{cyc}}$, then $\mathcal{S}(X) \in (U_1^0(-1,0))^{\text{cyc}}$ satisfies

$$\begin{aligned} \mathcal{S}(c_1 \dots c_m \hat{a}^*) &= \sigma^{-1} \sum_{j=1}^m (-1)^{|c_1 \dots c_{j-1}|} c_1 \dots c_{j-1} \hat{c}_j c_{j+1} \dots c_m \hat{a}^* \\ &= \sigma^{-1} S(c_1 \dots c_m) \hat{a}^*, \end{aligned}$$

where S is the operator $A \rightarrow M$ defined in Eq. 2.5 above. Furthermore,

$$\mathcal{S}^k(c_1 \dots c_m \hat{a}^*) = k! \sigma^{-k} \sum_{j_1 < \dots < j_k} (-1)^{\sum_{l=1}^k |c_1 \dots c_{j_l-1}|} c_1 \dots \hat{c}_{j_1} \dots \hat{c}_{j_k} \dots c_m \hat{a}^*.$$

2.1.E Symmetry breaking and contraction operations. We next effectively break the cyclic symmetry of U^{bal} . An *excited* monomial is a cyclic monomial $X \in U^{\text{bal}}$ with one of its factors from $\widehat{\mathcal{E}} \cup \widehat{\mathcal{E}}^* \cup \mathcal{U} \cup \mathcal{U}^*$ distinguished. We call the distinguished factor of an excited monomial X the *excited* generator of X . The K -vector space generated by excited monomials from U^{bal} will be denoted by $\mathbf{U}^{\text{bal}}$. Note that $\mathbf{U}^{\text{bal}}$ inherits decompositions from U^{bal} , $\mathbf{U}^{\text{bal}} = \bigoplus_{p,q} (\mathbf{U}_q^p)^{\text{bal}}$, etc. We set $(\mathbf{U}_0^0)^{\text{bal}} = 0$. Define the *excitation operator* $\mathcal{E}: U^{\text{bal}} \rightarrow \mathbf{U}^{\text{bal}}$ to be the linear map that maps a monomial X the sum of all its excitations.

Note that $\mathbf{U}^{\text{bal}}$ contain excited versions $\mathbf{M}^{\text{bal}} = (\mathbf{U}_1^0)^{\text{bal}}$ and $(\mathbf{M}^*)^{\text{bal}} = (\mathbf{U}_0^1)^{\text{bal}}$ of M^{bal} and $(M^*)^{\text{bal}}$, respectively, which are isomorphic to their nonexcited counterparts. [In the context of

$\mathbf{U}^{\text{bal}}$, we think of M^{bal} and $(M^*)^{\text{bal}}$ as generated by monomials in $(\mathbf{U}_1^0)^{\text{bal}}$ and $(\mathbf{U}_0^1)^{\text{bal}}$ *without* excited generator.]

We will use the following notation to identify excited generators in monomials: We underline the excited generator in a monomial in U^{bal} and write, e.g., $\underline{x}$, $\underline{x}^*$, $\underline{\hat{c}}$, and $\underline{\hat{c}}^*$.

We next define a product on $\mathbf{U}^{\text{bal}}$. Consider two excited cyclic monomials $X, Y \in \mathbf{U}^{\text{bal}}$. Fix a factor $u \in \widehat{\mathcal{E}}^* \cup \mathcal{U}^*$ from X and a factor $v \in \widehat{\mathcal{E}} \cup \mathcal{U}$ from Y . Write $X = \varepsilon_1 \llbracket X'_u u \rrbracket$ and $Y = \varepsilon_2 \llbracket v Y'_v \rrbracket$, where ε_1 is the sign that arises from cyclically rotating X so that u is its last factor and where ε_2 is the sign that arises from cyclically rotating Y so that v is its first factor. Define the *contraction* operation $\star: \mathbf{U}^{\text{bal}} \otimes \mathbf{U}^{\text{bal}} \rightarrow \mathbf{U}^{\text{bal}}$,

$$X \star Y := \sum_{(u,v)} \varepsilon_1 \varepsilon_2 u(v) \llbracket X'_u Y'_v \rrbracket, \quad [2.9]$$

where the sum is taken over all pairs (u,v) of factors $u \in \widehat{\mathcal{E}}^* \cup \mathcal{U}^*$ from X and $v \in \widehat{\mathcal{E}} \cup \mathcal{U}$ from Y , and where

$$u(v) = \begin{cases} \hbar & \text{if exactly one of } u \text{ and } v \text{ is excited, } u = v^*, \\ & \text{and } v \notin \mathcal{U}, \\ (-1)^{n-1} \hbar & \text{if } u = \underline{x_j^*} \text{ and } v = x_j, \text{ for some } j = 1, \dots, m, \\ \hbar & \text{if } u = \underline{x_j^*} \text{ and } v = \underline{x_j}, \text{ for some } j = 1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

If $Y \in (\mathbf{U}_q^1)^{\text{bal}}$ and $X \in \mathbf{U}^{\text{bal}}$, then we define, for $j = 1, \dots, q$, the *partial contraction* $X \star_j Y$ as follows:

$$X \star_j Y := \sum_u \varepsilon_1 \varepsilon_2 u(v_j) \llbracket X'_u Y'_v \rrbracket,$$

where the sum is taken over all $u \in \widehat{\mathcal{E}}^* \cup \mathcal{U}^*$ from X and where v_j is the j th factor from $\widehat{\mathcal{E}} \cup \mathcal{U}$ in Y counting *counterclockwise* from the unique factor from $\widehat{\mathcal{E}}^* \cup \mathcal{U}^*$. Similarly, if $X \in (\mathbf{U}_1^0)^{\text{bal}}$ and $Y \in \mathbf{U}^{\text{bal}}$ then we define, for $i = 1, \dots, p$, the *partial contraction* $X \star^i Y$:

$$X \star^i Y := \sum_v \varepsilon_1 \varepsilon_2 u_i(v) \llbracket X'_u Y'_v \rrbracket,$$

where the sum is taken over all $v \in \widehat{\mathcal{E}} \cup \mathcal{U}$ from Y and where u_i is the i th factor from $\widehat{\mathcal{E}}^* \cup \mathcal{U}^*$ in X counting *clockwise* from the unique factor from $\widehat{\mathcal{E}} \cup \mathcal{U}$.

Remark 2.3: Let $X, Y \in \mathbf{U}^{\text{bal}}$, then $X \star Y \in \mathbf{U}^{\text{bal}}$. Further, if $X \in (\mathbf{U}_q^1)^{\text{bal}}$, then $X \star_j Y \in \mathbf{U}^{\text{bal}}$, and if $Y \in (\mathbf{U}_1^p)^{\text{bal}}$, then $X \star^i Y \in \mathbf{U}^{\text{bal}}$.

Define the *bracket* or *commutator* $[\cdot, \cdot]: \mathbf{U} \otimes \mathbf{U} \rightarrow \mathbf{U}$ as

$$[X, Y] = X \star Y - (-1)^{|X||Y|} Y \star X. \quad [2.10]$$

Lemma 2.4. The bracket satisfies the graded Jacobi identity:

$$\begin{aligned} (-1)^{|X||Z|} [X, [Y, Z]] + (-1)^{|Z||Y|} [Z, [X, Y]] + (-1)^{|Y||X|} [Y, [Z, X]] \\ = 0. \end{aligned}$$

Let

$$\tau \mathbf{M}^{\text{bal}} = \mathbf{M}^{\text{bal}} \oplus M^{\text{bal}} \quad \text{and} \quad (\tau \mathbf{M}^*)^{\text{bal}} = (\mathbf{M}^*)^{\text{bal}} \oplus (M^*)^{\text{bal}},$$

and note that contractions and partial contractions acting on $\mathbf{M}^{\text{bal}}$ and $(\mathbf{M}^*)^{\text{bal}}$ extend to $\tau \mathbf{M}^{\text{bal}}$ and $(\tau \mathbf{M}^*)^{\text{bal}}$, respectively. We associate with $X \in (\mathbf{U}_1^p)^{\text{bal}}$ a multilinear operation $X^\dagger: (\tau \mathbf{M}^{\text{bal}})^{\otimes p} \rightarrow \tau \mathbf{M}^{\text{bal}}$ defined by

$$X^\dagger(A_1 \otimes \dots \otimes A_p) = (\dots ((X \star^1 A_1) \star^1 A_2) \star^1 \dots) \star^1 A_p. \quad [2.11]$$

Note that if $A_1, \dots, A_p \in \mathbf{M}^{\text{bal}}$, then $X^\dagger(A_1 \otimes \dots \otimes A_p) \in \mathbf{M}^{\text{bal}}$; that if exactly one of the arguments lies in M^{bal} , then the image lies in M^{bal} ; and that the operation vanishes on p -tuples with more than component in M^{bal} .

Similarly, $Y \in (\mathbf{U}_q^1)^{\text{bal}}$ defines a multilinear operation $Y^\dagger: ((\tau \mathbf{M}^*)^{\text{bal}})^{\otimes q} \rightarrow (\tau \mathbf{M}^*)^{\text{bal}}$,

$$Y^\dagger(B_1 \otimes \dots \otimes B_q) = B_q \star_1 (\dots \star_1 (B_2 \star_1 (B_1 \star_1 Y)) \dots), \quad [2.12]$$

if $B_1, \dots, B_q \in (\mathbf{M}^*)^{\text{bal}}$, then $Y^\dagger(B_1 \otimes \dots \otimes B_q) \in (\mathbf{M}^*)^{\text{bal}}$; if exactly one of the arguments lies in $(\mathbf{M}^*)^{\text{bal}}$, then the image lies in $(\mathbf{M}^*)^{\text{bal}}$; and the operation vanishes on q -tuples with more than one component in $(\mathbf{M}^*)^{\text{bal}}$.

In the special case $q = 2$, we associate with $Y \in (\mathbf{U}_2^1)^{\text{bal}}$ two more operations,

$$Y^{\uparrow\dagger}: (\tau\mathbf{M}^*)^{\text{bal}} \otimes \tau\mathbf{M}^{\text{bal}} \rightarrow \tau\mathbf{M}^{\text{bal}} \quad [2.13]$$

and

$$Y^{\downarrow\dagger}: \tau\mathbf{M}^{\text{bal}} \otimes (\tau\mathbf{M}^*)^{\text{bal}} \oplus \rightarrow \tau\mathbf{M}^{\text{bal}}, \quad [2.14]$$

where

$$Y^{\uparrow\dagger}(X, Z) = (X \star_1 Y) \star Z, \quad X \in (\tau\mathbf{M}^*)^{\text{bal}}, \quad Z \in \tau\mathbf{M}^{\text{bal}},$$

$$Y^{\downarrow\dagger}(Z, X) = (X \star_2 Y) \star Z, \quad Z \in \tau\mathbf{M}^{\text{bal}}, \quad X \in (\tau\mathbf{M}^*)^{\text{bal}}.$$

Similarly, $X \in (\mathbf{U}_0^2)^{\text{bal}}$ induces an operation $X^\odot: \tau\mathbf{M}^{\text{bal}} \rightarrow (\tau\mathbf{M}^*)^{\text{bal}}$,

$$X^\odot(Y) = X \star Y. \quad [2.15]$$

We observe that $X^\odot(\mathbf{M}^{\text{bal}}) \subset (\mathbf{M}^*)^{\text{bal}}$ and $X^\odot(\mathbf{M}^{\text{bal}}) \subset (\mathbf{M}^*)^{\text{bal}}$.

2.1.F The Hamiltonian. We define special elements $H_q^\dagger \in \mathbf{U}^{\text{bal}}$ that derive from the differential on A and that, in the spirit of SFT (see ref. 8), are called *Hamiltonians*.

Definition 2.5:

$$(h_1^\dagger)' := \sum_{c \in \mathcal{C}} \|S(dc) \hbar^{-1} \hat{c}^*\|$$

$$(h_1^\dagger)'' := \sum_{i,j=1}^m \sum_{c \in \mathcal{C}_{ij}} (\|cx_j \hbar^{-1} \hat{c}^*\| - \|x_i c \hbar^{-1} \hat{c}^*\|) \quad h_1^\dagger := (h_1^\dagger)' + (h_1^\dagger)''$$

$$h_2^\dagger := \frac{1}{2!} \mathcal{S}(h_1^\dagger)' + \mathcal{S}(h_1^\dagger)'' + (-1)^{n-1} \sigma^{-1} \sum_{j=1}^m \|x_j x_j \hbar^{-1} x_j^*\|$$

$$h_q^\dagger := \frac{1}{q!} \mathcal{S}^{q-1} h_1^\dagger, \quad \text{for } q > 2, \quad H_q^\dagger := \mathcal{E}(h_q^\dagger), \quad H^1 = \sum_{q \geq 1} H_q^\dagger.$$

We use the Hamiltonian $H_1^\dagger$ to define a differential on $\tau\mathbf{U}^{\text{bal}} := \mathbf{U}^{\text{bal}} \oplus \mathbf{M}^{\text{bal}} \oplus (\mathbf{M}^*)^{\text{bal}}$. First, define the *exterior differential* $d: \tau\mathbf{U}^{\text{bal}} \rightarrow \tau\mathbf{U}^{\text{bal}}$ as follows. Consider the map $d: U \rightarrow U$, which acts as the algebra differential $d: A \rightarrow A$ on generators of U from A , which maps all other generators to 0, and which satisfies the graded Leibniz rule. Then, d preserves U^{diag} , descends to U^{cyc} where it preserves U^{bal} . It then induces a map $d: \tau\mathbf{U}^{\text{bal}} \rightarrow \tau\mathbf{U}^{\text{bal}}$ as follows. The map is already determined on $\mathbf{M}^{\text{bal}} \oplus (\mathbf{M}^*)^{\text{bal}}$, and if $\llbracket uX \rrbracket$ is a monomial in $\mathbf{U}^{\text{bal}}$, where $u \in \mathcal{C} \cup \mathcal{X} \cup \widehat{\mathcal{C}} \cup \widehat{\mathcal{C}}^* \cup \mathcal{X}^*$, then $d(\llbracket uX \rrbracket) = (-1)^{|u|} \llbracket u(dX) \rrbracket$. Then, $d^2 = 0$, and we observe that H^1 satisfies the following *master equation*:

$$dH^1 + H^1 \star H^1 = 0. \quad [2.16]$$

Because $|H^1| = -1$, $H^1 \star H^1 = \frac{1}{2} [H^1, H^1]$ and Eq. 2.16 can be rewritten in Maurer–Cartan form:

$$dH^1 + \frac{1}{2} [H^1, H^1] = 0. \quad [2.17]$$

Decomposing $H^1 = \sum_{q \geq 1} H_q^\dagger$, Eq. 2.16 is equivalent to the following sequence of identities for the $(\mathbf{U}_q^1)^{\text{bal}}$ -components of $H^1 \star H^1$:

$$dH_q^1 + \sum_{k=1}^q \sum_{j=1}^k H_{q-k+1}^1 \star_j H_k^1 = 0, \quad q = 1, 2, 3, \dots \quad [2.18]$$

The first identity in this sequence is

$$dH_1^1 + H_1^1 \star H_1^1 = 0. \quad [2.19]$$

Definition 2.6: Define the differential $d_U: \tau\mathbf{U}^{\text{bal}} \rightarrow \tau\mathbf{U}^{\text{bal}}$,

$$d_U X := dX + [H_1^1, X].$$

(Note that Eq. 2.19 is equivalent to $d_U^2 = 0$.)

The differential interacts with the bracket in the following way.

Lemma 2.7. For $X, Y \in \tau\mathbf{U}^{\text{bal}}$,

$$d_U([X, Y]) = [d_U X, Y] + (-1)^{|X|} [X, d_U Y].$$

In particular, the operation $[,]$ descends to the homology $H(\tau\mathbf{U}^{\text{bal}}, d_U)$.

We observe that d_U leaves $\mathbf{M}^{\text{bal}}$, $\mathbf{M}^{\text{bal}}$, $(\mathbf{M}^*)^{\text{bal}}$, and $(\mathbf{M}^*)^{\text{bal}}$ invariant. For simple notation, we write $d_{\tau\mathbf{M}}$ for the restriction of d_U to any one of these subspaces. Consider the isomorphisms $\beta: \mathbf{M}^{\text{cyc}} \rightarrow \mathbf{M}^{\text{bal}}$ and $\underline{\beta}: \mathbf{M}^{\text{cyc}} \rightarrow \mathbf{M}^{\text{bal}}$,

$$\beta(X) = X \hbar^{-1} \sigma^{-1} \quad \text{and} \quad \underline{\beta} = \mathcal{E} \circ \beta, \quad \text{respectively.} \quad [2.20]$$

Lemma 2.8. The maps $\beta: (\mathbf{M}^{\text{cyc}}, d_M) \rightarrow (\mathbf{M}^{\text{bal}}, d_{\tau\mathbf{M}})$ and $\underline{\beta}: (\mathbf{M}^{\text{cyc}}, d_M) \rightarrow (\mathbf{M}^{\text{bal}}, d_{\tau\mathbf{M}})$ are chain isomorphisms.

The following result is a consequence of Lemma 2.7.

Corollary 2.9. If $X \in \mathbf{U}^{\text{bal}}$ satisfies $d_U X = 0$ and has the correct tensorial type to define one of the operations 2.11–2.15 (see Section 2.1.E), then this operation is a chain map. If, furthermore, $X = d_U Y$ for some $Y \in \mathbf{U}^{\text{bal}}$, then the operation is chain homotopic to 0.

2.2 Products. We introduce product operations on $\mathbf{M}^{\text{bal}}$ and $(\mathbf{M}^*)^{\text{bal}}$.

2.2.A The product on $H((\mathbf{M}^*)^{\text{bal}})$. Recall the operation $(H_2^1)^\dagger: (\tau\mathbf{M}^*)^{\text{bal}} \otimes (\tau\mathbf{M}^*)^{\text{bal}} \rightarrow (\tau\mathbf{M}^*)^{\text{bal}}$ defined by the formula

$$(H_2^1)^\dagger(X, Y) = Y \star (X \star_1 H_2^1)$$

(see Eq. 2.12). Define $X \diamond Y := (-1)^{|X|} (H_2^1)^\dagger(X, Y)$. Then, $(\mathbf{M}^*)^{\text{bal}} \diamond (\mathbf{M}^*)^{\text{bal}} \subset (\mathbf{M}^*)^{\text{bal}}$.

Proposition 2.10. The operation $\diamond: (\mathbf{M}^*)^{\text{bal}} \otimes (\mathbf{M}^*)^{\text{bal}} \rightarrow (\mathbf{M}^*)^{\text{bal}}$ descends to an associative product on the homology $H((\mathbf{M}^*)^{\text{bal}}, d_{\tau\mathbf{M}})$, which satisfies the following graded commutativity relation for the grading shifted by 1:

$$X \diamond Y = (-1)^{(|X|+1)(|Y|+1)} Y \diamond X.$$

The homology class of the cycle $E := (-1)^{n-1} \sum_{i=1}^k \hbar^{-1} x_i^* \sigma \in (\mathbf{M}^*)^{\text{bal}}$ is the unit for this product on $H((\mathbf{M}^*)^{\text{bal}})$. In fact, E is a unit for $\diamond$ already on the chain level, before passing to homology.

The above result is a consequence of the master equation 2.16 and the explicit formula for H_2^1 in Definition 2.5.

2.2.B The product on $H(\mathbf{M}^{\text{bal}})$. If there exists an element

$$H' = \sum_{p \geq 2} H^p \in \bigoplus_{p \geq 2} (\mathbf{U}^p)^{\text{bal}}$$

such that $H = H^1 + H'$ satisfies the master equation

$$dH + H \star H = dH + \frac{1}{2} [H, H] = 0, \quad [2.21]$$

then we say that H' is *compatible* with H^1 , and we call $H = H^1 + H'$ the *full (rational) Hamiltonian*. As we shall see later, when A is a Legendrian homology algebra, then we can always find H' compatible with H^1 using a relative SFT formalism associated with parallel copies of the Legendrian submanifold in the spirit of refs. 9 and 10.

Assume now that H , and in particular H_0^2 , which satisfies Eq. 2.21 exists and is given. Then, we define the operation $\square$: $\tau\mathbf{M}^{\text{bal}} \otimes \tau\mathbf{M}^{\text{bal}} \rightarrow \tau\mathbf{M}^{\text{bal}}$,

$$X \square Y = (H_0^2 \star_1 H_2^1)^{\downarrow}(X, Y) = ((H_0^2 \star_1 H_2^1) \star^1 X) \star Y.$$

Note that if $X, Y \in \mathbf{M}^{\text{bal}} \subset \tau\mathbf{M}^{\text{bal}}$ then $X \square Y \in \mathbf{M}^{\text{bal}}$.

Theorem 2.11.

1. The operation $\square$: $\mathbf{M}^{\text{bal}} \otimes \mathbf{M}^{\text{bal}} \rightarrow \mathbf{M}^{\text{bal}}$ descends to an operation on the homology $H(\mathbf{M}^{\text{bal}}, d_{\tau M})$, where it can be expressed by the following three equivalent expressions:

$$\begin{aligned} X \square Y &= -(H_0^2 \star_2 H_2^1)^{\downarrow}(X, Y) = (-1)^{|X|} (H_2^1)^{\uparrow 1} ((H_0^2)^{\odot}(X), Y) \\ &= (-1)^{|X||Y|+|X|+1} (H_2^1)^{\uparrow 1}(X, (H_0^2)^{\odot}(Y)) \end{aligned}$$

for $X, Y \in \mathbf{M}^{\text{bal}}$.

2. The operation $\square$ is associative and graded commutative on homology; i.e., if $X, Y \in H(\mathbf{M}^{\text{bal}})$, then $X \square Y = (-1)^{|X||Y|} Y \square X$.

The above result is a consequence of the master equation 2.21. The products $\square$ on $\mathbf{M}^{\text{bal}}$ and $\diamond$ on $(\mathbf{M}^*)^{\text{bal}}$ are related via the chain map $(H_0^2)^{\odot}$: $\mathbf{M}^{\text{bal}} \rightarrow (\mathbf{M}^*)^{\text{bal}}$ in the following way.

Proposition 2.12. *The map induced by $(H_0^2)^{\odot}$ on homology is a homomorphism of rings Φ : $(H(\mathbf{M}^{\text{bal}}), \square) \rightarrow (H((\mathbf{M}^*)^{\text{bal}}), \diamond)$. Moreover, if there is a unit e for $\square$ on $H(\mathbf{M}^{\text{bal}})$ [i.e., $e \square X = X \square e$, for all $X \in H(\mathbf{M}^{\text{bal}})$], then Φ is injective.*

The ring homomorphism property is a consequence of Eq. 2.21. To verify the injectivity of Φ , we use one of equivalent definitions of $\square$ from Theorem 2.11, part 1:

$$X = X \square e = (-1)^{|X|+1} (H_2^1)^{\uparrow 1}(\Phi(X), e),$$

and hence, $\Phi(X) = 0$ implies $X = 0$.

Using the isomorphism β : $M^{\text{cyc}} \rightarrow \mathbf{M}^{\text{bal}}$, $\beta(X) = \mathcal{E}(X \sigma^{-1} \hbar^{-1})$ defined above, we transport the product to M^{cyc} . Given $X, Y \in M^{\text{cyc}}$, we define

$$X \boxtimes Y := \beta^{-1}(\beta(X) \square \beta(Y)). \quad [2.22]$$

Note that the homomorphism β shifts the grading by $2 - n$, and hence the commutation equation for $\square$ translates to the equation

$$X \boxtimes Y = (-1)^{(|X|-n)(|Y|-n)} Y \boxtimes X.$$

Let us also point out that the degree of the product $\boxtimes$ is equal to $-n$.

3 Legendrian Algebra Constructions

In this section, we first reformulate the main result relating Legendrian homology algebra and symplectic homology (Corollary 5.7 in ref. 1) in the terminology of Section 2, and second give a brief sketch of the geometry that enters the construction of the full rational Hamiltonian in the framework of Legendrian SFT with focus on the parts of it that enter the expression of the product.

3.1 The Complex $LH^{\text{Ho}}(\Lambda)$ Expressed as M^{cyc} . Let $(\bar{X}_0, \lambda)$ be a Liouville domain with contact boundary $(Y_0, \ker(\lambda))$, and let $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_m \subset Y_0$ be a union of Legendrian spheres. Let X_0 be the Liouville manifold that is obtained by attaching the positive half of the symplectization, $Y_0 \times [0, \infty)$, of Y_0 to $\bar{X}_0$. Let $\mathcal{E}$ be the set of Reeb chords of Λ , and consider the decomposition $\mathcal{E} = \bigcup_{1 \leq i \leq m} \mathcal{E}_{ij}$, where a chord in $\mathcal{E}_{ij}$ starts on Λ_j and

ends on Λ_i . Let V be the graded module $K\langle \mathcal{E} \rangle$ generated by the set $\mathcal{E}$ with decomposition $V = \bigoplus_{1 \leq i, j \leq m} V^{(ij)}$, where $V^{(ij)}$ is generated by $\mathcal{E}_{ij}$. Then, the Legendrian homology algebra $LHA(\Lambda)$ is the tensor algebra A associated to V as in Eq. 2.1, and the differential d_{LHA} gives a differential graded algebra $(A, d) := (LHA(\Lambda), d_{LHA})$ (see Section 4.1 in ref. 1). Let $M = M(A)$ be the chain complex defined in Eq. 2.2. The following result is a consequence of Remark 7.4 in ref. 1.

Lemma 3.1. *The chain complex M^{cyc} is canonically isomorphic to the chain complex $LH^{\text{Ho}}(\Lambda)$ defined in Section 4.5 of ref. 1.*

In Corollary 5.7 of ref. 1, it was shown that if the symplectic homology of X_0 vanishes, $SH(X_0) = 0$ (e.g., if X_0 is a subcritical Weinstein manifold) and if X is obtained from X_0 by attaching Lagrangian n -handles along Λ , then there is a quasi-isomorphism

$$\Phi: SH(X) \rightarrow LH^{\text{Ho}}(\Lambda) = M^{\text{cyc}} \stackrel{\beta}{=} \mathbf{M}^{\text{bal}}, \quad [3.1]$$

where $SH(X)$ denotes a chain complex with homology equal to $SH(X)$.

3.2 Legendrian rational SFT. The differential in the Legendrian algebra $A = LHA(\Lambda)$ is defined by counting elements of moduli spaces of rigid holomorphic disks in $\mathbb{R} \times Y_0$, with boundary on $\mathbb{R} \times \Lambda$, and with exactly one positive and several negative punctures (see, e.g., Section 4.1 of ref. 1). The formalism of Legendrian rational SFT comes from algebraic structures associated with holomorphic disks with an arbitrary number of positive and negative punctures. As is well known, in order to construct such a theory one needs to handle boundary cusp-degenerations of holomorphic disks. The problem of boundary cusp-degeneration can be solved by incorporating string topology operations into the SFT formalism. The corresponding theory has not yet been constructed in sufficient generality for it to be applicable in the setting of this paper. (See, however, ref. 11 for the case of one-dimensional Legendrian knots in $\mathbb{R}^3$).

However, the algebraic formalism described in Section 2 requires a different version of the full rational Hamiltonian H , in the spirit of ref. 9. It arises naturally when one considers moduli spaces of holomorphic disks with boundaries on multiple parallel copies of $\mathbb{R} \times \Lambda$.

Let us first consider the term H_0^1 . This is the only term besides H_2^1 that is needed for defining the product on M^{cyc} (see Fig. 1). We recall here that H_k^1 for any $k \geq 1$ is determined by the differential on the Legendrian algebra A .

For each $i = 1, \dots, m$ fix a point $y_i \in \Lambda_i$. Let $Z = [\underline{u}_1 w_1 u_2 w_2] \in (\mathbf{U}_0^2)^{\text{bal}}$, where $w_j = c_j^1 \dots c_j^j \in A$ and $u_j \in \mathcal{E}^* \cup \mathcal{E}^*$, $j = 1, 2$, be an excited cyclic monomial. We associate to it a moduli space $\mathcal{M}(Z) = \mathcal{M}'(Z) \cup \mathcal{M}''(Z)$. Here, the moduli space $\mathcal{M}'(Z)$ consists of holomorphic disks anchored in X_0 (see Section 2.2 of ref. 1),

$$f: (D, \partial D \setminus \mathbf{z}) \rightarrow (\mathbb{R} \times Y_0, \mathbb{R} \times \Lambda),$$

where

$$\mathbf{z} = \{z_{u_1}, z_{c_1^1}, \dots, z_{c_1^{j_1}}, z_{u_2}, z_{c_2^1}, \dots, z_{c_2^{j_2}}\} \subset \partial D$$

Fig. 1. Holomorphic disks used in the expression of the product, Λ_i ; $j = 0, 1, 2$ are parallel copies of Λ , and only Reeb chords between different components are shown.

is a set of cyclically ordered boundary punctures on ∂D , with the following properties:

- (ch⁻) at the puncture $z_{c_i}^i$, f is asymptotic to the chord c_j^i at $-\infty$;
- (ch⁺) if $u_j = \hat{c}^* \in \hat{\mathcal{C}}^*$, then at the puncture z_{u_j} , f is asymptotic to the chord c at $+\infty$;
- (pt⁺) if $u_j = x_i^* \in \mathcal{X}^*$, then f extends continuously over z_{u_j} , and $f(z_{u_j}) \in \mathbb{R} \times \{y_i\}$.

The moduli space $\mathcal{M}''(Z)$ consists of holomorphic disks connected by Morse flow lines. To define it, choose auxiliary Morse functions $\phi_i: \Lambda_i \rightarrow \mathbb{R}$, $i = 1, \dots, m$, each with exactly two critical points. For $(s, t) \in \{0, 1, \dots, r_1\} \times \{0, 1, \dots, r_2\}$, define the auxiliary moduli space $\mathcal{M}_{(s,t)}(Z)$ of pairs (f_1, f_2) of holomorphic disks anchored in X_0 ,

$$f_\sigma: (D, \partial D \setminus \mathbf{z}_\sigma) \rightarrow (\mathbb{R} \times Y_0, \mathbb{R} \times \Lambda), \quad \sigma = 1, 2,$$

where

$$\begin{aligned} \mathbf{z}_1 &= \{z_{u_1}, z_{c_1^1}, \dots, z_{c_1^{r_1}}, z_{c_2^{r_1+1}}, \dots, z_{c_2^{r_2}}\} \subset \partial D, \\ \mathbf{z}_2 &= \{z_{u_2}, z_{c_2^1}, \dots, z_{c_2^{r_2}}, z_{c_1^{r_2+1}}, \dots, z_{c_1^{r_1}}\} \subset \partial D \end{aligned}$$

are sets of cyclically ordered boundary punctures on ∂D , with the following properties. At z_{u_σ} , f_σ satisfies condition (ch⁺) if $u_\sigma \in \hat{\mathcal{C}}^*$ or condition (pt⁺) if $u_\sigma \in \mathcal{X}^*$. At $z_{c_i}^i$, f_σ satisfies condition (ch⁻). At the puncture z_{c_σ} , $\sigma = 1, 2$, f_σ extends continuously and the following condition holds:

- (mo) The points $f_1(\zeta_1)$ and $f_2(\zeta_2)$ lie in the same component $\Lambda_i \subset \Lambda$, and there exists a trajectory of $-\nabla \phi_i$ that starts at $f_1(\zeta_1)$ and ends at $f_2(\zeta_2)$, where $\phi_i: \Lambda_i \rightarrow \mathbb{R}$ is the auxiliary Morse function on Λ_i . Here, the gradient is taken with respect to the metric induced by the symplectic form and the almost complex structure.

Finally, define

$$\mathcal{M}''(Z) := \bigcup_{(s,t) \in \{0, \dots, r_1\} \times \{0, \dots, r_2\}} \mathcal{M}_{(s,t)}(Z).$$

Given $u \in \hat{\mathcal{C}} \cup \hat{\mathcal{C}}^*$, let $\tilde{u} = a$ if $u = \hat{a}$ or $u = \hat{a}^*$. Define $|\tilde{x}_j| = |\tilde{x}_j^*| = -1$, $j = 1, \dots, m$. Then, we have the dimension formula (see ref. 12),

$$\dim(\mathcal{M}(Z)) = |\tilde{u}_1| + |\tilde{u}_2| - |w_1| - |w_2| - (n - 3).$$

In particular in terms of the grading on U , for the balanced monomial $\sigma^2 \hbar^{-1} Z$ we have

$$\dim(\mathcal{M}(Z)) = -|\sigma^2 \hbar^{-1} Z|.$$

Define $H_0^2 \in (U_0^2)^{\text{bal}}$ as $\mathcal{G}(h_0^2)$, where

$$h_0^2 = \sigma^2 \hbar^{-1} \sum_{|\sigma^2 \hbar^{-1} Z| = -1} \# \mathcal{M}(Z) Z, \quad [3.2]$$

where the sum is taken over all balanced monomials $\sigma^2 \hbar^{-1} Z$, which additively generate $(U_0^2)^{\text{bal}}$ and for which $|\sigma^2 \hbar^{-1} Z| = -1$. Here, $\# \mathcal{M}(Z)$ is the algebraic number of one-dimensional components of the moduli space $\mathcal{M}(Z)$. As will be briefly discussed below, the above definition of H_0^2 can be generalized to more than two positive punctures, and as a consequence we get the following result.

Proposition 3.2. *There exists an element $H'' \in U^{\text{bal}}$, with trivial component in $(U^1)^{\text{bal}} \oplus (U_0^2)^{\text{bal}}$, such that $H' = H_0^2 + H''$ complements H^1 to the full rational SFT Hamiltonian (i.e., if $H = H^1 + H_0^2 + H''$, then $dH + H \star H = 0$).*

Here, the components $H_q^p \in (U_q^p)^{\text{bal}}$, $p > 2$, of the full rational Hamiltonian H satisfies the relation $H_q^p = \frac{1}{q!} \mathcal{S}^q(h_0^p)$, $h_0^p = \frac{1}{p} \pi(H_0^p)$, and they are defined through a construction that generalizes that

Fig. 2. A generalized holomorphic disk in $Y_0 \times \mathbb{R}$ with boundary on $\mathbb{R} \times \Lambda$.

used in the definition of H_0^2 above. These higher-order components of the Hamiltonian are not needed for the definition of the product (but some of them are needed for proving its properties) and will not be defined in this paper. We give, however, a brief description: H_q^p is defined using moduli spaces $\mathcal{M}(Z)$ of so-called *generalized holomorphic disks*. Generalized holomorphic disks are combined objects consisting of holomorphic disks with boundary on one copy of Λ connected by gradient flow trees associated to auxiliary Morse functions on Λ ; an example is depicted in Fig. 2 (see ref. 13 for similar considerations). The elements of the moduli space $\mathcal{M}''(Z)$ defined above are examples of generalized holomorphic disks where the gradient flow tree consists of only one gradient trajectory. A more general configuration is shown in Fig. 2.

Remark 3.3: In this geometric context, the excitation operator $\mathcal{E}$ is related to an “abnormal” jump between parallel copies of Λ , and the “gluing sign” in the equations $x_j^*(x_j) = (-1)^{n-1} \hbar$, $j = 1, \dots, m$ arises because of a sign difference between the gluings at short Reeb chords between parallel copies corresponding to the maximum and minimum of a shifting Morse function.

4 The Product and the BV Operator via Legendrian Algebra

Let X_0 , $\Lambda \subset Y_0$, and X be as above. Let $A = LHA(\Lambda)$ and $M = M(A)$. Then, Eq. 3.1 gives a quasi-isomorphism $\Phi: SH(X) \rightarrow M^{\text{cyc}} \approx M^{\text{bal}}$. In this section, we present the operations on M^{cyc} that correspond to the product and the BV operator on $SH(X)$ under Φ .

4.1 The Product. Let $H \in U^{\text{bal}}$ be the full rational Hamiltonian determined by $\Lambda \subset Y_0$ (see Eq. 3.2). Then, according to Theorem 2.11 and Eq. 2.22, there exists a product $\boxtimes: M^{\text{cyc}} \otimes M^{\text{cyc}} \rightarrow M^{\text{cyc}}$ of degree $-n$ that is associative and commutative on homology and that is defined by conjugating the formula for the product $\boxtimes$ on M^{bal} with β .

Theorem 4.1. *If $S\mathbb{H}(X_0) = 0$ (e.g., if X_0 is subcritical) then, under the map on homology induced by the quasi-isomorphism $\Phi: SH(X) \rightarrow M^{\text{cyc}}$ (see Eq. 3.1), the pair-of-pants product on $S\mathbb{H}(X)$ corresponds to the product $\boxtimes$ on $H(M^{\text{cyc}})$.*

4.2 The BV Operator. Let $A = LHA(\Lambda)$, and consider the linear map $\Delta: M^{\text{cyc}} \rightarrow M^{\text{cyc}}$ of degree 1 defined as follows on monomials $\llbracket uw \rrbracket$, $u \in \hat{\mathcal{C}} \cup \mathcal{X}$, $w \in A$:

$$\Delta(\llbracket uw \rrbracket) = \begin{cases} S(w) & \text{if } u \in \mathcal{X}, \\ 0 & \text{if } u \in \hat{\mathcal{C}}. \end{cases}$$

Proposition 4.2. *The map Δ is a chain map, and if $S\mathbb{H}(X_0) = 0$, then, under the map on homology induced by the quasi-isomorphism $\Phi: SH(X) \rightarrow M^{\text{cyc}}$ (see Eq. 3.1), the BV operator on $S\mathbb{H}(X)$ corresponds to the operator Δ on $H(M^{\text{cyc}})$.*

5 Example: $SH(T^*S^n)$

In this section, we compute the symplectic homology with product and BV operator for cotangent bundles of spheres presented as the result of attaching a Lagrangian n -handle to the Legendrian unknot in the boundary of the ball.

Consider the Legendrian unknot in $\mathbb{R}^{2n-1}$ with coordinates $(q, p, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$ and contact form $dz - p \cdot dq$. To define the Legendrian unknot of dimension $n-1$ consider the one-dimensional unknot Λ with exactly one Reeb chord, over $q = p = 0$, and symmetric with respect to the rotation by a half turn around the z axis. Include Λ into the $q_1 p_1 z$ -subspace of $\mathbb{R}^{2n-1}$ and take the $(n-1)$ -dimensional unknot as the orbit of Λ under the action by $SO(n-1)$ given by $\sigma(q, p, z) = (\sigma \cdot q, \sigma \cdot p, z)$ for $\sigma \in SO(n-1)$. For simplicity, we let Λ denote the Legendrian unknot in $\mathbb{R}^{2n-1}$ for any n .

Consider $\Lambda \subset \mathbb{R}^{2n-1}$ and include it into a Darboux chart in the boundary of the $2n$ -ball B^{2n} . Then, attaching a Lagrangian n -handle to B^{2n} along Λ gives a symplectic manifold that is symplectomorphic to the cotangent bundle T^*S^n of the n -sphere. As explained in ref. 1, in order to determine $LHA(\Lambda)$ it suffices to study Reeb chords and holomorphic disks that are contained in the Darboux chart. The unknot Λ has only one Reeb chord, which we denote a and $|a| = n-1$.

The moduli space of holomorphic disks with positive puncture at a and a marked point on the boundary evaluates with degree ± 1 to Λ . To see this, note that for $n=2$, holomorphic disks that are rigid up to translation correspond to polygons with boundary on the knot diagram and with convex corners, where at positive punctures the boundary orientation goes from the lower to the upper strand and vice versa at negative punctures. There are thus two disks with evaluation maps that give a degree ± 1 map because we use the null-cobordant spin structure on the circle to orient moduli spaces. For $n > 2$, we note that we may take the almost complex structure to be invariant with respect to the $SO(n-1)$ action, and it follows that the moduli space of holomorphic disks with positive puncture at a is diffeomorphic to $S^{n-2} \times \mathbb{R}$ and that, when equipped with a marked point on the boundary, it evaluates with degree ± 1 .

The above description of moduli spaces implies in particular that the differential d on $A = LHA(\Lambda) = \mathbb{Q}\langle a \rangle$ is trivial; i. e., $d = 0$. Consequently, the differential on $\mathbf{M}^{\text{bal}}$ satisfies

$$d_M X = dX + [H_1^1, X] = [H_1^1, X],$$

where

$$H_1^1 = ([\underline{ax}\hat{h}^{-1}\hat{a}^*] - [\underline{xa}\hat{h}^{-1}\hat{a}^*]). \quad [5.1]$$

Let $k = 0, 1, \dots$. Then, for n even, $H(\mathbf{M}^{\text{bal}})$ is generated by the cycles $[\underline{x}] \hat{h}^{-1} \sigma^{-1}$, $[\underline{xa}^{2k+1}] \hat{h}^{-1} \sigma^{-1}$, and $[\underline{\hat{a}a}^{2k}] \hat{h}^{-1} \sigma^{-1}$ of degree $-(n-2)$, $2k(n-1)+1$, and $2k(n-1)+2$, respectively. For n odd, $H(\mathbf{M}^{\text{bal}})$ is generated by the cycles $[\underline{x}] \hat{h}^{-1} \sigma^{-1}$, $[\underline{xa}^{k+1}] \hat{h}^{-1} \sigma^{-1}$, and $[\underline{\hat{a}a}^k] \hat{h}^{-1} \sigma^{-1}$ of respective degrees $-(n-2)$, $k(n-1)+1$, and $k(n-1)+2$.

$H(M^{\text{cyc}})$ for an even n (recall the chain isomorphism $\beta: M^{\text{cyc}} \rightarrow \mathbf{M}^{\text{bal}}$; see Eq. 2.20) is generated by the cycles $[\underline{x}]$, $[\underline{xa}^{2k+1}]$, and $[\underline{\hat{a}a}^{2k}]$ of degree 0, $(2k+1)(n-1)$, and $(2k+1)(n-1)+1$, respectively. If n is odd, then $H(M^{\text{cyc}})$ is generated by the cycles $[\underline{x}]$, $[\underline{xa}^{k+1}]$, and $[\underline{\hat{a}a}^k]$ of the respective degrees 0, $(k+1)(n-1)$, and $(k+1)(n-1)+1$.

In order to compute the product we need to determine H_2^1 and H_0^2 . Consider first H_2^1 (see Definition 2.5):

$$\begin{aligned} h_2^1 &= \frac{1}{2} \mathcal{S} h_1^1 + \sigma^{-1} [\underline{xx}\hat{h}^{-1}x^*] \\ &= \sigma^{-1} ([\underline{\hat{a}x}\hat{h}^{-1}\hat{a}^*] + [\underline{x\hat{a}}\hat{h}^{-1}\hat{a}^*] + (-1)^{n-1} [\underline{xx}\hat{h}^{-1}x^*]); \quad [5.2] \end{aligned}$$

see Remark 5.2 for a discussion of the signs in this formula.

Remark 5.1: In our description of h in terms of generalized holomorphic disks, the disks that give rise to Eq. 5.2 are degenerate. The first two terms come from the trivial strip of a with an unconstrained Morse puncture lying in one of the two boundary components. The last term comes from the $\mathbb{R}$ -family of three-punctured constant disks lying in $\{y\} \times \mathbb{R}$, with one puncture constrained to lie in $\{y\} \times \mathbb{R}$ and the remaining two punctures unconstrained.

Next, consider h_0^2 . Because the evaluation map from the moduli space of holomorphic disks with positive puncture at a into Λ has degree ± 1 , there is algebraically ± 1 disk that satisfies a point constraint at $y \in \Lambda$, and we conclude from the description of generalized holomorphic disks that

$$h_0^2 = \sigma^2 [\underline{\hat{a}^*x^*}] \hat{h}^{-1}; \quad [5.3]$$

see Remark 5.2 for a discussion of the sign in this formula.

Remark 5.2: Orientations of moduli spaces are fixed by orienting capping operators at Reeb chords (see ref. 14 for details). The overall sign in Eq. 5.2 is independent of the choice of capping operator for a because variables associated to a appear quadratically in H_2^1 , the overall sign in Eq. 5.2 changes with the orientation data at x , and the overall sign in Eq. 5.3 changes, independently, with the orientations of capping operators at both x and a . It follows that we can choose any sign combination in the two equations and in particular the one under consideration.

To compute $H_0^2 \star_1 H_2^1$ acting on $\mathbf{M}^{\text{bal}} \otimes \mathbf{M}^{\text{bal}}$, we use the components of H_2^1 in which the second negative puncture as counted from the positive puncture in the counterclockwise direction is excited. We will denote the sum of these by $\hat{H}_2^1$. The relevant parts of the Hamiltonian are thus (see Eq. 5.3 and Eq. 5.2) (recall gradings: $|\hat{a}| = n$, $|\hat{a}^*| = n-3-n = -3$, $|x| = 0$, and $|x^*| = n-3$)

$$H_0^2 = \sigma^2 ([\underline{\hat{a}^*}\hat{h}^{-1}x^*] - [\underline{x^*}\hat{h}^{-1}\hat{a}^*]),$$

$$\hat{H}_2^1 = \sigma^{-1} ([\underline{\hat{a}x}\hat{h}^{-1}\hat{a}^*] + [\underline{x\hat{a}}\hat{h}^{-1}\hat{a}^*] + (-1)^{n-1} [\underline{xx}\hat{h}^{-1}x^*]),$$

and we get

$$\begin{aligned} H_0^2 \star_1 \hat{H}_2^1 &= -H_0^2 \star_1 ([\underline{\hat{a}x}\hat{h}^{-1}\hat{a}^*] + [\underline{x\hat{a}}\hat{h}^{-1}\hat{a}^*] \\ &\quad + (-1)^{n-1} [\underline{xx}\hat{h}^{-1}x^*]) \sigma^{-1} \\ &= \sigma^2 ([\underline{x^*}\hat{h}^{-1}\hat{a}^*] + (-1)^n [\underline{\hat{a}^*}\hat{h}^{-1}\hat{a}^*] - [\underline{\hat{a}^*}\hat{h}^{-1}x^*]) \sigma^{-1} \\ &= \sigma(-[\underline{x^*}\hat{h}^{-1}\hat{a}^*] + (-1)^{n-1} [\underline{\hat{a}^*}\hat{h}^{-1}\hat{a}^*] + [\underline{\hat{a}^*}\hat{h}^{-1}x^*]) \\ &= \sigma(-[\underline{x\hat{a}^*}\hat{h}^{-1}x^*] + (-1)^{n-1} [\underline{\hat{a}a^*}\hat{h}^{-1}\hat{a}^*] + [\underline{xx^*}\hat{h}^{-1}\hat{a}^*]). \end{aligned}$$

This then gives the following expression for the product $\square: H(\mathbf{M}^{\text{bal}}) \otimes H(\mathbf{M}^{\text{bal}}) \rightarrow H(\mathbf{M}^{\text{bal}})$, writing $\theta = \hat{h}^{-1} \sigma^{-1}$. If $n-1$ is odd, then

$$\begin{aligned} [\underline{x}] \theta \square [\underline{x}] \theta &= 0, \quad [\underline{x}] \theta \square [\underline{xa}^{2k+1}] \theta = 0, \\ [\underline{xa}^{2k+1}] \theta \square [\underline{xa}^{2l+1}] \theta &= 0, \quad [\underline{\hat{a}a}^{2k}] \theta \square [\underline{\hat{a}a}^{2l}] \theta = [\underline{\hat{a}a}^{2(l+k)}] \theta, \\ [\underline{\hat{a}}] \theta \square [\underline{x}] \theta &= [\underline{x}] \theta, \\ [\underline{\hat{a}a}^{2k}] \theta \square [\underline{x}] \theta &= [\underline{xa}^{2k}] \theta = \frac{1}{2} d_{\tau M} [\underline{\hat{a}a}^{2k}] \theta \simeq 0 \quad \text{if } k > 0, \\ [\underline{\hat{a}a}^{2k}] \theta \square [\underline{xa}^{2l+1}] \theta &= [\underline{xa}^{2(k+l)+1}] \theta, \end{aligned}$$

and if $n - 1$ is even, then for $k, l \geq 0$

$$\begin{aligned} \llbracket xa^k \rrbracket \theta \square \llbracket xa^l \rrbracket \theta &= 0, & \llbracket \hat{a}a^k \rrbracket \theta \square \llbracket xa^l \rrbracket \theta &= \llbracket xa^{k+l} \rrbracket \theta, \\ \llbracket \hat{a}a^k \rrbracket \theta \square \llbracket \hat{a}a^l \rrbracket \theta &= \llbracket \hat{a}a^{k+l} \rrbracket \theta. \end{aligned}$$

We conclude from this calculation that the commutative product $\boxtimes: M^{\text{cyc}} \otimes M^{\text{cyc}} \rightarrow M^{\text{cyc}}$, which corresponds to the pair-of-pants product on $S\mathbb{H}(T^*S^n)$, has unit represented by the class $\llbracket \hat{a} \rrbracket \in M^{\text{cyc}}$ and multiplication according to the above formulas after substituting θ with 1 everywhere.

Remark 5.3: A. Abbondandolo and M. Schwarz constructed in ref. 15 a ring isomorphism from $S\mathbb{H}(T^*S^n)$ with the pair-of-pants product to the homology of the free loop space $\Lambda(S^n)$ of S^n with the Chas–Sullivan product. This isomorphism matches the above computation of the product $\boxtimes$ with the computation of R.L. Cohen, J.D.S. Jones, and J. Yan in ref. 16 of the ring structure on $H_*(\Lambda(S^n))$.

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To finish the example, we determine also the operator $\Delta: H(M^{\text{cyc}}) \rightarrow H(M^{\text{cyc}})$, which corresponds to the BV operator on $S\mathbb{H}(T^*S^n)$. If $n - 1$ is odd, then

$$\begin{aligned} \Delta(\llbracket x \rrbracket) &= 0, & \Delta(\llbracket \hat{a}a^{2k} \rrbracket) &= 0, & k \geq 0, \\ \Delta(\llbracket xa^{2k+1} \rrbracket) &= \llbracket \hat{a}a^{2k} \rrbracket, & k \geq 0, \end{aligned}$$

and if $n - 1$ is even, then

$$\begin{aligned} \Delta(\llbracket x \rrbracket) &= 0, & \Delta(\llbracket \hat{a}a^{k-1} \rrbracket) &= 0, & k \geq 1, \\ \Delta(\llbracket xa^k \rrbracket) &= \llbracket \hat{a}a^{k-1} \rrbracket, & k \geq 1. \end{aligned}$$

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