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Mass Transfer in a Rigid Tube With Pulsatile Flow and Constant Wall Concentration

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Abstract

An approximate-analytical solution method is presented for the problem of mass transfer in a rigid tube with pulsatile flow. For the case of constant wall concentration, it is shown that the generalized integral transform (GIT) method can be used to obtain a solution in terms of a perturbation expansion, where the coefficients of each term are given by a system of coupled ordinary differential equations. Truncating the system at some large value of the parameter N , an approximate solution for the system is obtained for the first term in the perturbation expansion, and the GIT-based solution is verified by comparison to a numerical solution. The GIT approximate-analytical solution indicates that for small to moderate nondimensional frequencies for any distance from the inlet of the tube, there is a positive peak in the bulk concentration C_{1b} due to pulsation, thereby, producing a higher mass transfer mixing efficiency in the tube. As we further increase the frequency, the positive peak is followed by a negative peak in the time-averaged bulk concentration and then the bulk concentration C_{1b} oscillates and dampens to zero. Initially, for small frequencies the relative Sherwood number is negative indicating that the effect of pulsation tends to reduce mass transfer. There is a band of frequencies, where the relative Sherwood number is positive indicating that the effect of pulsation tends to increase mass transfer. The positive peak in bulk concentration corresponds to a matching of the phase of the pulsatile velocity and the concentration, respectively, where the unique maximum of both occur for certain time in the cycle. The oscillatory component of concentration is also determined radially in the tube where the concentration develops first near the wall of the tube, and the lobes of the concentration curves increase with increasing distance downstream until the concentration becomes fully developed. The GIT method proves to be a working approach to solve the first two perturbation terms in the governing equations involved.

1 Introduction

The problem of pulsatile flow in a tube has received considerable attention. The classical work of Womersley [1], Uchida [2], Atabek and Chang [3], and many others that followed led to analytical solutions and considerable information about the oscillatory flow field. Studies of the corresponding mass transfer problem via the diffusion equation are less numerous and existing results do not provide complete information about the concentration field and the rate of mass transfer in pulsatile flow.

The pulsatile flow field can be assumed to consist of a steady Poiseuille flow part and a purely oscillatory part and it is believed that the rate of mass transfer is changed because the pulsation alters the boundary layer near the wall. This view was first supported by

Richardson [4] who showed that the velocity profile for pulsating flow is steeper near the wall of the tube (i.e., the shear rate is higher) than it is in Poiseuille flow. It follows that the concentration profile will be affected in a similar way and the rate of mass diffusion should increase near the wall. Taylor [5] showed that when a small quantity of a diffusing substance is introduced into a fluid flowing along a circular tube, the ultimate spreading of the substance is enhanced by the flow of the fluid. This is due to the variation of the velocity of flow over the cross-section of the tube, which causes transverse diffusion to be effective in dispersing the substance (e.g., oxygen). A similar effect occurs for oscillating flow as shown by Bowden [6].

Watson [7] showed that the rate of mass transfer of a diffusing substance along a tube is augmented by an oscillatory motion of the ambient fluid in the tube. The increase of the flux of the substance across any plane z was evaluated for a circular tube. He showed results for a general cross-section in the limiting cases of slow and fast oscillations of the flow. His results correspond to the fact that the increase of flux arises from the whole flow at low frequency but only from the boundary layer at high frequency.

An exact method of analysis of unsteady convective diffusion systems with interphase mass transfer across their boundaries have been developed by Sankarasubramanian and Gill [8]. Dispersion of a nonuniform initial distribution in time-variable isothermal laminar flow in a tube with a first-order rate process at the tube wall is analyzed. As a specific example, miscible dispersion in laminar flow in a tube in the presence of interfacial transport due to an irreversible first-order reaction at the tube wall is analyzed by an exact procedure. The new concept of “exchange coefficients” is introduced and a general expression is derived showing the time-dependent nature of these coefficients. The exchange coefficient reflects the interphase process and it enables one to determine the average concentration distribution in terms of tabulated functions. The analysis conducted was confined to the case of dispersion in a fully developed steady flow.

A similar approach in solution method compared with Sankarasubramanian was conducted by Hazra et al. [9] who considered dispersion of a solute in oscillating flow through a channel. The longitudinal dispersion coefficient of a solute in a viscous incompressible fluid flowing in a parallel plate channel in the presence of a periodic pressure gradient with a nonzero mean was determined. The results of this investigation on solute dispersion coefficients, which are valid for all time, are likely to be of importance in blood flow in a cardiovascular system (e.g., in thoracic aorta) under pulsatile conditions.

Ng [10] conducted an asymptotic analysis for the advection-diffusion transport of a chemical species in flow through a small diameter tube (lined with a thin layer of retentive and reactive material) where the flow consists of steady and oscillatory components, and the species undergo linear reversible and irreversible reactions at the tube wall. The substance that undergoes Taylor dispersion in the fluid is subject to reversible phase exchange with the wall layer and irreversible absorption into the wall layer. The objective of the work was to examine the effects of such chemical reactions on the dispersion in steady and oscillatory flow. An asymptotic analysis was performed to obtain an overall effective transport equation containing advection and dispersion coefficients that are functions of the chemical reactions and flow oscillation.

There exist a large number of studies on Taylor dispersion under the sole influence of flow oscillation. Dispersion of inert matter in pulsating or oscillatory flow was studied by Chatwin [11], Smith [12], and Yasuda [13] while some more recent developments may include Mukherjee and Mazumder [14], Elad et al. [15], and Bandyopadhyay and Mazumder [16].

The effects of phase exchange, partitioning, boundary uptake, or chemical reaction on axial dispersion in steady flow were analyzed by Westhaver [17], Aris [18], Gupta and Gupta [19], and Sankarasubramanian and Gill [8].

This type of problem was then widely extended to applications in chromatography by Blackburn [20], in biological or physiological transport (e.g., Grotberg et al. [21] and Sarkar and Jayaraman [22]), and in environmental contaminant transport (e.g., Purnama [23]).

Kim et al. [24] conducted a study of the heat transfer characteristics of fully developed pulsatile flow in a channel. The effects of pulsatile frequency on heat transfer were found to be noticeable when the frequency is small or moderate but not when it is high. Moschandreou and Zamir [25] considered the problem of pulsatile flow in a tube with constant heat flux at the wall analytically to determine how pulsation affects the rate of heat transfer and how the phenomenon depends on the pulsation frequency. The results indicate that in the range of moderate values for the nondimensional frequency (~ 15) there is a positive peak in the effect of pulsation whereby the bulk temperature of the fluid and the Nusselt number are increased, whereas this effect is reversed for small (< 5) and large frequencies (> 25).

Cho and Hyun [26] conducted numerical studies of flow and heat transfer characteristics of a pulsating flow in a pipe. They found that in the downstream fully established region, the Nusselt number either increases or decreases over the steady-flow value depending on the frequency parameter, and the deviations from the steady-flow values are rather small for the parameter ranges computed. Also, the Nusselt number is amplified as the amplitude of oscillation increases and when the Prandtl number is low below unity. Faghri et al. [27] showed that the velocity pulsations induce harmonic oscillations in temperature, breaking the temperature field into a steady mean part and a harmonic part. The interaction between the velocity and temperature oscillations introduces an extra term in the energy equation, which reflects the effect of pulsations in producing higher heat transfer rates than steady flow.

More recently, numerical studies of flow and heat transfer in a circular tube under pulsating flow conditions were carried out by Chattopadhyay et al. [28] in the laminar regime. The flow was both thermally and hydrodynamically developing while the tube wall was kept at uniform temperature. The two-dimensional Navier–Stokes equation was solved and it was observed that in the frequency range of 0–20 Hz, for low amplitudes, pulsation has no effect on time-averaged heat transfer although the Nusselt number distribution varied in time in the near-entry region of the pipe.

The aims of the present paper are to describe an approximate-analytical solution for the mechanics of mass transfer in pulsatile flow in a pipe and to use this solution to show clearly how pulsation affects the bulk concentration and how the phenomenon depends on the Womersley number, which in turn depends on non-dimensional frequency. We use a convection-diffusion equation in cylindrical coordinates to describe mass transport in a tube. The radial profiles for a given axial length from the tube inlet are determined for the concentration of a species in the fluid such as oxygen. The mass transfer mixing efficiency in the tube is defined to be the bulk or mixing cup concentration, that is, the concentration obtained if the fluid is drained from the tube at a particular axial location and mixed. The mass transfer mixing efficiency is a measure of how well the concentration is dispersed across the lumen of the tube by the pulsatile flow. We choose the Dirichlet boundary condition with constant wall concentration at the tube wall. For steady flow, we refer to the solution of the Graetz problem. The oscillatory part of the solution is expressed as a perturbation sum of higher harmonics, where solutions are obtained via the generalized

integral transform technique [29]. The exact method of analysis of unsteady convective diffusion systems developed by Sankarasubramanian and Gill [8] is related to the generalized integral transform (GIT) technique in that a dimensionless area-average concentration is defined as the integral of the product of concentration and the radial coordinate of the tube. Using this number upon multiplication of the concentration equation by the radial coordinate and integration of the governing equation over the radius of the tube, it transforms the two-dimensional PDE into a system of one-dimensional PDEs, which are readily solvable. This is done by seeking a series solution for the concentration in terms of this dimensionless area-average concentration. We utilize a new method for the GIT technique, which allows an approximate solution up to any desired order and present sample calculations for varying frequencies and perturbation parameter β . To demonstrate the validity of the approximate solution (to the analytical solution), a numerical solution is also obtained for the corresponding problem.

2 Method of Solution

A formal solution of this nonhomogeneous problem is obtained through the use of the generalized integral transform technique, which is based on a Sturm–Liouville problem for which an analytical solution is known. We choose to follow the approach introduced by Cotta [29]. The nonhomogeneous complex partial differential equation we start with is obtained by a higher harmonic expansion on the original convection-diffusion equation with pulsatile flow.

A doubly infinite tube is considered with concentration prescribed at the entrance of the right half of the tube and a prescribed wall concentration (different from the entrance concentration) on the same half of the tube. A doubly infinite tube is considered so that the velocity distribution is fully developed pulsatile and independent of axial position, concentration being the only axially varying quantity. Refer to Fig. 1 for the geometry of the problem.

The convection-diffusion equation is

$$\frac{\partial C^*}{\partial t^*} + u^*(r^*, t^*) \frac{\partial C^*}{\partial x^*} = D \left(\frac{\partial^2 C^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial C^*}{\partial r^*} \right) + D \frac{\partial^2 C^*}{\partial x^{*2}} \quad (1)$$

From the Navier Stokes equations in cylindrical coordinates, the dimensional equation for pulsatile velocity u^* is the following classical solution [1]

$$u^*(r^*, t^*) = \frac{A_0}{4\mu} (r^{*2} - R^2) + \frac{A_1 R^2}{\mu i^3 \alpha^2} \left(1 - \frac{J_0(\alpha i^{3/2} r^*/R)}{J_0(\alpha i^{3/2})} \right) e^{i\omega^* t^*} \quad (2)$$

where R is the radius of the tube, ω^* is the dimensional frequency, r^* is the dimensional radius, t^* is the dimensional time, μ is the viscosity of blood, ν is the kinematic viscosity, $\omega = (R^2/D)\omega^*$ is the nondimensional frequency, $\alpha = R\sqrt{\omega/\nu}$ is the Womersley number, A_0 is the pressure gradient driving the steady part of the flow, and A_1 is the amplitude of the oscillatory pressure gradient driving the oscillatory part of the flow.

Dividing u^* by $\widehat{u}_0 = -A_0 R^2/4\mu$ gives us the following nondimensional equation:

$$u(r, t) = (1 - r^2) + F(r) e^{i\omega t}$$

where

$$F(r) = \frac{-4(A_1/A_0)}{\alpha^2 i^3} \left(1 - J_0(\alpha i^{3/2} r) / J_0(\alpha i^{3/2}) \right) = \frac{-\beta}{i^3} \left(1 - J_0(\alpha i^{3/2} r) / J_0(\alpha i^{3/2}) \right) \quad (3)$$

and $\beta = 4(A_1/A_0)/\alpha^2$

To nondimensionalize Eq. (1), we write the following variables:

$$r = r^*/R \\ t = t^*/T, \quad T = R^2/D$$

where D is the diffusion coefficient

$$x = x^*/L, \quad L = \widehat{u}_0 \cdot T$$

where $\widehat{u}_0$ is the Poiseuille centerline velocity

$$\text{Pe} = \widehat{u}_0 R/D$$

We also nondimensionalize C

$$C = (C^* - C_w) / (C_0 - C_w)$$

where C_w and C_0 are the wall and inlet concentrations, respectively.

This gives the following nondimensional equation for the concentration

$$\frac{\partial C}{\partial t} + u(r, t) \frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{1}{\text{Pe}^2} \frac{\partial^2 C}{\partial x^2} \quad (4)$$

where the boundary condition at the tube inlet $x=0$ is specified as $C=1$. We assume that the concentration field can be determined by the following sum:

$$C(r, x, t) = C_0(r, x) + \beta \widetilde{C}_1(r, x, t) + \beta^2 \widetilde{C}_2(r, x, t) + O(\beta^3) \quad (5)$$

where the second term on the RHS of Eq. (5) is the fluctuating or oscillatory concentration. Also, we assume that the Peclet number is large so that the axial diffusion term vanishes.

We proceed to solve Eq. (4) by substituting the sum given by Eq. (5) into Eq. (4) as well as substituting the velocity $u(r, t) = u_0(r) + \beta u_1(r, t)$ into Eq. (4). The first two boundary value problems we obtain for the unknowns C_0 and $\widetilde{C}_1$ are

$$\begin{aligned}
 u_0 \frac{\partial C_0}{\partial x} &= \frac{\partial^2 C_0}{\partial r^2} + \frac{1}{r} \frac{\partial C_0}{\partial r} \\
 \frac{\partial C_0}{\partial r}(x, 0) &= 0 \\
 C_0(0, r) &= 1 \\
 C_0(x, 1) &= 0
 \end{aligned} \tag{6a}$$

and

$$\begin{aligned}
 \frac{\partial \tilde{C}_1}{\partial t} + u_1 \frac{\partial \tilde{C}_0}{\partial x} + u_0 \frac{\partial \tilde{C}_1}{\partial x} &= \frac{\partial^2 \tilde{C}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{C}_1}{\partial r} \\
 \frac{\partial \tilde{C}_1}{\partial r}(x, 0) &= 0 \\
 \tilde{C}_1(0, r) &= 0 \\
 \tilde{C}_1(x, 1) &= 0
 \end{aligned} \tag{6b}$$

For the BVP given by Eq. (6b), we proceed to solve it by looking for a solution in the form:

$$\tilde{C}_1(r, x, t) = C_1(r, x) e^{i\omega t}. \text{ Equation (6b)}$$

$$i\omega C_1(r, x) + (1 - r^2) \frac{\partial C_1}{\partial x} + \frac{F(r)}{\beta} \frac{\partial C_0}{\partial x} = \frac{\partial^2 C_1}{\partial r^2} + \frac{1}{r} \frac{\partial C_1}{\partial r} \tag{7a}$$

becomes In the light of the method using the GIT approach [29], we introduce the following auxiliary problem, which uses boundary conditions at the wall of the tube and the center line of the tube

$$\begin{aligned}
 \frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi(\mu_i, r)}{dr} \right) + \mu_i^2 (1 - r^2) \psi(\mu_i, r) &= 0 \\
 \frac{d\psi(\mu_i, 0)}{dr} &= 0 \\
 \psi(\mu_i, 1) &= 0
 \end{aligned} \tag{7b}$$

Refer to the Appendix for the solution of Eq. (7b) and, hence, BVP (6a). The first few eigenfunctions of the auxiliary problem (Eq. (7b)) are shown in Fig. 2.

Operating on Eq. (7a) (see Ref. [29]) with $\int_0^1 r \psi(\mu_k, r) \times (1/N_k^{1/2}) dr$, where N_k is given by the following equation:

$$N_k = \int_0^1 r (1 - r^2) \psi^2(\mu_k, r) dr \tag{8a}$$

and using the following integral transform pair:

$$C_1(r, x) = \sum_{k=1}^{\infty} \frac{1}{N_k^{1/2}} \psi(\mu_k, r) \bar{C}_{1k}(x) \tag{8b}$$

$$\bar{C}_{1k}(x) = \int_0^1 r (1 - r^2) \frac{\psi(\mu_k, r)}{N_k^{1/2}} C_1(r, x) dr \tag{8c}$$

yields the following equation, which is a system of ode's [29]:

$$\frac{d\bar{C}_{1k}(x)}{dx} + \mu_k^2 \bar{C}_{1k}(x) + i\omega \sum_{k^*=1}^{\infty} a_{kk^*} \bar{C}_{1k^*}(x) = Q_k(x), \quad k=1, 2, 3, \dots \quad (9)$$

where

$$a_{kk^*} = \int_0^1 \frac{1}{N_k^{1/2} N_{k^*}^{1/2}} r \psi(\mu_k, r) \psi(\mu_{k^*}, r) dr$$

$$Q_k(x) = - \int_0^1 \frac{r \psi(\mu_k, r)}{N_k^{1/2}} \frac{F(r)}{\beta} \frac{\partial C_0(x, r)}{\partial x} dr$$

and $C_0(x, r)$ is given in the Appendix.

This is a system of complex coupled ordinary differential equations with the boundary condition transformed as $\bar{C}_{1k}(0)=0$ meaning that from the harmonics expansion and the integral transform pair above, $C_1(r,0)=0, C_2(r,0)=0, \dots$ implies that $C(r,0,t)=C_0(r,0)=1$. This condition is used in the Graetz problem, which is the zeroth-order problem in the present case.

The system of equations given by Eq. (9) is infinite and coupled. In order to obtain a solution, we must truncate the sum and solve a finite system of coupled ordinary differential equations. Equation (9) is then written as the following

$$\frac{d\bar{C}_{1k}(x)}{dx} + \mu_k^2 \bar{C}_{1k}(x) + i\omega \sum_{k^*=1}^N a_{kk^*} \bar{C}_{1k^*}(x) = Q_k(x) \quad k=1, 2, 3, \dots N \quad (10)$$

2.1 Diagonalization

We write this system in the following compact form:

$$\mathbf{x}' = A\mathbf{x} + Q(x) \quad (11)$$

where A is an $n \times n$ constant complex matrix given by the following:

$$A = \begin{cases} -\delta_{kk^*} \mu_k^2 - i\omega a_{kk^*} & k, k^*=1, 2, 3, \dots N \\ \delta_{kk^*} = \begin{cases} 0 & k \neq k^* \\ 1 & k = k^* \end{cases} \end{cases}$$

The vector $Q(x)=[Q_1(x), Q_2(x), Q_3(x), \dots, Q_N(x)]^T$. By diagonalizing the coefficient matrix A , we can transform Eq. (11) into a system of equations that is readily solvable. To do this we use the following procedure.

Let V be the matrix whose columns are the eigenvectors $\zeta^{(1)}, \dots, \zeta^{(n)}$ of A and define a new dependent variable y by $\mathbf{x} = V\mathbf{y}$. Then substituting for x into Eq. (11), we obtain

$$V\mathbf{y}' = AV\mathbf{y} + Q(x)$$

A is non-Hermitian and by assuming that it has a full set of N linearly independent eigenvectors, the existence of V^{-1} is guaranteed. (Note: `MATLAB` is used to give V) By multiplying by V^{-1} , it follows immediately that

$$\mathbf{y}' = (V^{-1}AV)\mathbf{y} + V^{-1}Q(x) = D\mathbf{y} + h(x), \quad (12)$$

where $h(x) = V^{-1}Q(x)$ and D is the diagonal matrix whose diagonal entries are the eigenvalues $r_1, r_2, \dots, r_n$ of A arranged in the same order as the corresponding eigenvectors $\zeta^{(1)}, \dots, \zeta^{(n)}$ that appear as columns of the matrix V .

Equation (12) is a system of n uncoupled equations for $y_1(x), \dots, y_n(x)$; as a consequence, the equations can be solved separately. In scalar form, Eq. (12) has the form

$$y_j'(x) = r_j y_j(x) + h_j(x), \quad j=1, \dots, N \quad (13)$$

where $h_j(x)$ is a certain linear combination of $Q_1(x), \dots, Q_N(x)$. Equation (13) is a first-order linear equation and can be solved by variation of parameters. We have the following:

$$y_j(x) = e^{r_j x} \int_{s=x_0}^x e^{-r_j s} h_j(s) ds + c_j e^{r_j x}, \quad j=1, \dots, N \quad (14)$$

where c_j are arbitrary constants (to be determined by boundary conditions). In this paper, we impose the condition $\bar{C}_{1k}(0) = 0$ at the tube inlet ($x=0$). We let $x_0=0$. This means that we will take the constant $c_j=0$.

Finally, the solution $\mathbf{x}'$ of Eq. (11) is obtained by the transformation equation $\mathbf{x} = V\mathbf{y}$. When multiplied by the transformation matrix V , the first term on the right side of Eq. (14) yields a particular solution of the nonhomogeneous system given by Eq. (11) while the second term on the right side of Eq. (14) produces the general solution of the homogeneous equation $\mathbf{x}' = A\mathbf{x}$. In this paper, we choose the coefficient of the general solution of the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ to be zero in order to satisfy the boundary condition $\bar{C}_{1k}(0) = 0$.

2.2 Higher Perturbation Terms

We substitute

$$C(r, x, t) = C_0(r, x) + \beta \bar{C}_1(r, x, t) + \beta^2 \bar{C}_2(r, x, t) + o(\beta^3)$$

and

$$u(r, t) = u_0(r) + \beta u_1(r, t)$$

into Eq. (4) and collect β^2 terms. Using the GIT gives us the governing equation for the case of higher perturbation terms in β^2 as

$$\frac{d\bar{C}_{2k}(x)}{dx} + \mu_k^2 \bar{C}_{2k}(x) + i\omega \sum_{k^*=1}^N a_{kk^*} \bar{C}_{2k^*}(x) = Q_k^1(x) \quad (15)$$

where

$$\begin{aligned} Q_k^1(x) &= -\int_0^1 \frac{r\psi(\mu_k, r)}{N_k^{1/2}} \frac{F(r)}{\beta} \frac{\partial C_1(x, r)}{\partial x} dr \quad \text{and} \quad C_1(x, r) \\ &= \sum_{k=1}^{\infty} \frac{1}{N_k^{1/2}} \psi(\mu_k, r) \bar{C}_{1k}(x) \end{aligned}$$

Equation (15) can be expressed in the form $\mathbf{x}' = A\mathbf{x} + Q^1(x)$, as in the first-order case given by Eq. (11) where $A = \{-\delta_{kk^*} \mu_k^2 - i\omega a_{kk^*}\}$ $k, k^* = 1, 2, 3, \dots, N$ as before. The only difference is in the right side of Eq. (15). The calculation of $Q^1(x)$ can be done using software packages such as MATLAB that allow symbolic differentiation. Note that

$Q^1(x) = [Q_1^1(x), Q_2^1(x), \dots, Q_N^1(x)]^T$. We diagonalize the coefficient matrix A and transform the equation $\mathbf{x}' = A\mathbf{x} + Q^1(x)$ into a system of equations that is readily solvable as in the first-order problem outlined above in Eqs. (12)–(14). In particular, we write

$$y_j(x) = e^{r_j x} \int_{s=x_0}^x e^{-r_j s} h_j(s) ds + c_j^1 e^{r_j x}, \quad j=1, \dots, N \quad (16)$$

where the term $h_j(s) = V^{-1} Q^1(s)$. Let $x_0 = 0$. Since $\bar{C}_{2k}(x_0) = 0$, we let $c_j^1 = 0$.

We can continue to solve the next higher order perturbation equations, which are in the form

$$\frac{d\bar{C}_{mk}(x)}{dx} + \mu_k^2 \bar{C}_{mk}(x) + i\omega \sum_{k^*=1}^N a_{kk^*} \bar{C}_{mk^*}(x) = Q_k^{m-1}(x), \quad m=3, 4, \dots \quad (17)$$

where $\bar{C}_{mk}(0) = 0$ and $Q_k^{m-1}(x) = -\int_0^1 \left((r\psi(\mu_k, r) / N_k^{1/2}) (F(r) / \beta) \times (\partial C_{m-1}(x, r) / \partial x) \right) dr$

Note that when performing symbolic calculations on a computer, these higher harmonic terms may take more computational time to calculate.

3 Results and Discussion

In this section, we solve Eq. (10) symbolically using MATLAB. The analytical solution is first verified by comparison to a numerical solution, and then the features of the solution are discussed in terms of the effect of flow oscillations on mass transport.

3.1 Numerical Verification of Analytical Method

A fourth-order Runge–Kutta method [30] was employed in the solution of the system of equations given by Eq. (10). This was compared with a MATLAB built-in variable step size Runge–Kutta method (ode 45 solver [31]). The local truncation error per step h will behave like Ch^5 for some constant C . Here, C is independent of h but dependent on the initial condition at $x=x_0$ and the fourth derivative of the vector $(C_{11}(x), C_{12}(x), \dots, C_{1N}(x))$ at $x=x_0$. A numerical solution was obtained in order to validate the analytical solution presented in

Sec. 2.1. In Fig. 3(a), the first nine solutions $\text{Re}(\bar{C}_{1k})$, $k=1, \dots, 9$ are plotted for the numerical solution of Eq. (10) for $0 \leq x \leq 0.31$ and sample frequency $\omega = 2$. In Fig. 3(b), the first nine solutions ... are plotted for the numerical solution of Eq. (10) for $0 \leq x \leq 1$ and sample frequency $\omega = 5$. In both cases it is observed that the most substantial term is the first term

C_{11} with higher C_{1k} shifting toward zero. The solutions $\text{Re}(\bar{C}_{1k})$, $k=1, \dots, 9$, are used in the series given by Eq. (8b) to obtain $\text{Re}(C_1(r,x))$, which is used in the definition of the bulk

concentration in Sec. 3.4. In Fig. 3(c), we plot $\text{Re}(\bar{C}_{1k})$, $k=1, \dots, 9$ close to the origin along the x -axis to show its variation clearly for frequency $\omega=5$. Equation (10) was solved for C_{1k} analytically with $N=9$ using MATLAB. In Figs. 3(d) and 3(e), the analytical solution is plotted versus the numerical solution for C_{1k} , $k=1,2,3,4$. It is observed that there is excellent

agreement between the two solutions. The solutions for $\text{Re}(\bar{C}_{1k}) + i \text{Im}(\bar{C}_{1k})$ were obtained for frequencies $\omega=2,5$ (results for $\text{Im}(\bar{C}_{1k})$ not shown). The solutions $\text{Im}(\bar{C}_{1k})$, $k=1, \dots, 9$ are used in the series given by Eq. (8c) to obtain $\text{Im}(C_1(r,x))$, which is also used in the definition of the bulk concentration. Other frequencies for both real and imaginary parts of C_{1k} were also tested, and there was good agreement between the numerical and analytical solutions. We have included the full range of frequencies used for C_{1k} in the solution of Eq. (10), and for this range of frequencies the terms $\text{Re}(C_1(x,r))$, $\text{Im}(C_1(x,r))$ are calculated, (see Sec. 3.4)

The numerical solutions to Eq. (10) and, therefore $C_1(r,x)$ (from Eq. (8c)) are still to be compared with the analytical solution, which is discussed next.

Analytical results in the next subsection are based on $N=30$ in Eq. (10). We have used MATLAB to calculate the eigenvalues r_j in Eq. (14) and the eigenvectors $\zeta^{(1)}, \dots, \zeta^{(N)}$ in Eqs. (12) and (14).

3.2 Concentration Fluctuations Versus Frequency and Axial Location

Corresponding to the bulk concentration, which we will define below, is the calculation of the fluctuating or oscillating concentration given by the second term of Eq. (5). This is

$C_{FI} = \beta C_1(r,x) e^{i\omega t}$. From Eq. (3), $\beta = \left(\sqrt{\omega/\nu} \cdot R \right)^{-2} \cdot 4A_1/A_0$. In the context of blood flow in the microcirculation, we choose a vessel of radius $R=0.005$ m and $A_1/A_0=1$ with the kinematic viscosity of blood at 0.038 Stokes at body temperature 37°C. One Stokes is 0.0001 m²/s. With this choice of radius and amplitude ratio, the parameter $\beta \ll 1$. In Table 1 below, computed values of β are given corresponding to the first seven frequencies.

For axial distance $x=0.07$ at sample frequency $\omega=2$, the numerical solution of Eq. (10) for $\bar{C}_{1k}$, $k=1, \dots, 9$ (see Fig. 3(a)) is used in Eq. (8b) to obtain an approximation to $C_1(r,x)$. The real part of this numerical solution is plotted versus the analytical solution $\text{Re}(C_{FI})(\omega t=2\pi k)$ in Fig. 4 for the same axial distance and frequency, and it is observed that there is excellent agreement between the analytical and numerical solution. Over a range of axial distances x , radial values r , and frequencies ω , the maximum error between the numerical and analytical solution for $C_1(x,r)$ was found to be less than 5×10^{-5} . With this bound in error, we conclude that our numerical and analytical solutions are almost identical, thus validating the analytical diagonalization approach given by Eq. (12).

In Fig. 5(a), we plot $\text{Re}(C_{FI})$ at $\omega t=2\pi k$, k is an integer, in a frequency range of $\omega=[2 \ 30]$ at $x=0.005$ distance from the inlet of the tube versus radius r of the tube. As the axial position moves in the downstream direction, the lobes of the C_{FI} curves broaden as can be seen in Fig. 5(b). These results are qualitatively consistent with the plots of the amplitudes of the fluctuating part of the temperature profiles θ_1 for moderate amplitudes in the work of Cho et al. in Fig. 12 of Ref. [26].

In Fig. 5(c), we plot $\text{Re}(C_{Fl})$ versus radius r for various β less than unity for fixed axial distance $x=0.07$ and at time $\omega t=2\pi$. In Fig. 6(a), $\text{Re}(C_{Fl})$ is plotted for axial distance $x=0.12$ at $\omega t=2\pi k$, and in Fig. 6(b), at $x=0.2$ the oscillating concentration becomes negative to a greater extent than in Fig. 6(a). According to Cho and Hyun [26], in the bulk of the tube apart from the entrance region, the time averaged Nusselt number associated with the pulsations of the fluid is generally lower than the steady Nusselt number for the majority of the frequency range. For some frequencies, the overall concentration C is lower than the steady concentration $C_0(r,x)$. For $\omega t=2\pi k$, we consider $\text{Re}(C_{Fl})$ at axial distance from the inlet $x=0.31$ in Fig. 6(c). Here, there is no positive concentration profile occurring in the full range of frequencies considered. For most frequencies, the total concentration is less than the corresponding steady solution given by the Graetz problem.

Note that the higher perturbation terms $\bar{C}_{2k}$, $K=1, \dots, 9$ from Eq. (15) were determined both analytically and numerically for $\omega=5$ (results not shown). When $\bar{C}_{2k}$, $K=1, \dots, 9$ are substituted into $C_2(x,r) = \sum_{k=1}^{\infty} (1/N_k^{1/2}) \psi(\mu_k, r) \bar{C}_{2k}(x)$ and $C_2(x,r)$ is multiplied by $\beta^2 \ll 1$, the values $\bar{C}_{2k}$ do not contribute any substantial amount to the total concentration. This is due to the fact that $\beta^2 C_2(x,r)$ is very small compared with the first two terms in the perturbation series for C .

3.3 Concentration Fluctuations Versus Time

As discussed previously, $\text{Im}(C_{1k}(x))$ is used in Eq. (8c) to obtain the solution for $\text{Im}(C_1)$. This is also used in the expression for $\text{Re}(C_{Fl})$. For frequencies $\omega=2, 5, 10, 30$ at $x=0.12$, $\text{Im}(C_1)$ is plotted versus radius r in Fig. 7(a). Other distances x were also considered. The numerical and analytical solutions for $\text{Im}(C_1)$ agree within the bound discussed previously in Sec. 3.2.

Let us now examine the concentration distribution throughout the time cycle for sample frequency $\omega=2$. In Figs. 7(b) and 7(c), we plot $\text{Re}(C_{Fl})=\text{Re}(\beta C_1(r,x)e^{i\omega t})$ at increments of $\pi/2$ through a cycle at $t=\pi/6, \pi/4, \pi/3, \pi/2, \pi, 2\pi$ at $x=0.01$ and 0.12 , respectively. It is interesting that for $x=0.12$ a maximum at $t=\pi/4, 5\pi/4$ occurs in phase with pulsatile velocity, which also has a maximum at $t=\pi/4, 5\pi/4$ compared with the rest of the time cycle. There is symmetry for the solution about multiples of $\pi/2$. At $t=3\pi/4, 7\pi/4$, a minimum occurs in phase with pulsatile velocity. The concentration $\beta \text{Re}(C_1(r,x)e^{i\omega t})$ for $A_1/A_0=1$, at $x=0.12$, $\omega=5$, $\omega=10$, and $\omega=30$, and higher, respectively, throughout the cycle was calculated, and the maximum values of $\beta \text{Re}(C_1(r,x)e^{i\omega t})$ occur at $t=\pi/2, \pi/3, 3\pi/4$, respectively. Minimum values were determined at $t=3\pi/2, 5\pi/6, \pi/6$, respectively. The maximum and minimum values for $\beta \text{Re}(C_1(r,x)e^{i\omega t})$ decrease toward zero as the frequency increases beyond $\omega=30$. It is worthy to note that the unique maximum value of $\beta \text{Re}(C_1(r,x)e^{i\omega t})$ occurring at nondimensional frequency $\omega=5$ is in phase with pulsatile velocity at $t=\pi/2$ (and at no other times in the cycle, see Fig. 7(d)). Here, the unique maximum value for pulsatile velocity $\text{Re}(F(r)e^{i\omega t})$ also occurs at $t=\pi/2$. For this frequency and only this frequency, there are distinct values for the concentration throughout the cycle. There are no multivalued curves for any time in the cycle $[0, 2\pi]$. For each time in the cycle, the pulsatile velocity and concentration have the same sign. The physics of this phenomenon indicates that there is an important relationship between pulsatile velocity and concentration for low to moderate frequencies. At frequencies much higher than $\omega=5$, the velocity and concentration are out of phase with respect to each other over the cycle with a decrease in the product of velocity and concentration. For example, at $\omega=10$ there exist times in the cycle for which the velocity and concentration are opposite in sign and thus, out of phase with each other. Similarly for $\omega=30$, the velocity and concentration are out of phase with respect to each other with velocity and concentration being opposite in sign for every time in the cycle. The previous

few paragraphs of this section are related in an important way to the next section, where we define the bulk concentration and show that at the frequency ($\omega=5$; $x=0.12$) the bulk concentration is maximum. It is clear that the $\omega t=2\pi k$ values from Sec. 3.2 are bounded across the radius of the tube by the maximum and minimum values over a cycle as can be seen in Fig. 7(c). In the following subsection, a way of integrating out the time-dependent contributions for concentration over a cycle is given.

3.4 Bulk Concentration Versus Frequency and Axial Location

Introducing the following quantity:

$$C_{\text{Avgt}}(r) = \overline{\text{Re}(u_1) \text{Re}(\bar{C}_1)}$$

where $u_1 = (F(r)/\beta)e^{i\omega t}$ and the overbar indicates time average over one cycle, we define the bulk concentration as

$$C_{1b} = \int_0^1 \overline{\text{Re } u(r, t) * \text{Re } \bar{C}_1(x, r, t)} r dr / \int_0^1 \text{Re}(\bar{u}(r)) r dr$$

so that $C_{1b} = \int_0^1 C_{\text{Avgt}} r dr / \int_0^1 \text{Re}(\bar{u}(r)) r dr$.

With $u(r, t) = (F(r)/\beta)e^{i\omega t}$ and $\bar{C} = C_1(r, x)e^{i\omega t}$ becomes

$$C_{1b} = \int_0^1 \frac{4}{2} \left[\text{Re} \frac{F(r)}{\beta} \text{Re}(C_1) + \text{Im} \frac{F(r)}{\beta} \text{Im}(C_1) \right] r dr$$

The profiles for C_1 (both real and imaginary parts) were obtained in the previous sections and are used in the bulk concentration C_{1b} , which is plotted in Fig. 8 versus frequency ω at $x=0.01, 0.07, 0.12, 0.24$, and 0.48 . The `MATLAB` `trapz` function was used to calculate the integral involved in the definition of the bulk concentration. It is observed that in a range of small to moderate nondimensional frequencies there is a positive peak in the bulk concentration C_{1b} due to the pulsations, thereby, producing a higher mass transfer mixing efficiency in the tube. For all x values, there is an oscillating between positive and negative values and a dampening effect as ω increases. The dampening effect for bulk concentration occurs for smaller and smaller values of frequency as the axial distance from the tube inlet increases. Also, the peak in the bulk concentration is attained for smaller frequency as we increase the axial distance from the inlet of the tube. Finally there are more nodal points where maximum and minimum points occur with increases in x . It can be observed that as x gets large downstream in the tube the peak near zero frequency approaches zero.

Recalling that

$$C_b = \frac{\int_0^1 \text{Re } u(r, t) \text{Re } \bar{C}(r, x, t) r dr}{\int_0^1 \text{Re } \bar{u}(r) r dr} \quad (18)$$

we first have that

$$\int_0^1 \text{Re } \bar{u}(r) r dr = \int_0^1 u_0(r) dr \int_0^1 (1-r^2) r dr = 1/4$$

and then substituting the approximating series

$$C(r, x, t) = C_0(r, x) + \beta C_1(r, x) e^{i\omega t}$$

into Eq. (18) together with the expression for velocity

$$u(r, t) = (1 - r^2) + \beta u_1(r, t)$$

we obtain the following for the total bulk concentration C_b :

$$C_b = 4 \int_0^1 u_0 C_0 r dr + \beta^2 4 \int_0^1 u_1 \bar{C}_1 r dr = 4C_{0b} + 4\beta^2 C_{1b}$$

For different axial distances from the inlet, $x=0.01, 0.07, 0.12, 0.24$, and 0.48 , we calculate C_b , the total bulk concentration for the total concentration C and velocity u . C_b has a similar trend as C_{1b} with an oscillating behavior about the steady value $4C_{0b}$ for increasing frequency ω . From Moschandreou and Zamir [25], if we define the local Sherwood numbers

for pulsatile and steady flow as $\text{Sh} = 2 / \left(\text{Re } \bar{C}(1) - C_b \right)$ and $\text{Sh}_0 = 2 / (C_0(1) - 4C_{0b})$, respectively, then the relative Sherwood number $(\text{Sh} - \text{Sh}_0) / \text{Sh}_0 \times 100\%$ will be opposite in sign to C_{1b} . This will result in the same behavior in relative Sherwood number as in Fig. 4 of Ref. [14] for a corresponding heat transfer problem, where the relative Nusselt number is first negative for small frequencies (beginning at zero) and positive for slightly higher frequencies and then becomes negative again for even higher frequencies so that there is a band of frequencies $0 < \omega_1 < \omega_2$, where the relative Nusselt number is positive. Thus, the relative Sherwood number becomes negative first for small frequencies and then becomes positive for a band of frequencies $0 < \omega_1 < \omega_2$. At low frequencies and near to the entrance region, there is a negative relative Sherwood number indicating that the effect of pulsation tends to reduce the mass transfer.

4 Conclusions

A new approximate solution for mass transfer in a tube with pulsatile flow has been presented and numerically validated. The approximate-analytical solution for $N=30$ to Eq. (10) indicates that for small to moderate nondimensional frequencies there is a positive peak in the time-averaged bulk concentration due to the pulsation, thereby producing a higher mass transfer mixing efficiency in the tube. As we further increase the frequency, the positive peak is followed by a negative peak in the time-averaged bulk concentration and then the bulk concentration C_{1b} oscillates and dampens to zero. Initially for small frequencies, the relative Sherwood number is negative indicating that the effect of pulsation tends to reduce the mass transfer. Also, there is a band of frequencies where the relative Sherwood number is positive indicating that the effect of pulsation tends to increase the mass transfer. Fluctuating concentration profiles are obtained for various axial distances from the tube inlet. The positive peak in bulk concentration corresponds to a matching of the phase of the pulsatile velocity and the concentration, respectively, where the unique maximum of both occur for certain time in the cycle. The oscillatory component of

concentration using the GIT-based solution proved to be in excellent agreement with the numerical solution used to verify the approximate-analytical results. The GIT solution method described here can be applied to other similar problems and could be used, for example, to estimate how more general flow waveforms affect mass transport (of oxygen or other diffusible substances) in various size pipes or blood vessels. Also, a future consideration for Eq. (7a) would be to use a Green's function approach to the solution.

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Appendix

We transform Eq. (7b) by the following similarity transformation:

$$W(X) = \exp\left(\frac{X}{2}\right) \psi(r) \quad (\text{A1})$$

where $X = \mu_i r^2$. The transformed partial differential equation for $W(X)$ is given by the following,

$$X \frac{d^2 W}{dX^2} + (1 - X) \frac{dW}{dX} + \left(\frac{\mu_i}{4} - \frac{1}{2}\right) W(X) = 0 \quad (\text{A2})$$

which is a second-order ordinary differential equation. This differential equation is identified as the confluent hypergeometric equation or Kummer's equation [32] where W is the confluent hypergeometric function of the first or second kind. These functions are put together to obtain the general solution of the problem. We immediately obtain

$$W(X) = c_1 M(a, 1, X) + b_1 U(a, 1, X)$$

where c_1 and b_1 are real constants and the confluent hypergeometric function of the first kind is proved to be

$$M(a, 1, X) = 1 + \frac{a}{1} X + \frac{a(a+1)}{1(2)} \frac{X^2}{2!} + \dots + \frac{a(a+1)\dots(a+n-1)}{n!} \frac{X^n}{n!}$$

The constant a is given by

$$a = \frac{1}{2} - \frac{\mu_i}{4} \quad \text{for } i=1, 2, 3, \dots$$

The infinitely many eigenvalues μ_i are the solutions following equation:

$$M\left(\frac{1}{2} - \frac{\mu_i}{4}, 1, \mu_i\right) = 0$$

In general $\mu_i = 4 \times (i-1) + 8/3$, $i=1,2,3,\dots$ where we have referred to Sellars et al. [33] for the formula of eigenvalues μ_i .

The second general solution for Kummer's equation is given by the following:

$$U(a, b, X) = \frac{\pi}{\sin \pi b} \left[\frac{M(a, b, X)}{\Gamma(1+a-b)\Gamma(a)} - X^{1-b} \frac{M(1+a-b, 2-b, X)}{\Gamma(a)\Gamma(2-b)} \right]$$

This second solution is unbounded at $b=1$, which is the value for b in our problem. We are left with one solution to Kummer's equation, which is given by the following:

$$W(X) = M(a, 1, X) \quad (\text{A3})$$

where a is given by $a = (1/2) - (\mu_i/4)$

Upon solving for W from Eq. (A2), Eq. (A1) is used to obtain $\psi(\mu_n; r)$. The solution for BVP (6a) for $C_0(r, x)$ is obtained by writing it as an infinite series

$$C_0(r, x) = \sum_{n=1}^{\infty} A_n e^{-\mu_n^2 x} \psi_n(\mu_n; r)$$

where $A_n = \int_0^1 \psi_n(r) r (1-r^2) dr / \int_0^1 \psi_n^2(r) r (1-r^2) dr$.

Glossary

Nomenclature

A₁	oscillatory amplitude
A₀	steady amplitude
C	concentration
C_{Avgt}(r)	time-averaged concentration
C_{1b}	time-averaged oscillatory bulk concentration
C_{0b}	steady bulk concentration
C_b	overall bulk concentration
$\tilde{C}_1$	time-dependent oscillatory concentration
C₁	spatial oscillatory concentration
$\bar{C}_{1k}, \bar{C}_{2k}, \dots$	GIT basis functions
D	diffusion coefficient
Im	imaginary part of complex quantity
i	$\sqrt{-1}$
J₀	Bessel function of the first kind of order 0
L	$\widehat{u_0} T$

N	number of GIT eigenfunctions
N_k	normalization integral
R	tube radius
Re	real part of complex quantity
r	radial coordinate measured from tube axis
r_i	eigenvalue of matrix $A = \{-\delta_{kk^*} \mu_k^2 - i\omega a_{kk^*}\}$
s	integration parameter
T	R^2/D
t	time
u	pulsatile velocity component in axial direction
$\widehat{u}_0$	$-A_0 R^2/4\mu$, Poiseuille centerline velocity
V	matrix whose columns are eigenvectors of $A = \{-\delta_{kk^*} \mu_k^2 - i\omega a_{kk^*}\} k, K^*=1,2, \dots, N$
W	Kummer's function
x	axial coordinate measured from tube entrance

Greek Symbols

α	$R \sqrt{\omega/\nu}$, Womersley number
β	$= (4A_1/A_0)/\alpha^2$, perturbation parameter
δ_{kk^*}	Kronecker delta function
μ	viscosity of the fluid
μ_i	eigenvalues for S-L problem
ν	kinematic viscosity of the fluid
$\zeta^{(i)}$	i th eigenvector of matrix A
ψ	eigenfunction for S-L problem of GIT method
ω	frequency
*	denotes a dimensional quantity

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Fig. 1.
Geometry of problem considered

Fig. 2.
Eigenfunctions ψ associated with Graetz problem

Fig. 3.

(a)–(c) First nine solutions $\text{Re}(\bar{C}_{1k})$, $K=1, \dots, 9$, of Eq. (10) for $0 \leq x \leq 0.31$ and $\omega=2$, $0 \leq x \leq 1$ and $\omega=5$, and $0 \leq x \leq 0.1$ and $\omega=5$, respectively. (d)–(e) Numerical and analytical plots of C_{1k} for $k=1,2$ and $k=3,4$, respectively.

Fig. 4. Numerical solution of Eq. (7) compared with the analytical solution for $x=0.07$ and $\omega=2$

Fig. 5.
Concentration $\beta \text{Re}(C_{Fl})$ for $A_1/A_0=1$ and $\omega t=2\pi k$ at (a) $x=0.005$, (b) $x=0.01$, and (c) $x=0.07$

Fig. 6. Concentration $\beta \text{Re}(C_F)$ for $A_1/A_0=1$ and $\omega t=2\pi k$ at (a) $x=0.12$, (b) $x=0.2$, and (c) $x=0.31$

Fig. 7.

Radial variation of $C_1(x, r)$: (a) $\text{Im}(C_1)$ at $x=0.12$. (b–d) Variation over an oscillation period of $\beta \text{Re}(C_1(r, x)e^{i\omega t})$ for $A_1/A_0=1$ at $x=0.01$ and $\omega=2$, $x=0.12$ and $\omega=2$, and $x=0.12$ and $\omega=5$, respectively.

Fig. 8. Bulk concentration C_{1b} versus frequency ω at (a) $x=0.07$, (b) $x=0.12$, (c) $x=0.24$, (d) $x=0.48$, and (e) $x=0.01$

Table 1Frequencies ω corresponding to perturbation parameter β

ω	β
2	0.304000
3	0.202666
4	0.151999
5	0.121600
6	0.101333
7	0.086857