Published in final edited form as:

Automatica (Oxf). 2011 June 1; 47(6): 1236–1242. doi:10.1016/j.automatica.2011.01.074.

On influences of global and local cues on the rate of synchronization of oscillator networks

Yongqiang Wang^a and Francis J. Doyle III^a

Yongqiang Wang: wyqthu@gmail.com; Francis J. Doyle: doyle@engineering.ucsb.edu

^aDepartment of Chemical Engineering, University of California, Santa Barbara, California 93106 USA

Abstract

Synchronization of connected oscillator networks under global and local cues is ubiquitous in both science and engineering. Over the last few decades, enormous attention has been paid to study synchronization conditions of connected oscillators in chemistry, physics, mechanics, and particularly in biology. However, the influences of global and local cues on the rate of synchronization have not been fully studied. It is widespread that synchronization is achieved in the simultaneous presence of both global and local cues, such as intercellular coupling signals and external entrainment signals in terms of biological oscillators, and inter-neighbor coupling signals between follower nodes and central guiding signals in terms of groups of mobile autonomous agents. We prove in this paper that strength of the global cue is the only determinant of the rate of synchronization. More specifically, we prove that a stronger global cue means a faster rate of synchronization whereas a stronger local cue does not necessarily make the synchronization rate faster. Our results not only apply to the noise free case, but also apply to the case that the oscillator natural frequencies are subject to white noise. The analysis does not require the interplay to be symmetric or balanced. Simulation results are given to illustrate the proposed results.

Keywords

Rate of synchronization; global cue; local cue; oscillators; circadian rhythm; entrainment

1 Introduction

Synchronization of oscillating dynamical systems is ubiquitous in science, nature, engineering and social life. Over the last several decades, it has been intensively studied in chemical processes, mechanical systems, physics and telecommunications, etc [1–3]. Recently, with the development of the emerging field of 'systems biology', synchronization phenomena in bioscience are also attracting considerable attention in the control community. Elucidation of the essential principles or fundamental mechanisms of synchronization has been greatly expedited by systems and control theory over the last few decades (c.f. [4–13] and the references therein).

© 2011 Elsevier Ltd. All rights reserved.

*Corresponding author Francis J. Doyle III. Tel. +1 805 893 8133. Fax 805-893-4731.

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

The interplay of global cues and local cues (or alternatively the interplay of external cues and local coupling) is an important feature in the achievement of synchronization. For example, in mammalian circadian systems, the circadian pacemaker controlling daily rhythms is located in the suprachiasmatic nucleus (SCN), in which about 20,000 neuronal oscillators produce an extremely accurate 24-hour rhythm utilizing intercellular interplay among individual oscillators while at the same time receiving a global driving cue such as the sunlight via the retinal input [14,15]. In the reproduction of certain species of corals, spawning is found to occur synchronously with each other within a few hours once a year, which is guided by global drivers such as ocean water temperature and phase of the moon, and simultaneously by local protein-based interactions [16]. There are many other examples of synchronization in biology, such as somatogenesis, cardiac function, respiration and insulin secretion, glycolytic oscillations and eukaryotic cell cycle, to name only a few [17]. In engineering applications, such as coordination of groups of mobile autonomous agents (e.g. UAV-unmanned aerial vehicles and MANET-mobile ad hoc network), the signal from central resources (e.g. leader node and satellite) acts as the global cue and the interplay between the follower nodes acts as the local cue, both of which are essential to the synchronization of the collective motion [4,18]. In clock synchronization of wireless sensor networks, different time references in a real-time network are synchronized through internal interplay between different nodes and external coordination from the global time base such as Global Positioning System (GPS) [19]. Therefore studying the influences of global and local cues in the achievement of synchronization is not only important in scientific research, but also of crucial importance in engineering applications.

Although synchronization of coupled oscillators has been studied for years, compared with the rich results on synchronization condition analysis (e.g., [20–22] discussed the influences of global and local cues on synchronization conditions), research on the rate of synchronization is relatively sparse. This paper studies the influences of global and local cues on the rate of synchronization. We prove analytically that the rate of synchronization is only determined by the strength of the global cue, and the strength of the local cue is not directly relevant. In other words, a stronger global cue means a faster rate of synchronization, whereas a stronger local cue does not necessarily make the rate of synchronization faster.

2 Problem formulation

2.1 Model in the noise free case

We first consider the phase model in the absence of noise. Suppose the overall system is composed of a network of N oscillators, which will henceforth be referred to as 'nodes'. Note that interpreted as a leader node, the global cue can be regarded as a node in the network. We denote the phase dynamics of the global cue as

$$\dot{\varphi}_1 = w_1 \quad (1)$$

and phase dynamics of the isolated oscillators as

$$\dot{\varphi}_i = w_i, i = 2, 3, \dots, N \quad (2)$$

where φ_1 and φ_i ($i = 2, 3, \dots, N$) denote the phases of the global cue and the rest oscillators, and w_1 and w_i ($i = 2, 3, \dots, N$) denote their natural frequencies, respectively.

After taking into consideration the interplay between different nodes, the overall dynamics can be rewritten as

$$\begin{aligned}\dot{\varphi}_1 &= w_1, \\ \dot{\varphi}_i &= w_i + \sum_{1 \leq j \leq N, j \neq i} a_{ij}(\varphi_j - \varphi_i), \quad 2 \leq i \leq N\end{aligned}\quad (3)$$

where $a_{ij} \geq 0$ denotes the coupling strength between node i and node j . We do not require the interplay to be symmetric, i.e., a_{ij} and a_{ji} are not necessarily equal. Note that the global cue influences all the rest nodes but is not conversely influenced by them. That is, a_{1j} ($j = 2, 3, \dots, N$), which denote the influences of the follower nodes on the leader node, are identically zero, but a_{j1} ($j = 2, 3, \dots, N$), which denote the influences of the leader node on the follower nodes, are not zero.

In a biological context such as in SCN, sunlight is a global cue, and the vasoactive intestinal peptide (VIP) is a local cue [14]. Therefore a_{j1} ($j = 2, 3, \dots, N$) represent the influences of sunlight on circadian neurons, and a_{ij} ($i, j = 2, 3, \dots, N$) represent the interplay between circadian neurons through the VIP [14]. And since the rhythm of sunlight is not influenced by circadian neurons, a_{1j} ($j = 2, 3, \dots, N$) are all zero.

2.2 Model in the presence of noise

When the natural frequencies w_i ($i = 2, 3, \dots, N$) are subject to noise, the dynamics can be denoted by

$$\begin{aligned}\dot{\varphi}_1 &= w_1, \\ \dot{\varphi}_i &= w_i + \sigma_i n_i(t) + \sum_{1 \leq j \leq N, j \neq i} a_{ij}(\varphi_j - \varphi_i), \quad 2 \leq i \leq N\end{aligned}\quad (4)$$

where $n_i(t)$ ($i = 2, 3, \dots, N$) are Gaussian white noise terms, $\sigma_i > 0$ ($i = 2, 3, \dots, N$) are constants. Since Gaussian white noise is a formal derivative of a Wiener process [23], (4) can be rewritten as a stochastic differential equation

$$\begin{aligned}d\varphi_1 &= w_1 dt, \\ d\varphi_i &= w_i dt + \sigma_i dB_i(t) + \sum_{1 \leq j \leq N, j \neq i} a_{ij}(\varphi_j - \varphi_i) dt\end{aligned}\quad (5)$$

for $2 \leq i \leq N$ where $B_i(t)$ is the Wiener process.

Based on the above models, we will analyze the influences of global cue, i.e., a_{j1} ($2 \leq j \leq N$) and local cue, i.e., a_{ij} ($2 \leq i \leq N, j \neq 1$) on the rate of synchronization.

3 Main results

3.1 Results in the noise free case

In the noise free case, we study the influences of global cue a_{j1} ($2 \leq j \leq N$) and local cue a_{ij} ($2 \leq j \leq N, 2 \leq i \leq N$) on rate of synchronization based on model (3).

Assumption 1 *In the following of the paper, we assume that the influences of the global cue on the rest nodes are the same, i.e., $a_{21} = a_{31} = \dots = a_{N1} = g$, where g is a positive constant. In the case of circadian systems, the assumption means that every neuronal oscillator is identically entrained to the same light-dark cycle [14]. In the case of technological systems, this assumption represents the situation in which the central server (such as GPS) influences*

each node in the same way in the clock synchronization of overall systems (such as computer networks [24] and wireless sensor networks [25]).

Assumption 2 Moreover, we also assume that all the oscillators have identical natural frequencies, i.e., $w_1 = w_2 = \dots = w_N = w$, where w is a positive constant.

For convenience in the analysis, we first transform (3) into a rotating reference frame via the transformation $\varphi_j = \omega t + \theta_j$ ($j = 1, 2, \dots, N$):

$$\begin{aligned} \dot{\theta}_1 &= 0, \\ \dot{\theta}_i &= \sum_{1 \leq j \leq N, j \neq i} a_{ij}(\theta_j - \theta_i), \quad 2 \leq i \leq N \end{aligned} \quad (6)$$

It can be verified that (3) and (6) have the same synchronization properties, such as synchronization condition and the rate of synchronization. Eqn. (6) can be rearranged into vector form as follows

$$\dot{\Theta} = A\Theta \quad (7)$$

with Θ and A given in (8) on the top of the next page.

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ g & -g - \sum_{2 \leq i \leq N, i \neq 2} a_{2i} & a_{23} & \dots & a_{2N} \\ g & a_{32} & -g - \sum_{2 \leq i \leq N, i \neq 3} a_{3i} & \dots & a_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g & a_{N2} & a_{N3} & \dots & -g - \sum_{2 \leq i \leq N, i \neq N} a_{Ni} \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{bmatrix} \quad (8)$$

$$\tilde{A} = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -g - \sum_{2 \leq i \leq N, i \neq 2} a_{2i} & a_{23} & \dots & a_{2N} \\ 0 & a_{32} & -g - \sum_{2 \leq i \leq N, i \neq 3} a_{3i} & \dots & a_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{N2} & a_{N3} & \dots & -g - \sum_{2 \leq i \leq N, i \neq N} a_{Ni} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_{2-N} \end{bmatrix} \quad (9)$$

It can be easily verified that all row-sums of matrix A are zero and '0' is one of the eigenvalues of A with associated eigenvector $[1 \ 1 \ \dots \ 1]^T$. From consensus theory [26-27], it can be proved that if all the remaining eigenvalues of matrix A have negative real parts, then all θ_i will asymptotically achieve agreement, i.e., the ultimate system response is $\theta_1 = \theta_2 = \dots = \theta_N$, which leads to $\varphi_1 = \varphi_2 = \dots = \varphi_N$, meaning that all the oscillators are synchronized. In this paper, we will go one step further, i.e., we want to study the impact of global cues and local cues on the rate of synchronization.

According to dynamical systems theory, the rate for all θ_i to achieve agreement is determined by the eigenvalues of A . If the oscillators can be synchronized, all the real parts of A 's eigenvalues should be negative except for the unique eigenvalue '0', so the rate of synchronization is determined by the non-zero eigenvalue with the largest real part. Therefore, we only need to study the influences of global and local cues on the non-zero eigenvalue with the largest real part, which is given in Theorem 1.

Theorem 1 *For an oscillator network with a global cue and an arbitrary local cue (not necessarily symmetric or balanced), the oscillators will always synchronize, and the rate of synchronization is only determined by the strength of the global cue.*

Proof: Multiply matrix A on the right by P (given in (8) on the top of the next page) and on the left by P^{-1} , which leads to (9) on the top of the next page with $\tilde{A}_{2-N}$ an $(N-1) \times (N-1)$ dimensional matrix. Therefore the eigenvalues of A are given by $\{0, \lambda_2, \lambda_3, \dots, \lambda_N\}$ where $\lambda_2, \lambda_3, \dots, \lambda_N$ are the eigenvalues of $\tilde{A}_{2-N}$.

According to the Gershgorin Circle Theorem [28], all the eigenvalues of $\tilde{A}_{2-N}$ are no larger than $-g$. So the real parts of the eigenvalues of A are no larger than $-g$ except for the eigenvalue '0'. Then according to consensus theory, the oscillators will always synchronize.

Notice that in fact $-g$ is an eigenvalue of A associated with eigenvector $[0 \ 1 \ \dots \ 1]^T$, which in combination with the above derivation means that $-g$ is the non-zero eigenvalue with the largest real part and it determines the synchronization rate of the oscillator network. Recall that g is the strength of the global cue, thus the rate of synchronization is only determined by the strength of the global cue, which completes the proof.

Remark 1 *Theorem 1 means that a stronger local cue does not necessarily bring a faster rate of synchronization, which is somewhat counterintuitive. In fact, our numerical simulations in Sec. 4 will confirm this point.*

Remark 2 *When there is no global cue, i.e., $g = 0$, $\tilde{A}_{2-N}$ in (9) reduces to the additive inverse of the graph Laplacian in [26]. Therefore, we have from [26] that the network will still synchronize if the graph of the local cue is connected. Moreover, noting the fact that the maximal eigenvalue of $\tilde{A}_{2-N}$ multiples if every a_{ij} ($2 \leq i, j \leq N$) multiples, so the rate of synchronization will increase if the strength of the local cue increases.*

Remark 3 *The results in Theorem 1 are equivalent to Perron-Frobenius theory (which is limited to non-negative matrices), which asserts that a real square matrix with positive entries (also applicable to certain classes of non-negative matrices) has a unique largest real eigenvalue [29].*

Remark 4 *When Assumption 1 is not satisfied, i.e., $a_{21}, a_{31}, \dots, a_{N1}$ are not identical, the largest real part of the non-zero eigenvalue of A is not only determined by g any more. However, using a derivation similar to the proof of Theorem 1, we can obtain that the largest real part of the non-zero eigenvalue of A is no larger than $-g_{\min}$ with $g_{\min}$ given by*

$$g_{\min} = \min\{a_{21}, a_{31}, \dots, a_{N1}\} \quad (10)$$

So we have that the rate of synchronization will still increase with an increase in the global cue.

3.2 Results in the presence of noise

Next we consider the synchronization problem in the presence of white noise in the natural frequencies. Oscillators with noise in the natural frequencies are widespread, particularly in biological oscillator networks such as circadian rhythms [14] and quorum sensing [30]. Noise emerges from both molecular scale kinetics as well as environmental perturbations. Due to the presence of noise, we need to define synchronization in the stochastic framework.

Definition 1 *In the presence of noise, system (5) is synchronized if the mathematical expectations of all oscillator phases asymptotically achieve agreement, and meanwhile the variances of all the oscillator phases are finite.*

Similar to the noise free case, we retain Assumption 1 and 2, and transform (5) into a rotating reference frame first by using the transformation $\varphi_j = \omega t + \theta_j$ ($j = 1, 2, \dots, N$):

$$\begin{aligned} d\theta_1 &= 0, \\ d\theta_i &= \sigma_i dB_i(t) + \sum_{1 \leq j \leq N, j \neq i} a_{ij} (\theta_j - \theta_i) dt, \quad 2 \leq i \leq N \end{aligned} \quad (11)$$

Eqn. (11) can be rearranged into vector form as follows

$$\dot{\Theta} = A\Theta + \Sigma dB(t) \quad (12)$$

with

$$\Sigma = \text{diag}\{0, \sigma_2, \dots, \sigma_N\}, \quad B(t) = [0 \ B_2(t) \ \dots \ B_N(t)]^T$$

and the rest parameters are the same as those in (7).

For system (12), we can also prove that the rate of synchronization is only determined by the strength of the global cue, which is given in Theorem 2.

Theorem 2 *For an oscillator network with a global cue and an arbitrary local cue (not necessarily symmetric or balanced) formulated in (12), the oscillators will always synchronize in the sense of Definition 1, and the rate of synchronization is only determined by the strength of the global cue.*

Proof: We first prove that the mathematical expectations of all θ_i will achieve agreement asymptotically, and then prove that their variances are finite.

According to [23], (12) is an N-dimensional Ornstein-Uhlenbeck process, and the solution is given by

$$\Theta(t) = e^{At} \Theta(0) + \int_0^t e^{A(t-\tau)} \Sigma dB(\tau) \quad (13)$$

The mathematical expectation $E\{\Theta(t)\}$ and variance $V\{\Theta(t)\}$ of $\Theta(t)$ are given by

$$E\{\Theta(t)\} = e^{At} E\{\Theta(0)\} \quad (14)$$

$$V\{\Theta(t)\} = e^{At} V\{\Theta(0)\} e^{A^T t} + \int_0^t e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} d\tau \quad (15)$$

First we show the mathematical expectation $E\{\Theta(t)\}$ will achieve agreement and the convergence rate is only determined by the strength of the global cue.

$E\{\Theta(t)\}$ can be regarded as the solution to

$$\frac{dE\{\Theta\}}{dt} = AE\{\Theta\} \quad (16)$$

By using a derivation similar to that of Sec. 3.1, we can obtain that the N elements in vector $E\{\Theta(t)\}$ will achieve agreement $E\{\theta_1(t)\} = E\{\theta_2(t)\} = \dots = E\{\theta_N(t)\}$ and the rate of convergence is only determined by the strength of the global cue g .

In the following, we will show the variance $V\{\Theta(t)\}$ is finite and will converge with convergence rate determined by the strength of the global cue. Here our derivation is divided into two steps. In the first step, we will show the first item in (15), i.e., $e^{At} V\{\Theta(0)\} e^{A^T t}$ will converge to zero with convergence rate determined by the strength of the global cue g , and then in the second step, we will prove the second item in (15), i.e., $\int_0^t e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} d\tau$ is always finite and will converge to a constant with convergence rate determined by the strength of the global cue g .

Since θ_1 is noise free, it can be easily obtained that $V\{\Theta(0)\}$ is of the following form

$$V\{\Theta(0)\} = \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix}$$

with $V_{2-N}(0)$ an $(N-1) \times (N-1)$ dimensional matrix. Making use of the fact that

$$P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} P^{-T} = \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix}$$

where P is given in (8), we have

$$\begin{aligned} e^{At} V\{\Theta(0)\} e^{A^T t} &= e^{P \tilde{A} P^{-1} t} \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} e^{P^{-T} \tilde{A}^T P^T t} \\ &= P \begin{bmatrix} e^0 & 0 \\ 0 & e^{\tilde{A}_{2-N} t} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & V_{2-N}(0) \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^{\tilde{A}_{2-N}^T t} \end{bmatrix} P^T \\ &= P \begin{bmatrix} 0 & 0 \\ 0 & e^{\tilde{A}_{2-N} t} V_{2-N}(0) e^{\tilde{A}_{2-N}^T t} \end{bmatrix} P^T \end{aligned} \quad (17)$$

According to the analysis in Sec. 3.1, the real parts of the eigenvalues of $\tilde{A}_{2-N}$ are all no larger than $-g$, so we know $e^{\tilde{A}_{2-N} t} V_{2-N}(0) e^{\tilde{A}_{2-N}^T t}$ is finite and will converge to zero with

convergence rate $2g$. Therefore $e^{At}V\{\Theta(0)\}e^{A^T t}$ is finite and will converge to zero with convergence rate $2g$.

Now we will prove that the second item in (15), i.e., $\int_0^t e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} d\tau$ is finite and will converge to a constant with convergence rate determined by the strength of the global cue g .

By using (9) and the fact that $P^{-1} \Sigma \Sigma^T P = \Sigma \Sigma^T$, we have

$$e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} = e^{P \tilde{A} P^{-1}(t-\tau)} \Sigma \Sigma^T e^{P^{-T} \tilde{A}^T P^T(t-\tau)} = P e^{\tilde{A}(t-\tau)} \Sigma \Sigma^T e^{\tilde{A}^T(t-\tau)} P^T \quad (18)$$

Denote $\Sigma = \text{diag}(0, \Sigma_{2-N})$ with

$$\Sigma_{2-N} = \text{diag}(\sigma_2, \dots, \sigma_N)$$

then we have

$$\begin{aligned} & P e^{\tilde{A}(t-\tau)} \Sigma \Sigma^T e^{\tilde{A}^T(t-\tau)} P^T \\ &= P \begin{bmatrix} e^0 & 0 \\ 0 & e^{\tilde{A}_{2-N}(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_{2-N} \Sigma_{2-N}^T \end{bmatrix} \\ & \times \begin{bmatrix} e^0 & 0 \\ 0 & e^{\tilde{A}_{2-N}^T(t-\tau)} \end{bmatrix} P^T \\ &= P \begin{bmatrix} 0 & 0 \\ 0 & e^{\tilde{A}_{2-N}(t-\tau)} \Sigma_{2-N} \Sigma_{2-N}^T e^{\tilde{A}_{2-N}^T(t-\tau)} \end{bmatrix} P^T \end{aligned} \quad (19)$$

Therefore it follows

$$\int_0^t e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} d\tau = P \begin{bmatrix} 0 & 0 \\ 0 & Q_t \end{bmatrix} P^T, \quad Q_t = \int_0^t e^{\tilde{A}_{2-N}(t-\tau)} \Sigma_{2-N} \Sigma_{2-N}^T e^{\tilde{A}_{2-N}^T(t-\tau)} d\tau \quad (20)$$

According to the analysis in Sec. 3.1, all the real parts of the eigenvalues of $\tilde{A}_{2-N}$ are negative, so it follows that Q_t is determined by [23]

$$Q_t = Q - e^{\tilde{A}_{2-N} t} Q e^{\tilde{A}_{2-N}^T t}, \quad t > 0 \quad (21)$$

where Q is a finite constant and is given by

$$Q \triangleq \int_0^\infty e^{\tilde{A}_{2-N} t} \Sigma_{2-N} \Sigma_{2-N}^T e^{\tilde{A}_{2-N}^T t} dt$$

Recall that the real parts of the eigenvalues of $\tilde{A}_{2-N}$ are no larger than $-g$, we have that Q_t is finite and will converge to Q with convergence rate $2g$.

Concluding the derivations above, we know the mathematical expectations will asymptotically achieve agreement with convergence rate determined by g , and the variances

will always be finite, and converge to a finite constant with convergence rate determined by g as well, which completes the proof.

Remark 5 *It is interesting to note that Q_t is in fact the controllability Gramian of the subsystem $\theta_2, \dots, \theta_N$.*

Next, we estimate the variance upper-bound Q .

From [23], Q can be obtained by solving the following linear matrix equation

$$\tilde{A}_{2-N}Q + Q\tilde{A}_{2-N}^T = -\Sigma_{2-N}\Sigma_{2-N}^T \quad (22)$$

If influences of the local cue are bidirectional, i.e.,

$$a_{ij} = a_{ji}, j \in \{2, 3, \dots, N\}$$

which yields

$$\lambda_{\max}(\tilde{A}_{2-N} + \tilde{A}_{2-N}^T) = -2g \quad (23)$$

or balanced [26], i.e.,

$$\sum_{2 \leq j \leq N} a_{ij} = \sum_{2 \leq j \leq N} a_{ji} \quad \forall 2 \leq i \leq N$$

which from the Gershgorin Circle Theorem yields

$$\lambda_{\max}(\tilde{A}_{2-N} + \tilde{A}_{2-N}^T) \leq -2g \quad (24)$$

then according to [31], we have

$$Q \leq -\lambda_{\max}^{-1}(\tilde{A}_{2-N} + \tilde{A}_{2-N}^T) \lambda_{\max}(\Sigma_{2-N}\Sigma_{2-N}^T) I_{N-1} \leq \frac{1}{2g} \lambda_{\max}(\Sigma_{2-N}\Sigma_{2-N}^T) I_{N-1} \quad (25)$$

which leads to

$$\int_0^\infty e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} d\tau \leq \frac{1}{2g} \lambda_{\max}(\Sigma_{2-N}\Sigma_{2-N}^T) P P^T \quad (26)$$

where $\lambda_{\max}(\bullet)$ denotes the maximal eigenvalue and I_{N-1} is the identity matrix of dimension $N-1$.

Therefore, when the local cue is symmetric or balanced, the variances are finite and upper-bounded by a function of the strength of the global cue.

When the local cue is neither symmetric nor balanced, suppose the maximal imbalance [32] of the local cue is

$$\Lambda = \max_{2 \leq i \leq N} \left| \sum_{2 \leq j \leq N, j \neq i} a_{ij} - \sum_{2 \leq j \leq N, j \neq i} a_{ji} \right| \quad (27)$$

then by using the Gershgorin Circle Theorem, we have

$$\lambda_{\max} (\tilde{A}_{2-N} + \tilde{A}_{2-N}^T) \leq -2g + \Lambda \quad (28)$$

When the maximal imbalance is moderate ($\Lambda < 2g$), we can still estimate the upper-bound of the variances:

$$\int_0^t e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} d\tau \leq \frac{\lambda_{\max} (\Sigma_{2-N} \Sigma_{2-N}^T) P P^T}{2g - \Lambda} \quad (29)$$

So in this case, the upper-bound of the variances is determined by the strength of the global cue and the imbalance of the local cue.

Remark 6 If Assumption 1 is not satisfied, i.e., $a_{21}, a_{31}, \dots, a_{N1}$ are not identical, then by using the Gershgorin Circle Theorem and a derivation similar to the proof of Theorem 1, we can obtain that

$$\lambda_{\max} (\tilde{A}_{2-N} + \tilde{A}_{2-N}^T) \leq -2g_{\min} \quad (30)$$

where $g_{\min}$ is given in (10). Accordingly we have

$$Q \leq \frac{1}{2g_{\min}} \lambda_{\max} (\Sigma_{2-N} \Sigma_{2-N}^T) I_{N-1} \quad (31)$$

and

$$\int_0^t e^{A(t-\tau)} \Sigma \Sigma^T e^{A^T(t-\tau)} d\tau \leq \frac{\lambda_{\max} (\Sigma_{2-N} \Sigma_{2-N}^T) P P^T}{2g_{\min}} \quad (32)$$

which means that the variances are still upper-bounded by a function of the strength of the global cue.

4 Simulation results

In this section, simulation results are given to illustrate the analytical results. In the simulation, we considered an oscillator network composed of ten oscillators and one of them is the leader node (acting as the global cue). The strength of the global cue g is constant and the strength of the local cue a_{ij} ($i, j \in \{2, 3, \dots, N\}$) are randomly chosen from the interval $[0, 1]$. As in most papers, we use the synchronization index to measure the degree of synchrony [13]

$$SI = \left| \frac{1}{N} \sum_{i=1}^N e^{j\varphi_i} \right|$$

$SI \in [0, 1]$ reflects the degree of synchrony and will be 1 if the network is perfectly synchronized and 0 if the phases are randomly distributed.

We first considered the noise free case. In order to compare influences of global and local cues on the rate of synchronization, we simulated the oscillator network under different strengths of global and local cues. Synchronization is defined to be achieved when SI exceeds 0.9. In our simulations, the synchronization index SI converged to 1 as shown in Fig. 1, which confirms the synchronization claim in Theorem 1. The synchronization times under different circumstances are given in Table 1. In the table, $k \times a_{ij}$ ($k = 1, 2, \dots, 10$) denotes that we made the strength of the local cue a_{ij} ($\forall i, j \in \{2, 3, \dots, N\}$) k times larger than the original ones. So each row gives the synchronization times under a constant strength of global cue and different strengths of local cues, and meanwhile each column gives the synchronization times under different global cues and a constant local cue. When there is global cue ($g \neq 0$), from Table 1, it is clear that the rate of synchronization will increase with the increase of the strength of the global cue. However, under a constant strength of the global cue, the strongest local cue may not produce the fastest rate of synchronization, namely the shortest synchronization time, which are emphasized in red in each row with a non-zero g in Table 1. When where is no global cue ($g = 0$), the rate of synchronization will increase with an increase in the local cue, which confirms Remark 2.

The simulation results corroborate the results on the synchronization of the mammalian circadian clock in [15]: VIP input acts as the local cue, external entrainment light-dark cycles act as the global cue, and it is shown that changing the VIP input from pulsed input to constant input (making the local coupling stronger) will impair synchronization, whereas introducing light-dark cycles will enhance synchronization. Moreover, in molecular-dynamics simulations of bimolecular systems [33], it is found that the fastest rate of synchronization is achieved when the local coupling strength is moderate (neither too weak nor too strong), which can also be regarded as a confirmation of our theoretical results.

We also considered the case where the natural frequencies were subject to Gaussian white noise. The simulation setup is the same with noise free case except that noise terms were added with $\sigma_i = 1$ ($i = 2, 3, \dots, N$). The results are given in Table 2. Under a constant strength of the global cue, the shortest synchronization times corresponding to the fastest rates of synchronization are also emphasized in red in each row. As in the noise free case, it can be seen that the strength of the global cue determines the rate of synchronization, and a stronger local cue does not necessarily bring a faster rate of synchronization. It is worthwhile to note that when there is no global cue ($g = 0$), the network will not synchronize (with synchronization indices converging to a value near 0.8, as shown in Fig. 2), so the synchronization times in the $g = 0$ cases are represented as '—'.

It is worthwhile to mention that when the local coupling strength is strong, the synchronization rate in the global-cue-free case is even faster than the case with a weak global cue (the first and second entries of the last column in Table 1). This shows that when there are both global and local cues, the global cues dominate the synchronization process. Moreover, when there is no global cue, it has been established in some cases that local coupling plays an important role in ensuring robust synchronization with respect to model uncertainties. But under the formulation in the paper, our simulation results demonstrate that in the presence of Gaussian white noise in the natural frequencies, the global cue is much

more important than the local cue in ensuring synchronization of the whole system. (As shown in the first row of Table 2, synchronization is lost when there are only local cues but no global cue.)

5 Conclusions

The influence of global and local cues on the rate of synchronization is studied. This has important ramifications for the study of biological oscillators. It is proved through analysis and numerical simulations that only the strength of the global cue, but not the strengths of global and local cues together, determines the rate of synchronization. More specifically, a stronger global cue means a faster rate of synchronization but a stronger local cue does not necessarily bring a faster rate of synchronization. This points out a way to elucidate complex synchronization properties in connected oscillator networks, such as biological oscillator networks. The results apply both to the noise free case and to the case with Gaussian white noise in oscillators' natural frequencies.

Acknowledgments

The work was supported in part by Institute for Collaborative Biotechnologies under grant W911NF-09-D-0001, U.S. Army Research Office under Grant W911NF-07-1-0279, and National Institutes of Health under Grant GM078993.

References

1. Kuramoto, Y. Chemical oscillations, waves, and turbulence. New York: Dover Publications; 1984.
2. Wu, CW. Synchronization in coupled chaotic circuits and systems. New Jersey: World Scientific; 2002.
3. Pikovsky, A.; Rosenblum, M.; Kurths, J. Synchronization: A universal concept in nonlinear sciences. New York: Cambridge University Press; 2001.
4. Jadbabaie A, Lin J, Morse AS. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*. 2003; 48:988–1000.
5. Xiao F, Wang L, Chen J, Gao Y. Finite-time formation control for multi-agent systems. *Automatica*. 2009; 45:2605–2611.
6. Liu C, Duan Z, Chen G, Huang L. L_2 norm performance index of synchronization and LQR control synthesis of complex networks. *Automatica*. 2009; 45:1879–1885.
7. Zhou J, Lu JA, Lu JH. Adaptive synchronization of an uncertain complex dynamical network. *IEEE Transactions on Automatic Control*. 2006; 51:652–656.
8. Angeli D, Sontag ED. Oscillations in I/O monotone systems under negative feedback. *IEEE Transactions on Automatic Control*. 2008; 53:166–176.
9. Li X, Wang XF, Chen GR. Pinning a complex dynamical network to its equilibrium. *IEEE Transactions on Circuits and Systems-I: Regular Papers*. 2004; 51:2074–2087.
10. Nedić A, Olshevsky A, Ozdaglar A, Tsitsiklis J. On distributed averaging algorithms and quantization effects. *IEEE Transactions on Automatic Control*. 2009; 54:2506–2517.
11. Arcak M. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*. 2007; 52:1380–1390.
12. Stan G, Sepulchre R. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*. 2007; 52:256–270.
13. Kim, JS.; Allgöwer, F. Nonlinear synchronization of coupled oscillators: the polynomial case. In: Astolfi, A.; Marconi, L., editors. *Analysis and design of nonlinear control systems*. Berlin: Springer; 2007. p. 339–351.
14. Herzog E. Neurons and networks in daily rhythms. *Nature Review Neuroscience*. 2007; 8:790–802.
15. To T, Henson MA, Herzog E, Doyle FJ. A molecular model for intercellular synchronization in the mammalian circadian clock. *Biophysical Journal*. 2007; 92:3792–3803.

16. Guest J, Baird A, Clifton K, Heyward A. From molecules to moonbeams: spawning synchrony in coral reef organisms. *Invertebrate Reproduction and Development*. 2008; 51:145–149.
17. Goldbeter, A. *Biochemical oscillations and cellular rhythms: the molecular bases of periodic and chaotic behaviour*. New York: Cambridge University Press; 1997.
18. Sepulchre R, Paley D, Leonard N. Stabilization of planar collective motion: all-to-all communication. *IEEE Transactions on Automatic Control*. 2007; 52:811–824.
19. Kopetz H, Ochsenreiter W. Clock synchronization in distributed real-time systems. *IEEE Transactions on Computers*. 1987; C-36:933–940.
20. Ren W. Multi-vehicle consensus with a time-varying reference state. *Systems and Control Letters*. 2007; 56:474–483.
21. Wieland, P. PhD thesis. University of Stuttgart; 2010. From static to dynamic couplings in consensus and synchronization.
22. Wieland P, Kim J, Allgöwer F. On topology and dynamics of consensus among linear high-order agents. *International Journal of Systems Science*. 2010 In press.
23. Karatzas, I.; Shreve, SE. *Brownian motion and stochastic calculus*. Berlin: Springer-Verlag; 1991.
24. Mills DI. Improved algorithms for synchronizing computer network clocks. *IEEE/ACM Transactions on Networking*. 1995; 3:245–254.
25. Elson, JE. PhD thesis. Los Angeles: University of California at Los Angeles; 2003. Time synchronization in wireless sensor networks.
26. Olfati-Saber R, Fax JA, Murray RM. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*. 2007; 95:215–233.
27. Ren W, Beard RW, Atkins EM. Information consensus in multivehicle cooperative control. *IEEE Control Systems Magazine*. 2007; 27:71–82.
28. Golub, GH.; Van Loan, CF. *Matrix Computations*. Baltimore: Johns Hopkins University Press; 1996.
29. Farina, L.; Rinaldi, S. *Positive linear systems: theory and applications*. New York: John Wiley & Sons; 2000.
30. Wang R, Li C, Chen L, Aihara K. Modeling and analyzing biological oscillations in molecular networks. *Proceedings of the IEEE*. 2008; 96:1361–1385.
31. Komaroff N. Upper bounds for the eigenvalues of the solution of the Lyapunov matrix equation. *IEEE Transactions on Automatic Control*. 1990; 35:737–739.
32. Henning MA, Rautenbach D. On the irregularity of bipartite graphs. *Discrete Mathematics*. 2007; 307:1467–1472.
33. Karplus M, McCammon J. Molecular dynamics simulations of biomolecules. *Nature Structural Biology*. 2002; 9:646–652.

Biography

Yongqiang Wang was born in Shandong, China, in 1982. He received the B.E. degree in Electrical Engineering & Automation, the B.E. degree in Computer Science & Technology from Xi'an Jiaotong University, Shanxi, China, in 2004. From 2007–2008, he was with the University of Duisburg-Essen, Germany, as a visiting student. He received the M.Sc. and the Ph.D. degrees in Control Science & Engineering from Tsinghua University, Beijing, China, in 2009. He is now with the University of California, Santa Barbara as a postdoctoral research scholar. His research interests are model based fault diagnosis, systems modeling and analysis of circadian rhythm networks, reverse engineering of synchrony as an algorithmic strategy for wireless communication network control.

Mr. Wang is the recipient of 2008 Young Author Prize from IFAC Japan Foundation for a paper presented at the 17th IFAC World Congress in Seoul.

Francis J. Doyle, III, received his B.S.E. degree from Princeton in 1985, C.P.G.S. degree from Cambridge in 1986, and Ph.D. degree from Caltech in 1991, all in chemical engineering. He is the associate dean for research in the College of Engineering at UCSB, and he is the director of the Army Institute for Collaborative Biotechnologies. He holds the Duncan and Suzanne Mellichamp Chair in process control in the Department of Chemical Engineering as well as appointments in the Electrical Engineering Department and the Biomolecular Science and Engineering Program. He is a Fellow of the IEEE and a Fellow of the IFAC. He served as the editor-in-chief of IEEE Transactions on Control Systems Technology from 2004 to 2009 and currently holds the associate editor positions with Journal of Process Control, SIAM Journal on Applied Dynamical Systems, and Royal Society's Interface. His research interests are in systems biology, network science, modeling

and analysis of circadian rhythms, drug delivery for diabetes, model-based control, and control of particulate processes.

Fig. 1.
Tendency of SI in the noise free case and SI in the noisy case with $g \neq 0$

Fig. 2.
Tendency of SI in the noisy case with $g = 0$ (the network is not synchronized)

Table 1

Synchronization times under different strengths of global and local cues-Noise free case

SLC ^b	$1 \times a_{ij}$	$2 \times a_{ij}$	$3 \times a_{ij}$	$4 \times a_{ij}$	$5 \times a_{ij}$	$6 \times a_{ij}$	$7 \times a_{ij}$	$8 \times a_{ij}$	$9 \times a_{ij}$	$10 \times a_{ij}$
SGC ^a										
$g = 0$	1.3167	0.6584	0.4390	0.3292	0.2634	0.2195	0.1882	0.1647	0.1464	0.1318
$g = 1$	0.4827	0.4671	0.4674	0.4675	0.4676	0.4676	0.4676	0.4676	0.4676	0.4676
$g = 2$	0.2807	0.2414	0.2344	0.2336	0.2337	0.2337	0.2338	0.2338	0.2338	0.2338
$g = 4$	0.1719	0.1404	0.1268	0.1207	0.1182	0.1172	0.1169	0.1169	0.1169	0.1169
$g = 8$	0.1028	0.0860	0.0764	0.0703	0.0662	0.0635	0.0616	0.0604	0.0596	0.0592
$g = 16$	0.0585	0.0514	0.0466	0.0430	0.0403	0.0382	0.0365	0.0352	0.0341	0.0332

^aSGC-Strength of the global cue

^bSLC-Strength of the local cue

Table 2
Synchronization times under different strengths of global and local cues-*Noisy case*

SLC ^b	1 × <i>a</i> _{ij}	2 × <i>a</i> _{ij}	3 × <i>a</i> _{ij}	4 × <i>a</i> _{ij}	5 × <i>a</i> _{ij}	6 × <i>a</i> _{ij}	7 × <i>a</i> _{ij}	8 × <i>a</i> _{ij}	9 × <i>a</i> _{ij}	10 × <i>a</i> _{ij}
	SGC ^a									
<i>g</i> = 0	– <i>c</i>	–	–	–	–	–	–	–	–	–
<i>g</i> = 1	0.4836	0.4669	0.4670	0.4670	0.4670	0.4670	0.4670	0.4670	0.4670	0.4670
<i>g</i> = 2	0.2815	0.2430	0.2359	0.2352	0.2352	0.2352	0.2354	0.2354	0.2354	0.2354
<i>g</i> = 4	0.1720	0.1403	0.1267	0.1206	0.1179	0.1170	0.1167	0.1166	0.1166	0.1166
<i>g</i> = 8	0.1028	0.0860	0.0764	0.0702	0.0661	0.0634	0.0617	0.0605	0.0598	0.0593
<i>g</i> = 16	0.0585	0.0515	0.0467	0.0431	0.0404	0.0383	0.0366	0.0352	0.0341	0.0332

^aSGC-Strength of the global cue
^bSLC-Strength of the local cue
c, '–' represents that the network is not synchronized