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The Product Operator Formalism: A Physical and Graphical Interpretation

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Abstract

The product-operator formalism is the most commonly used tool for describing and designing multidimensional NMR experiments. In spite of its relative simplicity and sound theoretical underpinnings, however, students and practitioners often find it difficult to relate the mathematical manipulations to a physical picture. In an effort to address this pedagogical challenge, the present paper begins with a quantum-mechanical treatment of pure populations of scalar-coupled spin-pairs, rather than the equilibrium population of spin-pairs in different quantum states, which is the usual starting point for treatments based on the density matrix and product operators. In the context of pure populations, the product operators are shown to represent quantum correlations between the nuclei in individual molecules, and a new variation on the classical vector diagram is introduced to represent these correlations. The treatment is extended to mixed populations that begin at thermal equilibrium, and the density matrix is introduced as an efficient means of carrying out quantum calculations for a mixed population. Finally, it is shown that the operators for observable magnetization and correlations can be used as a basis set for the density matrix, providing the formal justification for the widely-used rules of the product-operator treatment. Throughout the discussion, the vector diagrams are used to help maintain a connection between the mathematics and the sometimes subtle physical principles. An electronic supplement created with the *Mathematica* computer program is used to provide additional mathematical details and the means to carry out further calculations.

Keywords

product operators; scalar coupling; quantum correlations; vector diagrams

INTRODUCTION

Any description of NMR spectroscopy requires, at some level, the theoretical tools of both quantum mechanics and statistical mechanics. On one hand, the absorption of electromagnetic radiation by nuclei and the subsequent observable effects are quantum mechanical phenomena. At the same time, statistical mechanics must be employed because the samples used in NMR contain large numbers of nuclei in different quantum mechanical states, and the observation of a spectroscopic signal depends on an excess population of nuclei in the lower energy states at equilibrium. For most one-dimensional NMR experiments with spin-1/2 nuclei, the quantum and statistical phenomena can be dealt with using relatively simple treatments. The effects of radiation on the nuclei can be described using a vector representing the net magnetization of the sample, and the behavior of this vector can be accounted for using an analogy with the action of torques on a spinning object

(1,2). The statistical mechanics are accounted for by considering only the initial excess of nuclei in the lower energy state.

The treatment of multidimensional NMR experiments, especially those involving scalar coupling, is considerably more difficult both conceptually and practically. With just two coupled spins, there are four energy states with distinct properties, and it is no longer sufficient to consider only the state with the lowest energy. In addition, the interaction between the spins leads to new features in the time-dependent evolution of the irradiated nuclei that are more difficult to account for using an analogy with classical angular momentum.

Some of the difficulties in describing scalar-coupled spins using traditional vector diagrams emerge from examining relatively simple heteronuclear NMR experiments (3), such as shown in Figure 1. In these experiments, the sample is assumed to be composed of molecules containing a single nucleus of one type, labeled *S*, covalently bound to a nucleus of a second type, *I*. For the actual experiments, chloroform (CHCl_3) was used, with *S* representing ^{13}C (at natural abundance), and *I* representing the single ^1H nucleus. To simplify the drawings and analysis, the resonance frequencies of the two spins (in the absence of scalar coupling) are set to match the rotating-frame frequencies, and all pulses are aligned with the x' -axis. The vector diagrams in the figure represent the expected magnetization components immediately before the data acquisition period. All of the vector diagrams in this paper are drawn using a right-handed three-dimensional coordinate system, and rotational motions of the vectors are drawn following the “left-hand rule”, so that motion in the positive direction appears clockwise when viewed from the positive end of the axis of rotation. Pulses are designated by the rotation angle, in radians, with a subscript indicating the axis and, optionally, the irradiated nucleus. Thus, the pulse represented in Figure 1a is designated $(\pi/2)_{x,S}$.

In experiment (a) of the figure, a single $\pi/2$ -pulse is applied to the *S*-spin, and a free-induction decay (FID) is recorded from that nucleus either immediately after the pulse ($\tau = 0$) or after a delay of length $\tau = 1/(2J)$, where *J* is the scalar coupling constant. In the absence of the delay, the expected doublet is observed, with the two components displaying the same phase. This result can be readily accounted for by noting that each *S*-nucleus in the sample is coupled to an *I*-spin, which can be aligned either with or against the external field, thereby generating two slightly different local fields and two precession frequencies for the *S*-spins. If a delay of $1/(2J)$ is introduced between the pulse and the data-acquisition period, the phases of the resulting signals are altered so that the two peaks are $\pi/2$ radians out of phase with those observed with $\tau = 0$, and π radians out of phase with one another. The phase differences are accounted for by the different precession frequencies of the *S*-magnetization in the two populations, which lead to two components pointing in opposite directions at the end of the delay period, as illustrated by the vector diagram.

Experiment (b) includes a second $\pi/2$ -pulse, which is applied to the *I*-spins immediately before the FID is recorded. If the time interval between the first and second pulses is 0, the resulting spectrum is the same as seen in experiment (a) with $\tau = 0$. This result is somewhat more difficult to explain, since the orientation of the dipoles from the *I*-spins have presumably been rotated so that they no longer are aligned either with or against the external field, yet the splitting is unaffected. The result obtained is even more surprising if a delay of $\tau = 1/(2J)$ is introduced between the first pulse and the second. Now, the signal is almost completely eliminated, and the small residual peaks (due to timing errors) have the same phase as when $\tau = 0$. The vector diagrams offer no help in explaining the disappearance of the observable *S*-magnetization when a $\pi/2$ -pulse is applied to the *I*-spin just before the FID is recorded.

In experiment (c), an initial $(\pi/2)_x$ -pulse to the I -spin is added, so that the I -magnetization lies in the transverse plane during the delay period. If the delay time is zero, the second $(\pi/2)_{x,I}$ -pulse rotates the I magnetization further, to the negative z -axis, with no apparent effect on the spectrum. With the delay of $\tau = 1/(2J)$, however, the second $(\pi/2)_{x,I}$ -pulse eliminates the signal from the S -spin, just as in experiment (b). In this case, the I -spin magnetization is expected to split into two components during the delay period, so that they are pointed in opposite directions along the x' -axis at the time of the second $(\pi/2)_{x,I}$ -pulse. It thus appears that the the second $(\pi/2)_{x,I}$ -pulse should have no effect on the I -magnetization, and yet this pulse still somehow eliminates the signal from the S -spin.

These examples suggest that there are properties, distinct from the observable magnetization components, of the coupled spin-pair that can be manipulated by irradiating one of the nuclei but are manifest in the magnetization of the other. These effects form the basis for many widely used NMR experiments, but are notoriously difficult to explain. As will be demonstrated here, the underlying phenomena are quantum correlations between the nuclei in individual molecules, but predicting their effects in an NMR experiment also requires consideration of the distribution of molecules in different states. The standard tool for treating systems of this type is the density matrix, which essentially combines the quantum mechanical and statistical description of the system in a compact form (4,5). The changes in the system caused by radio-frequency (rf) pulses and evolution periods are described by changes in the density matrix, which are generated mathematically by multiplication by appropriate matrices. The observable net magnetization components from the spins are calculated from the density matrix by multiplication by operator matrices. Unfortunately, the underlying physical phenomena are not easily discerned from the matrix elements, and the matrix multiplications can quickly become unwieldy.

The density matrix calculations can be significantly simplified by representing the matrix as a linear combination of orthogonal matrices. Particularly when the nuclei of interest satisfy the weak-coupling condition, manipulations of the matrices are defined by relatively simple rules, and this treatment, commonly referred to as the “product-operator formalism”, has become a primary tool for describing and designing solution NMR experiments (6–8). Although using the product operator formalism is relatively straight forward, the physical significance of the mathematical manipulations can be elusive, leading several authors to suggest graphical representations based on vector diagrams analogous to those used to describe simpler systems (9–14). In this paper, an approach is presented that leads to a similar graphical representation, but that begins by considering the basic elements of the formalism, the operators, from the perspective of their original use in quantum-mechanical calculations. The treatment begins with calculations for idealized pure populations described by single wavefunctions, from which the special properties of coupled spins emerge, as does the physical significance of the product operators and the proposed graphical representation. This interpretation is then extended to populations composed of molecules described by different wavefunctions, as found in a real NMR experiment. The goal of this approach is to explicitly separate the quantum-mechanical and statistical aspects of the NMR experiment, which are normally merged in the density-matrix and product-operator treatments, in the hopes of clarifying the important features of both.

The treatment presented here depends on quantum-mechanical calculations. Although these calculations are not intrinsically difficult to carry out, a full derivation of each result would be quite lengthy. In order to keep the level of mathematical detail in the paper manageable, all of the calculations are presented separately in the form of an appendix prepared using the computer program *Mathematica*. The appendix can be viewed using a free reader program, or it can be used to carry out additional calculations with the full *Mathematica* program.

As this paper was being reviewed for publication, another paper with a similar theme and title appeared in this journal (15). The two papers employ different and complementary approaches, and readers are encouraged to study both.

ISOLATED SPIN-1/2 NUCLEI

Although the major focus of this paper is the system composed of a pair of scalar-coupled spin-1/2 nuclei, it is helpful to begin with a brief review of the quantum mechanics of an isolated spin. The basic mathematical elements of the quantum treatment are:

- The wavefunction, which defines the state of individual spins, but cannot, in general be determined experimentally.
- Operators, which are used to calculate the *average* outcome of an experimental measurement, such as the orientation of the magnetic dipole along a particular direction, when applied to a population with a given wavefunction.
- The Hamiltonian operator, which is used to calculate the average energy of a population of particles with a given wavefunction.
- The time-dependent Schrödinger equation, which is used together with the Hamiltonian operator to calculate the change in the wavefunction with time under different conditions.

With these elements in hand, it is possible to predict the average outcome of any experiment that begins with a population of spins with a defined wavefunction. After treating some idealized experiments with pure populations, an equilibrium mixture of spins with different wavefunctions will be considered.

Wavefunctions and operators

The classic 1922 experiment of Stern and Gerlach provides a useful starting point for a discussion of the quantum mechanics of spin, although the relationship between their observations and the concept of spin, introduced in 1926 by Goudsmit and Uhlenbeck, did not become clear for some time (16). In brief, Stern and Gerlach directed a beam of vaporized silver atoms through a non-uniform magnetic field designed so that the individual atoms would be deflected if they possessed a magnetic moment. The paths of the atoms were detected by their accumulation on a glass plate, as illustrated in Figure 2. Rather than being spread out continuously along the vertical axis of the plate, there was a clear concentration of deposited silver atoms in two spots. It is now understood that this separation was due to the magnetic dipole of the unpaired electron of each silver atom.

In the language of quantum mechanics, we say that each of the beams leaving the magnetic field contains particles described by one of two wavefunctions, written $|\alpha\rangle$ and $|\beta\rangle$ using Dirac notation. However, these are not the only possible wavefunctions for a spin-1/2 particle, as a limitless number of others can be written as linear combinations, or superpositions, of $|\alpha\rangle$ and $|\beta\rangle$. The general form of the wavefunction can be written as:

$$|\Psi\rangle = c_\alpha |\alpha\rangle + c_\beta |\beta\rangle \quad [1]$$

where c_α and c_β are complex coefficients. The wavefunction is, in general, a complex function, and its complex conjugate is written in Dirac notation as:

$$\langle\Psi| = c_\alpha^* \langle\alpha| + c_\beta^* \langle\beta| \quad [2]$$

where c_α^* and c_β^* are the complex conjugates of c_α and c_β , respectively, and $\langle\alpha|$ and $\langle\beta|$ are the complex conjugates of $|\alpha\rangle$ and $|\beta\rangle$. The wavefunctions $|\alpha\rangle$ and $|\beta\rangle$ form a basis set for all possible wavefunctions for the single spin.

The Stern-Gerlach filter represents a physical measurement, in this case the orientation of the magnetic dipoles along the vertical (z) axis. The result of the measurement is called I_z and can take on only the values of $1/2$ and $-1/2$, corresponding to upward and downward deflection by the magnetic field. The measurement is associated with an operator, $\hat{I}_z$, a mathematical object that is used to predict the *average* outcome of the measurement from the wavefunction. Operators, in general, convert one mathematical object into another; in this case the operator converts the wavefunction into another wavefunction. The average outcome of the experiment is calculated by first applying the operator to the wavefunction and then multiplying the resulting wavefunction by the complex conjugate of the wavefunction, $\langle\Psi|$. This is written in Dirac notation as:

$$\langle I_z \rangle = \langle \Psi | \hat{I}_z | \Psi \rangle \quad [3]$$

where the brackets on the left-hand side indicate an average value expected if the measurement is made on a large number of spins, all with the same wavefunction. Unlike ordinary algebra, these manipulations are carried out from right to left.

If the $\hat{I}_z$ operator is used to calculate the average I_z -magnetization for spins with the $|\alpha\rangle$ wavefunction, the result will be $1/2$, indicating that the particles are always deflected upwards. For spins with the $|\beta\rangle$ wavefunction, $\langle I_z \rangle = -1/2$, implying that I_z is always $-1/2$. The two wavefunctions $|\alpha\rangle$ and $|\beta\rangle$ are said to be eigenfunctions of the $\hat{I}_z$ operator. For spins with other wavefunctions, representing superpositions of $|\alpha\rangle$ and $|\beta\rangle$, each particle will be deflected either upward or downward, but the outcome for an individual particle cannot be predicted. Once the measurement of I_z has been made for a particle with a superposition wavefunction, the wavefunction is converted to $|\alpha\rangle$ or $|\beta\rangle$, depending on the result obtained. Thus, if the particle is deflected upwards, its wavefunction is converted to $|\alpha\rangle$, and if it is directed through a second Stern-Gerlach filter it will always be deflected upwards again. This process is described as a “collapse” of the wavefunction.

The orientation of the magnetic dipole of a spin- $1/2$ particle can, in principle, be measured along any direction. Thus, there are operators $\hat{I}_x$ and $\hat{I}_y$, which are associated with measurements along the x - and y -axes, respectively. Like I_z , the outcome of measuring I_x or I_y for a single particle will always be $1/2$ or $-1/2$. Measurements along the three axes are not independent, however, so that the overall orientation of a single spin cannot, in general, be defined. But, if the three measurements are each independently performed on a large subpopulation of particles with the same wavefunction, the average magnetization in each direction can be determined. The results of these measurements can be described by a vector, $\mathbf{I}$, defined as:

$$\mathbf{I} = \langle I_x \rangle \mathbf{X} + \langle I_y \rangle \mathbf{Y} + \langle I_z \rangle \mathbf{Z} \quad [4]$$

where $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are each vectors one unit long aligned with the corresponding Cartesian axis. The vector $\mathbf{I}$ is usually referred to, rather loosely, as either the magnetization or angular momentum vector. This leads to the familiar vector diagram that summarizes the average magnetization components for a population of spins, as illustrated in Figure 3a.

In NMR experiments, the sample is composed of a mixture of spins with different wavefunctions, which begin in an equilibrium distribution defined by the energies of interaction with the external magnetic field. Though the equilibrium mixture can contain a vast number of different superposition states and cannot be uniquely defined, it can be treated *as if* it were composed of only the $|\alpha\rangle$ and $|\beta\rangle$ states with their relative populations determined by the Boltzmann law:

$$\frac{N_\alpha}{N_\beta} = e^{(E_\beta - E_\alpha)/(kT)} \quad [5]$$

where N_α and N_β are the numbers of spins in the $|\alpha\rangle$ and $|\beta\rangle$ states; k is the Boltzmann constant; T is the temperature, and E_α and E_β are the energies of the $|\alpha\rangle$ and $|\beta\rangle$ states.

The fractional excess population in the $|\alpha\rangle$ state at thermal equilibrium can be defined as:

$$\Delta P = \frac{N_\alpha - N_\beta}{N_\alpha + N_\beta} \quad [6]$$

If the energy difference between the two states is much smaller than kT , as in conventional NMR experiments, this difference can be well approximated by:

$$\Delta P = \Delta E / (2kT) = \gamma B / (4\pi kT) \quad [7]$$

where $\Delta E = E_\beta - E_\alpha$, γ is the gyromagnetic ratio, and B is the strength of the external field. For simple systems in which the spins behave independently, *i.e.* where there is no coupling interaction among spins in the same molecule, the observable properties can be calculated by considering only the small excess of spins in the $|\alpha\rangle$ state when the experiment begins. As the wavefunctions of the spins are manipulated by rf irradiation and are allowed evolve with time, as discussed in the next section, the average magnetization components for the spins that begin in the $|\alpha\rangle$ and $|\beta\rangle$ states always move in opposite directions. As a consequence, the magnetization from the spins that begin in the $|\beta\rangle$ state always cancels out the magnetization of all but the small excess that began in the $|\alpha\rangle$ state. A similar cancellation occurs in the case of a coupled spin-pair, but, as will be shown, the result is not quite so simple.

The effects of pulses and time-evolution

The change in the properties of a quantum system are described by the change in the underlying wavefunction that determines the outcomes of possible measurements. This change is calculated using the time-dependent Schrödinger equation and the Hamiltonian operator, which is also the operator that is used to calculate the energy of a quantum system, providing a fundamental link between time and energy. The general mathematical relationships describing these effects are provided in the electronic appendix, and the treatment here is limited to a qualitative description illustrated by a few examples.

In the presence of a constant uniform magnetic field the relative contributions of the $|\alpha\rangle$ and $|\beta\rangle$ basis functions to the wavefunction remain constant, but the phases of the complex coefficients (*i.e.* the relative magnitudes of the real and imaginary parts) change. On the other hand, the fluctuating magnetic field associated with rf irradiation leads to a change in the moduli (or total magnitude) of the coefficients. As in the classical description of pulses, in which a torque acting on a spinning object is envisioned, it is convenient to shift into a rotating reference frame to describe the quantum-mechanical effects, using the primed

symbols x' - and y' - to represent the transverse axes of the rotating coordinate system. This transformation is implicit in the equations used to describe the change in wavefunction due to a pulse.

As an example, a $\pi/2$ -pulse applied to the $|\alpha\rangle$ state converts the wavefunction to:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} |\alpha\rangle + \frac{i}{\sqrt{2}} |\beta\rangle \quad [8]$$

In this wavefunction, the moduli of the two coefficients are equal, meaning that $|\alpha\rangle$ and $|\beta\rangle$ make equal contributions. If the $\hat{I}_z$ -operator is used to calculate the average expected value of the z -magnetization, the result $\langle I_z \rangle = 0$ is obtained, indicating that if the magnetization of individual spins are measured, the values $1/2$ and $-1/2$ will be observed with equal probability. The average x' -magnetization is also zero. However, the wavefunction is an eigenfunction of the $\hat{I}_y$ -operator, and each measurement of I_y will give the result $1/2$. Thus, the change in wavefunction corresponds to the rotation of the average magnetization from the z -axis to the positive y' -axis, as represented by the vector diagrams in Figure 3b. A second $(\pi/2)_x$ -pulse will change the wavefunction to $|\Psi\rangle = |\beta\rangle$, corresponding to a rotation to the negative z -axis.

In a constant magnetic field, the time-dependent change in the wavefunction can be written as:

$$|\Psi(t)\rangle = e^{-it(E_\alpha/\hbar)} c_\alpha(0) |\alpha\rangle + e^{-it(E_\beta/\hbar)} c_\beta(0) |\beta\rangle \quad [9]$$

where $\hbar$ is Planck's constant divided by 2π , and $c_\alpha(0)$ and $c_\beta(0)$ are the initial values of the complex coefficients. The energies of the $|\alpha\rangle$ and $|\beta\rangle$ states are given by:

$$E_\alpha = -\frac{\hbar}{2}\gamma B \quad \text{and} \quad E_\beta = \frac{\hbar}{2}\gamma B \quad [10]$$

From the expression for the wavefunction and the operators described in the previous section, the average magnetization components, $\langle I_x \rangle$, $\langle I_y \rangle$ and $\langle I_z \rangle$, can be readily predicted as a function of time. Taking as the starting state the wavefunction generated by application of a $(\pi/2)_x$ -pulse to the $|\alpha\rangle$ wavefunction, the time-dependent wavefunction is:

$$|\Psi(t)\rangle = e^{-it(E_\alpha/\hbar)} \frac{1}{\sqrt{2}} |\alpha\rangle + e^{-it(E_\beta/\hbar)} \frac{i}{\sqrt{2}} |\beta\rangle \quad [11]$$

Using the $\hat{I}_z$ operator to calculate the average I_z -magnetization gives the following result:

$$\langle I_z(t) \rangle = 0 \quad [12]$$

Thus, the z -magnetization does not change with time. This result is quite general; for a population of spins with any wavefunction in a constant magnetic field, the average z -magnetization remains constant with time (in the absence of relaxation).

For the x - and y -magnetization, the following relationships can be derived:

$$\begin{aligned}\langle I_x(t) \rangle &= \frac{1}{2} \sin(t(E_\beta - E_\alpha)/\hbar) \\ \langle I_y(t) \rangle &= \frac{1}{2} \cos(t(E_\beta - E_\alpha)/\hbar)\end{aligned}\quad [13]$$

These equations describe a cyclic motion of the average magnetization vector in the transverse plane, as illustrated in Figure 3c. This motion can be detected by the induction of an electric current in the spectrometer coils, and Fourier transformation of this signal gives rise to the spectrum. The time required for a complete cycle of the magnetization is calculated as:

$$\begin{aligned}t_{\text{cycle}} &= \frac{2\pi\hbar}{E_\beta - E_\alpha} \\ &= \frac{2\pi}{\gamma B}\end{aligned}\quad [14]$$

and the frequency of the cycle is given by the reciprocal of this time, to give the familiar Larmor equation:

$$\nu = \frac{\gamma B}{2\pi} \quad [15]$$

This picture is derived from a purely quantum-mechanical treatment, but is entirely consistent with the classical picture of precession due to a torque acting on a spinning object.

If we begin with the equilibrium state, with a small excess of spins in the $|\alpha\rangle$ state relative to the $|\beta\rangle$ state, the pulse rotates the magnetization for both populations in the same direction, resulting in a small excess of magnetization along the y' -axis. After the pulse, the magnetization vectors will precess in the transverse plane, as described above, but the contributions from the two populations will always point in opposite directions, so that only the original excess is detectable. In some descriptions of NMR, it is implied that only population differences are affected by pulses. It is probably better to say that all of the spins are affected, but, for simple systems, the pulses maintain the original differences in magnetization, as does evolution in a constant magnetic field (in the absence of relaxation).

WAVEFUNCTIONS AND OPERATORS FOR A COUPLED SPIN-PAIR

Having laid out the basic elements of a quantum-mechanical treatment of NMR, the case of two spin-1/2 nuclei that interact by scalar coupling can be considered. While two basis wavefunctions, $|\alpha\rangle$ and $|\beta\rangle$, were sufficient to form all of the possible wavefunctions for a single spin, four are now required for the coupled spin-pair. For this treatment, the most convenient basis set is composed of the four eigenfunctions for the measurements of the z -magnetization components of the two spins. These are written as $|\alpha\alpha\rangle$, $|\alpha\beta\rangle$, $|\beta\alpha\rangle$ and $|\beta\beta\rangle$. The Greek symbols represent the z -magnetization of the individual spins, with α , as before, indicating 1/2 and β indicating -1/2. The general form of the wavefunction is written as:

$$|\Psi\rangle = c_{\alpha\alpha} |\alpha\alpha\rangle + c_{\alpha\beta} |\alpha\beta\rangle + c_{\beta\alpha} |\beta\alpha\rangle + c_{\beta\beta} |\beta\beta\rangle \quad [16]$$

where $c_{\alpha\alpha}$, $c_{\alpha\beta}$, $c_{\beta\alpha}$ and $c_{\beta\beta}$ are complex coefficients.

The energetic effect of the coupling between the two spins is usually expressed in terms of a frequency, J , so that the interaction energy is given by $E_j = hJ$. If, in the presence of a given

stationary magnetic field, J is much smaller than the difference between the Larmor frequencies of the two nuclei, then the system is said to satisfy the “weak-coupling” condition. When this condition is satisfied, the eigenfunctions for the z -magnetization operators ($|\alpha\alpha\rangle$, $|\alpha\beta\rangle$, $|\beta\alpha\rangle$ and $|\beta\beta\rangle$) are also the eigenfunctions for the Hamiltonian, meaning that they have defined energies and do not change with time in a stationary field. The energies of the eigenstates are:

$$\begin{aligned} E_{\alpha\alpha} &= -h(\nu_I + \nu_S)/2 + hJ/4 \\ E_{\alpha\beta} &= -h(\nu_I - \nu_S)/2 - hJ/4 \\ E_{\beta\alpha} &= h(\nu_I - \nu_S)/2 - hJ/4 \\ E_{\beta\beta} &= h(\nu_I + \nu_S)/2 + hJ/4 \end{aligned} \quad [17]$$

where ν_I and ν_S are the intrinsic Larmor frequencies of the uncoupled spins. In the absence of scalar coupling, the energies of the two spins would be independent of one another. When $J > 0$, however, the energy difference between the α and β states for each spin is greater when the other spin is in the β state. This gives rise to four frequencies in a simple NMR spectrum, *i.e.* the familiar “splitting” of peaks.

Even if the weak-coupling condition is not satisfied, $|\alpha\alpha\rangle$, $|\alpha\beta\rangle$, $|\beta\alpha\rangle$ and $|\beta\beta\rangle$, are still eigenfunctions of the z -magnetization operators, discussed below, and $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$ are both eigenfunctions of the Hamiltonian. Under strong-coupling conditions, however, the other two eigenfunctions of the Hamiltonian are linear superpositions of $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$. The distinction between weak and strong coupling is expressed in the frequencies and intensities of the observable transitions in NMR experiments and in the time-dependent evolution of the magnetization components. Because the treatment of strong coupling is considerably more complex than for the weak case, the product operator formalism is usually applied only to the latter, a simplification that will be assumed throughout this paper. Fortunately, the weak-coupling condition is always satisfied for heteronuclear pairs and nearly always is for homonuclear pairs in the high magnetic fields used in modern spectrometers.

Magnetization operators

The z -magnetization components for two coupled nuclei in a volatile molecule could, in principle, be measured using a Stern-Gerlach filter, as illustrated in Figure 4a for the case where the two spins are of the same type. Because the two spins have the same gyromagnetic ratio, the net z -magnetization for the pairs in the $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$ states is zero, and these pairs form a single beam that is not deflected by the magnet. If a molecule in which the nuclei have different gyromagnetic ratios is used, then the $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$ molecules will be deflected differently by the magnet, as shown in Figure 4b for the case where γ_I is about twice as large as γ_S . In this case, it is possible to simultaneously determine the z -magnetization component of each of the spins. Though it isn't so obvious how to do it with a Stern-Gerlach type experiment, any one of the three orthogonal magnetization components for one spin can, in principle, be determined independently of one of the components for the other spin. (It should be noted that the expected behavior in the Stern-Gerlach filter does not depend on whether or not the weak-coupling condition is satisfied, since the filter measures the z -magnetization components, for which $|\alpha\alpha\rangle$, $|\alpha\beta\rangle$, $|\beta\alpha\rangle$ and $|\beta\beta\rangle$ are eigenfunctions.)

The ability to measure the magnetization components of the individual spins implies that there are three independent magnetization operators for each spin: $\hat{I}_x$, $\hat{I}_y$ and $\hat{I}_z$; and $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$. As for the single-spin case, the operators represent rules for the conversion of the wavefunction of interest into another wavefunction. Each operator has eigenfunctions, which represent quantum states for which the corresponding measurement gives a predictable value, either $1/2$ or $-1/2$. For $\hat{I}_z$ and $\hat{S}_z$, these are the four basis functions, which

are eigenfunctions for both. The average outcome for each measurement can be calculated from the wavefunction using the appropriate operator, just as for the one-spin case. Measuring the z -magnetization components with a Stern-Gerlach filter collapses the wavefunction of each of the spin-pairs into one of the eigenfunctions.

To illustrate the relationships between the wavefunction and magnetization components, and how both change with time, some specific wavefunctions will be used as examples. The first of these are two of the basis functions, $|\alpha\alpha\rangle$ and $|\beta\alpha\rangle$. Applying the magnetization operators to $|\alpha\alpha\rangle$ gives the following results:

$$\begin{aligned}\langle I_x \rangle &= 0 & \langle I_y \rangle &= 0 & \langle I_z \rangle &= 1/2 \\ \langle S_x \rangle &= 0 & \langle S_y \rangle &= 0 & \langle S_z \rangle &= 1/2\end{aligned}$$

These results are consistent with the definition of $|\alpha\alpha\rangle$ as the eigenfunction of the I_z and S_z operators associated with positive z -magnetization. As with the case of the isolated spin, the average magnetization components that are equal to zero imply that the two possible outcomes, $1/2$ and $-1/2$, are expected with equal probability.

If, starting with the $|\alpha\alpha\rangle$ wavefunction, a selective $\pi/2$ -pulse along the x' -axis is applied to the S -spin, the following wavefunction, labeled $|\Psi_A\rangle$, is generated:

$$|\Psi_A\rangle = \frac{1}{\sqrt{2}} |\alpha\alpha\rangle + \frac{i}{\sqrt{2}} |\alpha\beta\rangle \quad [18]$$

When the magnetization operators are used to calculate the average magnetization components for a population of spins with the wavefunction $|\Psi_A\rangle$, the following results are obtained:

$$\begin{aligned}\langle I_x \rangle &= 0 & \langle I_y \rangle &= 0 & \langle I_z \rangle &= 1/2 \\ \langle S_x \rangle &= 0 & \langle S_y \rangle &= 1/2 & \langle S_z \rangle &= 0\end{aligned}$$

These values represent the rotation of the S -magnetization from the z -axis to the y' -axis, as illustrated in Figure 5a.

For $|\beta\alpha\rangle$ the only non-zero magnetization components are $\langle I_z \rangle = -1/2$ and $\langle S_z \rangle = 1/2$. A selective $(\pi/2)_{x'}$ -pulse applied to the S -spin of $|\beta\alpha\rangle$ generates the following wavefunction:

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}} |\beta\alpha\rangle + \frac{i}{\sqrt{2}} |\beta\beta\rangle \quad [19]$$

Again, the S -magnetization is rotated to the positive y' -axis, while the I -magnetization is unaffected, in this case remaining aligned with the negative z -axis, as shown in Fig 5b. If a $\pi/2$ -pulse is applied along the x' -axis to both the I - and S -spin of the $|\alpha\alpha\rangle$ wavefunction, the following wavefunction is generated:

$$|\Psi_C\rangle = \frac{1}{2} |\alpha\alpha\rangle + \frac{i}{2} |\alpha\beta\rangle + \frac{i}{2} |\beta\alpha\rangle - \frac{1}{2} |\beta\beta\rangle \quad [20]$$

In this case all four of the basis functions are represented in the superposition state. The corresponding magnetization vectors are both aligned with the y' -axis (Figure 5c).

Lest all of this appear too obvious, consider now the wavefunction:

$$|\Psi_D\rangle = \frac{1}{\sqrt{2}} |\alpha\beta\rangle + \frac{1}{\sqrt{2}} |\beta\alpha\rangle \quad [21]$$

Applying the individual magnetization operators to this wavefunction gives the following results:

$$\begin{aligned} \langle I_x \rangle &= 0 & \langle I_y \rangle &= 0 & \langle I_z \rangle &= 0 \\ \langle S_x \rangle &= 0 & \langle S_y \rangle &= 0 & \langle S_z \rangle &= 0 \end{aligned}$$

This is quite odd! It appears as though neither of the spins points in any direction. If a Stern-Gerlach filter were used to measure the magnetization in the z -direction for the I -spin, for instance, either $1/2$ or $-1/2$ would be observed with equal probability. Similarly, each of the other possible measurements will give the two possible outcomes with equal probability. It is important to emphasize that this result does not represent a canceling out of contributions from different spins in a heterogeneous population, such as generated by saturation in a bulk NMR experiment. Instead, this result is obtained for a pure population of spin-pairs, all with the same wavefunction. An equivalent result *cannot* be obtained from a population of single spins. The properties of the $|\Psi_D\rangle$ wavefunction will be considered further in the next section.

Product operators and correlations

There are special properties of a coupled spin-pair that can be revealed using operators that are generated by multiplying a magnetization operator for one of the spins by an operator for the other, for instance the product $\hat{I}_x \hat{S}_z$. The expression $\hat{I}_x \hat{S}_z |\Psi\rangle$ represents the result of applying the $\hat{I}_x$ operator to the result of applying $\hat{S}_z$ to the wavefunction $|\Psi\rangle$. Multiplication of quantum-mechanical operators does not, in general, satisfy the commutative property of ordinary algebra. For instance, the expression $\hat{I}_x \hat{I}_y |\Psi\rangle$ is not equal to $\hat{I}_y \hat{I}_x |\Psi\rangle$ (except for special cases of $|\Psi\rangle$). However, because the magnetization components of the I - and S -spins can be measured independently, their operators do commute with one another, so that $\hat{I}_x \hat{S}_y |\Psi\rangle$ and $\hat{S}_y \hat{I}_x |\Psi\rangle$ are equivalent. Operators that are generated in this way are referred to as “product operators”

Application of the product operators to a wavefunction corresponds to the physical process of making the two indicated measurements and then multiplying the results. For a single spin-pair, the results of each independent measurement will be either $1/2$ or $-1/2$, so that the product of the two measurements must be either $1/4$ or $-1/4$. Just as for the magnetization operators, the product operators can be used to calculate the average of the products for a population of spin-pairs described by a wavefunction. For instance the average value of the product $I_x S_y$ is calculated as:

$$\langle I_x S_y \rangle = \langle \Psi | \hat{I}_x \hat{S}_y | \Psi \rangle \quad [22]$$

If this procedure is applied to the wavefunction $|\alpha\alpha\rangle$, the following values are obtained for the nine products:

$$\begin{aligned}\langle I_x S_x \rangle &= 0 & \langle I_y S_x \rangle &= 0 & \langle I_z S_x \rangle &= 0 \\ \langle I_x S_y \rangle &= 0 & \langle I_y S_y \rangle &= 0 & \langle I_z S_y \rangle &= 0 \\ \langle I_x S_z \rangle &= 0 & \langle I_y S_z \rangle &= 0 & \langle I_z S_z \rangle &= 1/4\end{aligned}$$

This result is not so surprising, since it is consistent with the definition of $|\alpha\alpha\rangle$ as the wavefunction that describes a spin-pair with both spins aligned with the positive z -axis. For this wavefunction, I_z and S_z are both always $1/2$, so that $I_z S_z$ must be $1/4$. On the other hand, for the I_x and S_x measurements, the outcomes $1/2$ or $-1/2$ are expected with equal probability for each, and the product $I_x S_x$ has equal probability of being $1/4$ or $-1/4$. Thus, the average of $I_x S_x$ is 0.

For the superposition wavefunction, $|\Psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\alpha\rangle + \frac{i}{\sqrt{2}}|\alpha\beta\rangle$, the only non-zero product average is $\langle I_z S_y \rangle = 1/4$. Again, this is entirely consistent with the orientations of individual magnetization components; I along the z -axis and S along the y' -axis. There does not seem to be any additional information provided by using the product operators.

Consider now, however, the wavefunction $|\Psi_D\rangle = \frac{1}{\sqrt{2}}|\alpha\beta\rangle + \frac{1}{\sqrt{2}}|\beta\alpha\rangle$. As shown earlier, all six of the average magnetization components are zero for this wavefunction. It might be anticipated that the average products would also be zero. However, applying the product operators to this wavefunction gives:

$$\begin{aligned}\langle I_x S_x \rangle &= 1/4 & \langle I_y S_x \rangle &= 0 & \langle I_z S_x \rangle &= 0 \\ \langle I_x S_y \rangle &= 0 & \langle I_y S_y \rangle &= 1/4 & \langle I_z S_y \rangle &= 0 \\ \langle I_x S_z \rangle &= 0 & \langle I_y S_z \rangle &= 0 & \langle I_z S_z \rangle &= -1/4\end{aligned}$$

This result shows that the wavefunction is an eigenfunction for three product operators, $\hat{I}_x \hat{S}_x$, $\hat{I}_y \hat{S}_y$ and $\hat{I}_z \hat{S}_z$. As a consequence, if $\hat{I}_x$ and $\hat{S}_x$ are independently measured for individual molecules, the product of the measurements will always be $1/4$. But, we also know that the average values of I_x and S_x are zero, so that the two possible outcomes of the individual measurements, $1/2$ and $-1/2$, have equal probabilities. The product operator result implies, then, that whenever the result of measuring I_x for a pair is $1/2$, the value of S_x must also be $1/2$, and whenever I_x is $-1/2$, S_x must be $-1/2$. An outcome where $I_x = 1/2$ and $S_x = -1/2$ or where $I_x = -1/2$ and $S_x = 1/2$ will never be seen. Similar relationships hold for I_y and S_y . On the other hand, the result $\langle I_z S_z \rangle = -1/4$ implies that whenever $I_z = 1/2$, $S_z = -1/2$, and vice versa.

It may help in thinking about this to imagine the outcome of a hypothetical Stern-Gerlach experiment applied to a homogeneous population of molecules described by $|\Psi_D\rangle$, as shown in Figure 4c. In this case, only two outcomes of the simultaneous measurement of I_z and S_z are seen, corresponding to $I_z = 1/2$ and $S_z = -1/2$ or $I_z = -1/2$ and $S_z = 1/2$. The outer two beams, corresponding to the two states in which the spins are oriented in the same direction, are now absent. Pairs of spins that are related in this way are said to be *entangled*, and the defining character of such states is that the individual measurements are correlated, even when the outcome of either measurement is unpredictable. The study of entangled pairs, in both real and *gedanken* experiments, is an important thread in the history of quantum mechanics and has helped illuminate some of its most mysterious aspects (17–21).

These examples show that the physical significance of the product operators lies in the correlation between the magnetization components of two coupled spins. In all that follows, the term “correlation” is used to refer to non-zero values of the calculated magnetization products, *i.e.* the results of calculations such as that indicated by Equation [22]. In some cases, a correlation may be associated with non-zero average magnetization, but it need not be. Although all of the magnetization components may be zero, as shown by the example of $|\Psi_D\rangle$, it can be shown that there is no wavefunction for which all of the average correlations are zero. Thus, the correlations are a fundamental feature of two coupled spins, and a full description of the state of a spin-pair requires that the correlations, as well as the average magnetization components be defined.

It is important to emphasize that the correlations are properties that are defined by the wavefunctions of individual spin-pairs, but they can only be detected by making measurements of many particles. In NMR spectroscopy, we must deal with samples made up of spin-pairs with different wavefunctions. Just as the individual magnetization components of individual spins in a heterogeneous population can give rise to a net magnetization for the sample, under some circumstances correlations between spin-pairs in a population can add together so that one or more of the average products takes on a non-zero value. The average correlations cannot be detected directly in an NMR signal, but they can be manipulated in ways that affect the subsequent evolution of net magnetization. Indeed, these effects form the basis of most multi-dimensional NMR experiments involving scalar coupling interactions.

A graphical representation of the average magnetization products

Like the observable magnetization components, the average correlations calculated from the product operators evolve with time in the presence of a stationary magnetic field and are altered by rf pulses. With care, a special form of vector diagram may be introduced to illustrate the correlations and the changes they undergo. As an example, consider the wavefunction $|\Psi_A\rangle$, which is associated with the non-zero magnetization components $\langle I_z \rangle = 1/2$ and $\langle S_y \rangle = 1/2$, as well as the correlation $\langle I_z S_y \rangle = 1/4$. The magnetization components can each be represented as simple vectors, as illustrated in Figure 6a. Although the two vectors have a common origin, no correlation between the spins should be implied by these vectors alone.

The correlation associated with the wavefunction $|\Psi_A\rangle$, $\langle I_z S_y \rangle = 1/4$, is illustrated by a separate diagram, in Figure 6b, which consists of two pairs of vectors. The diagram is meant to indicate that, for a single spin-pair, $I_z = 1/2$ is always correlated with $S_y = 1/2$, and $I_z = -1/2$ is correlated with $S_y = -1/2$. Although the positive value of $\langle S_z I_y \rangle$ could, in principle, be represented by a single vector pair, such as used in panel (a), the use of two pairs helps distinguish the diagrams for correlations from those representing the observable magnetization components. In addition, because the vectors in panel (b) cancel one another, the diagram helps convey the important idea that the correlation, by itself, does not imply a net magnetization in any direction.

For the wavefunction described by $|\Psi_D\rangle$, all of the average magnetization components are zero, but there are three non-zero average operator products: $\langle I_x S_x \rangle = 1/4$, $\langle I_y S_y \rangle = 1/4$ and $\langle I_z S_z \rangle = -1/4$. Diagrams representing these correlations are shown in Figure 7. In the diagram representing $\langle I_x S_x \rangle = 1/4$, the positive I_x -component is linked to the positive S_x -components, and the two negative components are linked together. A similar diagram is drawn to represent $\langle I_y S_y \rangle = 1/4$. For the negative correlation, $\langle I_z S_z \rangle = -1/4$, each positive component is linked with a negative component, with the links crossing one another at the origin. As in the previous figure, each correlation is represented by two vector pairs, and the correlations do not imply a net magnetization from either spin.

The vector diagrams proposed here are similar to those previously introduced by others (9,10,13), in that they utilize pairs of arrows to represent the product operators. A major difference, however, is that the diagrams such as shown in Figs. 6 and 7 are designed to make a clear distinction between observable magnetization components, which are represented by single arrows for each of the spins (or as split vectors, discussed below), and correlations between the two spins, which are each represented by two pairs of linked arrows drawn to indicate that no net magnetization is implied. In this regard, the vectors used here more closely resemble those proposed by Wei and Snider (12), but may be somewhat easier to draw and visualize.

The diagrams illustrating correlations, together with those representing the observable magnetization components, can be used to illustrate the effects of any combination of rf pulses and time evolution, as discussed below.

PULSES AND TIME-EVOLUTION FOR A WEAKLY-COUPLED SPIN-PAIR

Pulses

A few examples of the effects of pulses on the wavefunction and observable magnetization components have already been introduced and illustrated in Figure 5. In general, pulses change the wavefunction by altering the relative contributions of the four basis functions ($|\alpha\alpha\rangle$, $|\alpha\beta\rangle$, $|\beta\alpha\rangle$ and $|\beta\beta\rangle$), and the equations for calculating these changes are provided in the *Mathematica* supplement. In addition to the expected rotations of the magnetization components, pulses can also change the correlations between spins, even if these correlations are not associated with observable net magnetization, and these effects can be described using the vector diagrams introduced in the previous section.

For instance, starting with the $|\alpha\alpha\rangle$ state, a selective $(\pi/2)_x$ pulse to the S -spin rotates the S -magnetization to the y' -axis, and the correlation $\langle I_z S_z \rangle = 1/4$ is converted into $\langle I_z S_y \rangle = 1/4$. This is represented by a rotation of the S -vectors in the correlation diagram, as shown in Figure 8. If this pulse is immediately followed by a $(\pi/2)_x$ pulse to the I -spin, its magnetization is also rotated to the y' -axis, and the correlation is now $\langle I_y S_y \rangle = 1/4$.

Even if there is no average observable magnetization, as seen with the wavefunction $|\Psi_D\rangle$, the components of the correlations will be rotated by pulses. These rotations will not, in themselves, generate observable magnetization, but they will affect the subsequent evolution of correlations and magnetization.

Time evolution in a constant magnetic field

The time-dependent evolution of the wavefunction is determined by the Hamiltonian operator and the energies of its eigen functions. As discussed earlier, the situation is simplified considerably when the weak-coupling condition is satisfied, *i.e.* when $J \ll |\nu_I - \nu_S|$. Under these conditions, the eigen functions for the Hamiltonian are $|\alpha\alpha\rangle$, $|\alpha\beta\rangle$, $|\beta\alpha\rangle$ and $|\beta\beta\rangle$, with energies as given in Equation [17]. If the starting wavefunction is written as:

$$|\Psi(0)\rangle = c_{\alpha\alpha}(0) |\alpha\alpha\rangle + c_{\alpha\beta}(0) |\alpha\beta\rangle + c_{\beta\alpha}(0) |\beta\alpha\rangle + c_{\beta\beta}(0) |\beta\beta\rangle \quad [23]$$

where $c_{\alpha\alpha}(0)$, $c_{\alpha\beta}(0)$, $c_{\beta\alpha}(0)$ and $c_{\beta\beta}(0)$ are the coefficients at time 0, then at time t , the wavefunction will be:

$$|\Psi(t)\rangle = e^{(-itE_{\alpha\alpha}/\hbar)} c_{\alpha\alpha}(0) |\alpha\alpha\rangle + e^{(-itE_{\alpha\beta}/\hbar)} c_{\alpha\beta}(0) |\alpha\beta\rangle + e^{(-itE_{\beta\alpha}/\hbar)} c_{\beta\alpha}(0) |\beta\alpha\rangle + e^{(-itE_{\beta\beta}/\hbar)} c_{\beta\beta}(0) |\beta\beta\rangle \quad [24]$$

From the wavefunction, the average values of all of the magnetization components and the average correlations can be calculated as a function of time, as illustrated below for a few examples.

Evolution of observable magnetization

As a first example, consider the wavefunction $|\Psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\alpha\rangle + \frac{i}{\sqrt{2}}|\alpha\beta\rangle$. For this wavefunction, the I -magnetization is initially aligned with the z -axis, and the S -magnetization is aligned with the y' -axis, as shown in Figure 9a. The wavefunction changes with time according to:

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{(-itE_{\alpha\alpha}/\hbar)}|\alpha\alpha\rangle + \frac{i}{\sqrt{2}}e^{(-itE_{\alpha\beta}/\hbar)}|\alpha\beta\rangle \quad [25]$$

If the appropriate operators are used to calculate the average I -magnetization in the three directions, it is found that they do not change with time and remain $\langle I_x \rangle = 0$, $\langle I_y \rangle = 0$ and $\langle I_z \rangle = 1/2$. In addition, the mean S_z magnetization remains constant, just as for a single spin. But, the magnetization components of the S -spin along the x' - and y' -axes change with time according to:

$$\begin{aligned} \langle S_x(t) \rangle &= (1/2) \sin(2\pi(\nu_S - J/2)t) \\ \langle S_y(t) \rangle &= (1/2) \cos(2\pi(\nu_S - J/2)t) \end{aligned} \quad [26]$$

As seen with the single-spin, functions of this form describe the precession of the S magnetization in the x - y plane, as shown Figure 9a. As before, separate arrows are used to represent the net magnetization of the I and S spins. Note that the frequency of precession is $\nu_{S,1} = (\nu_S - J/2)$, corresponding to the α - β transition of the S -spin with the I -spin in the α state.

Consider next the wavefunction $|\Psi_B\rangle = \frac{1}{\sqrt{2}}|\beta\alpha\rangle + \frac{i}{\sqrt{2}}|\beta\beta\rangle$, for which the initial S -magnetization is again aligned with the positive y' -axis, but the I -magnetization is aligned with the negative z -axis. For this wavefunction, the same pattern of changes in the average magnetization is predicted, except that the frequency is $\nu_{S,2} = \nu_S + J/2$, as illustrated in Figure 9b. This and the previous result are consistent with a simple picture in which the precession frequency of one spin is determined by the alignment of the other with the external field.

But what if neither spin is aligned with the z -axis? To address this question, consider the

wavefunction $|\Psi_C\rangle = \frac{1}{2}|\alpha\alpha\rangle + \frac{i}{2}|\alpha\beta\rangle + \frac{i}{2}|\beta\alpha\rangle - \frac{1}{2}|\beta\beta\rangle$, which is generated by applying a non-selective $(\pi/2)_x$ -pulse to the $|\alpha\alpha\rangle$ state. As shown in Figure 10, the average magnetization vectors for both the I - and S -spins are initially aligned with the positive y -axis. With time, the S -magnetization components change according to:

$$\begin{aligned} \langle S_x(t) \rangle &= (1/4) (\sin(2\pi(\nu_S - J/2)t) + \sin(2\pi(\nu_S + J/2)t)) \\ \langle S_y(t) \rangle &= (1/4) (\cos(2\pi(\nu_S - J/2)t) + \cos(2\pi(\nu_S + J/2)t)) \end{aligned} \quad [27]$$

There are now two frequency terms in the expressions for $\langle S_x(t) \rangle$ and $\langle S_y(t) \rangle$. Together, the two expressions can be thought of as describing the precession of two magnetization vectors in the transverse plane, as illustrated of Figure 10. There are also two frequency components for the evolution of the I -magnetization, which, for clarity, are shown on a separate set of coordinate axes in the figure. This drawing is somewhat misleading, however, because it suggests that there are two populations of S -spins (and two populations of I -spins) with different precession frequencies. This is not the case: The magnetization is coming from a single population of spin-pairs, all with the same wavefunction. For each spin, however, the experimentally-determined average magnetization evolves as if there were two small magnets revolving about the z -axis with different frequencies. When detected by the spectrometer coil and Fourier transformed, the signal will give rise to two peaks for each spin. For the S -spin, these will represent the same two frequencies as arose from the separate populations after a selective S -pulse.

The generation of two peaks for the S -spin (or the I -spin) from a population with a single wavefunction is a quite surprising result that is not at all obvious from a classical picture. Since there is no net magnetization from the I -spin along the z -axis, it might have been expected that the S -magnetization would precess at a single frequency, as if the I -spin were not there at all. But, the I -spin is still there, though in a superposition state. If the z -magnetization of single I -spins from this population could be measured, the results $1/2$ and $-1/2$ would be obtained with equal probabilities. When we ask what the precession frequency of the S -spin is, we also are indirectly asking what the z -magnetization of the I -spin is, and, as with other quantum measurements, the two quantized answers are obtained.

Evolution of correlations

The average correlations described with the product operators also evolve with time.

Consider again the wavefunction $|\Psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\alpha\rangle + \frac{i}{\sqrt{2}}|\beta\alpha\rangle$, for which the average magnetization components and the correlations are diagrammed in Figure 6. As shown earlier, the S -magnetization precesses with the frequency $\nu_{S,1} = (\nu_S - J/2)$. At the same time, the initial average correlation, $\langle I_z S_y \rangle = 1/4$, also evolves, with the same frequency, into a mixture of two correlations:

$$\begin{aligned} \langle I_z S_x(t) \rangle &= (1/4) \sin(2\pi(\nu_S - J/2)t) \\ \langle I_z S_y(t) \rangle &= (1/4) \cos(2\pi(\nu_S - J/2)t) \end{aligned} \quad [28]$$

This change in correlation can be readily accounted for by the precession of the S -magnetization from the y' -axis towards the x' -axis. Initially, $\langle I_z \rangle = 1/2$ and $\langle S_y \rangle = 1/2$, so that, for each spin-pair, the product $I_z S_y$ is always equal to $1/4$. As the S -spin precesses, the probability that $S_y = 1/2$ decreases, so that the average value of $I_z S_y$ also decreases. When the S -magnetization has precessed to the x' -axis, the two possible outcomes of measuring the S_y -magnetization have equal probability, so that there is no correlation with the I_z -magnetization. But, the probability that $S_x = 1/2$ is now 1, so that $I_z S_x = 1/4$ for each spin-pair. This cycle repeats as the I -spin continues to precess.

A somewhat more subtle relationship between the observable magnetization and correlations emerges from a situation in which both magnetization components lie in the transverse plane. This is illustrated by the case in which the starting positions of both the I - and S -magnetization vectors are aligned along the y' -axis, *i.e.* the state described by $|\Psi_C\rangle$. As discussed above, the magnetization components associated with the two spins each evolves with two frequencies, even though there is only a single wavefunction. To simplify the

resulting analysis and diagrams, it is convenient to set the average frequencies of both spins to zero. This is equivalent to using a separate rotating frame for each spin and setting the frequencies of these frames to those of the corresponding spins (in the absence of coupling). With this simplification, the evolution of the magnetization components becomes:

$$\begin{aligned}\langle I_y(t) \rangle &= (1/2) \cos(\pi Jt) \\ \langle S_y(t) \rangle &= (1/2) \cos(\pi Jt)\end{aligned}\quad [29]$$

With v_I and v_S set to zero, the evolution of the magnetization depends only on J , and the only net magnetization is along the y' -axis. This can be visualized as the precession of two vectors (for each spin) in opposite directions, as illustrated in Figure 11. The x' -components of the two vectors always cancel one another, and the y' -components cancel one another when $t = 1/(2J)$, so that the net magnetization disappears.

During this evolution period, the non-zero product operator averages are $\langle I_x S_z \rangle$, $\langle I_y S_y \rangle$ and $\langle I_z S_x \rangle$, which change with time according to:

$$\begin{aligned}\langle I_x S_z(t) \rangle &= - (1/4) \sin(\pi Jt) \\ \langle I_y S_y(t) \rangle &= 1/4 \\ \langle I_z S_x(t) \rangle &= - (1/4) \sin(\pi Jt)\end{aligned}\quad [30]$$

Note first that the correlation that is present initially, $\langle I_y S_y \rangle = 1/4$, remains constant with time, while two new correlations appear and change cyclically. Also note that the cyclic patterns of $\langle I_x S_z \rangle$ and $\langle I_z S_x \rangle$ have the same frequency as the cycles for observable y' -magnetization for each spin, but are $\pi/2$ radians out of phase with the magnetization cycles.

The cycles for the observable magnetization and the correlations can be viewed as two manifestations of the same phenomenon, as illustrated in Figure 12a using vector diagrams for the S_y -magnetization and the $I_z S_x$ correlation. In the leftmost part of the figure, a new variation on the vector diagrams is introduced to represent the initial S_y -magnetization. This diagram resembles those used to represent correlations, with two vector pairs showing a coupling between the I - and S -spins. Now, however, the two S -vectors point in the same direction, along the y' -axis, indicating the presence of a net S -magnetization, while the vectors for the I -spin point in opposite directions along the z -axis. Importantly, this diagram is designed to represent the *absence* of a correlation between I_z and S_y components, rather than its presence, since positive S_y -magnetization is associated equally with positive and negative I_z -magnetization. The two S_z -vectors in the drawing can be thought of as representing a *potential* correlation, which emerges as the two vectors associated with the S -magnetization precess in opposite directions.

As the two S -vectors precess away from the y' -axis, the x' -magnetization component of each is correlated with an I_z component of opposite sign. When $t = 1/(2J)$, the S -magnetization disappears, and the negative correlation between S_x and I_z reaches a maximum. With further evolution, the $I_z S_x$ -correlation is converted back to a net S -magnetization. A similar set of diagrams can be drawn to represent the conversion of net I_y -magnetization into a correlation between I_x and S_z .

If the average precession frequency, v_S , does not match the reference frequency, a more complicated pattern of magnetization components and correlations evolves from the S_y -magnetization, as illustrated by the vector diagrams in Figure 12b. In this case, the two vectors representing the S -components precess at different rates relative to the coordinate

system, giving rise to a mixture of S_y - and S_x -magnetization, as well as both $I_z S_x$ - and $I_z S_y$ -correlations. As in the simpler case, however, there is an interconversion between observable magnetization and a net correlation. After a period of $t = 1/(2J)$, the two vectors representing the S -nucleus point in opposite directions, indicating the absence of net observable magnetization and the maximum correlation between I_z -magnetization and transverse S -magnetization. At this point, both $\langle I_z S_x \rangle$ and $\langle I_z S_y \rangle$ are, in general, non-zero, with the relative values of the two determined by the precession frequencies. With additional time, the correlations are converted back into observable magnetization.

What about the $I_y S_y$ -correlation that was initially associated with the I_y - and S_y -magnetization? It may seem surprising that Equation [30] indicates that $\langle I_y S_y \rangle$ remains constant with time, but this reflects the choice to set the rotating frame frequencies to the average frequencies of the I - and S -spins. If ν_I and ν_S are not constrained to be zero, $I_y S_y$ is found to interconvert with three other correlations as follows:

$$\begin{aligned}\langle I_x S_x(t) \rangle &= (1/4) \sin(2\pi\nu_I t) \sin(2\pi\nu_S t) \\ \langle I_x S_y(t) \rangle &= (1/4) \sin(2\pi\nu_I t) \cos(2\pi\nu_S t) \\ \langle I_y S_x(t) \rangle &= (1/4) \cos(2\pi\nu_I t) \sin(2\pi\nu_S t) \\ \langle I_y S_y(t) \rangle &= (1/4) \cos(2\pi\nu_I t) \cos(2\pi\nu_S t)\end{aligned}\quad [31]$$

These equations can be visualized as arising from the precession of independent I - and S -components of the initial $I_y S_y$ -correlation, as illustrated in Figure 13. Each expression represents the product of the projections of the two precessing vectors onto the indicated axes. Initially, the I - and S components are both aligned with the positive (or negative) y' -axis, and the $I_y S_y$ -correlation is maximal. As both vectors precess towards the x' -axis, the projections onto the y' -axis decrease, and the average $I_x S_x$ -correlation increases. The evolution of each correlation depends on the precession frequency of both spins. However, these expressions do not depend on the scalar coupling constant, J , because there is no z -component in any of the correlations. An important consequence is that these correlations do not interconvert with observable magnetization, since the individual I -components, for instance, always precess at the same rate and balance one another, never combining to generate a net magnetization component.

Although the $I_x S_x$, $I_x S_y$, $I_y S_x$ and $I_y S_y$ correlations do not give rise to observable magnetization during free precession, pulses can convert any of these correlations into one of the correlations that do interconvert with net magnetization ($I_x S_z$, $I_y S_z$, $I_z S_x$ or $I_z S_y$). The magnitude of the resulting magnetization will depend on the relative orientations of the correlation vectors and the direction of the pulse. If the evolution period is incremented to generate a two-dimensional spectrum, the positions of peaks in the indirectly-detected dimension will be determined by the frequencies associated with the evolution of the correlations. Because the expressions in Equation [31] are written as products of trigonometric functions, they do not reveal the actual frequencies for the evolution of the correlations. Using trigonometric identities, however, these equations can be rewritten as:

$$\begin{aligned}\langle I_x S_x(t) \rangle &= (1/8) (\cos(2\pi(\nu_I - \nu_S)t) - \cos(2\pi(\nu_I + \nu_S)t)) \\ \langle I_x S_y(t) \rangle &= (1/8) (\sin(2\pi(\nu_I - \nu_S)t) + \sin(2\pi(\nu_I + \nu_S)t)) \\ \langle I_y S_x(t) \rangle &= (1/8) (-\sin(2\pi(\nu_I - \nu_S)t) + \sin(2\pi(\nu_I + \nu_S)t)) \\ \langle I_y S_y(t) \rangle &= (1/8) (\cos(2\pi(\nu_I - \nu_S)t) + \cos(2\pi(\nu_I + \nu_S)t))\end{aligned}\quad [32]$$

These equations show that the evolution of each of the correlations is determined by two frequencies, which represent the sum and difference of ν_I and ν_S . These frequencies also correspond to the energies associated with the transitions between the $|\alpha\alpha\rangle$ and $|\beta\beta\rangle$ states and between $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$. Although these transitions cannot be stimulated by absorption of single photons with the corresponding energies, spectral peaks at the frequencies $(\nu_I + \nu_S)$ and $(\nu_I - \nu_S)$ can be generated in the indirect dimension of an appropriately designed two-dimensional experiment.

Because the correlations I_xS_x , I_xS_y , I_yS_x and I_yS_y evolve with frequencies associated with the forbidden double- and zero-quantum transitions, they are often referred to as “multiple-quantum coherences”. As Freeman (14) has noted, however, this phrase does not offer a great deal of insight into the “nature of the beast”. As an alternative, the following terminology is suggested to describe different classes of correlations:

1. A unique longitudinal correlation, involving only z -components: I_zS_z . Like longitudinal magnetization, this correlation does not change with time in a static magnetic field.
2. Transverse correlations, which involve only x' - or y' - components: I_xS_x , I_xS_y , I_yS_x and I_yS_y . These correlations interconvert among themselves with time in a stationary field, but, in the absence of pulses (or relaxation), do not interconvert either with other classes of correlations or with observable magnetization. The evolution of these correlations is determined by the sum and difference of ν_I and ν_S , but is independent of the coupling constant, J .
3. Longitudinal-Transverse correlations, or “mixed correlations” each of which involves a z -component and an x' - or y' -component: I_xS_z , I_yS_z , I_zS_x and I_zS_y . In a static magnetic field, each of these correlations interconverts cyclically with observable magnetization components. The frequency of this cycle is determined by the coupling constant, J . The correlations I_xS_z and I_yS_z also interconvert with one another, as do I_zS_x and I_zS_y , as determined by the precession rates of the transverse components.

It is also common practice to describe the states associated with the mixed correlations as possessing “anti-phase magnetization”, both because the vectors representing the transverse magnetization components point in opposite directions and because these states can lead to anti-phase peaks in the resulting spectrum, as in Experiment *a* of Figure 1 when $\tau = 1/(2J)$. The most important feature of these states, however, is the correlation between the transverse and longitudinal magnetization components. As is stressed in the following sections of this paper, the key to analyzing and understanding NMR experiments involving coupled spins is that the observable magnetization components *and* the correlations must be treated explicitly at each step. Although the correlations cannot be directly observed, they are quite “real” in the sense that they can be influenced by pulses and evolution periods, and they can evolve into observable magnetization.

REAL NMR SAMPLES: POPULATIONS WITH DIFFERENT WAVEFUNCTIONS

So as to highlight the quantum mechanical phenomena associated with coupled spin-pairs, the treatment so far has been restricted to pure populations in which all of the molecules have the same wavefunction. As noted at the outset, however, a treatment of NMR experiments requires that we consider the distribution of molecules with different wavefunctions and how this distribution affects the average magnetization components of the sample. For a sample of single spins, it is sufficient to consider only the small excess of spins initially in the $|\alpha\rangle$ state at thermal equilibrium. For the case of a coupled spin-pair,

however, the properties of the equilibrium state are *not* equivalent to those of the lowest-energy state, $|\alpha\alpha\rangle$. As a consequence, all four of the eigen states of the Hamiltonian must be considered in analyzing the equilibrium population and the populations that are generated by pulses and time-evolution periods.

The equilibrium state

At thermal equilibrium, the relative concentrations of the four states for a two-spin system can be calculated from the Boltzmann equation and can be expressed as:

$$\begin{aligned} f_{aa} &= \frac{N_{aa}}{N_T} = \frac{1}{4} (1 + \Delta P_I + \Delta P_S) \\ f_{ab} &= \frac{N_{ab}}{N_T} = \frac{1}{4} (1 + \Delta P_I - \Delta P_S) \\ f_{ba} &= \frac{N_{ba}}{N_T} = \frac{1}{4} (1 - \Delta P_I + \Delta P_S) \\ f_{bb} &= \frac{N_{bb}}{N_T} = \frac{1}{4} (1 - \Delta P_I - \Delta P_S) \end{aligned} \quad [33]$$

where N_T is the total number of spin-pairs in the total population, and the terms N_{aa} , *etc.* are the number of spins in each of the eigenstates; ΔP_I and ΔP_S represent the excess population of the α states of the I - and S -spins, respectively, as defined in Equation [6] for the single spin system. The contribution of the scalar-coupling constant to the energy terms are ignored for this calculation, since it does not significantly affect the relative populations.

To calculate the average observable magnetization components for the population, the components of the individual states are weighted by the corresponding fractions and summed. Since only the longitudinal components, $\langle I_z \rangle$ and $\langle S_z \rangle$ have non-zero values for any of the eigenstates, they are the only components that must be considered for the equilibrium population. The average I_z magnetization is calculated as:

$$\begin{aligned} \bar{I}_z &= f_{aa}\langle I_z \rangle_{aa} + f_{ab}\langle I_z \rangle_{ab} + f_{ba}\langle I_z \rangle_{ba} + f_{bb}\langle I_z \rangle_{bb} \\ &= f_{aa} \left(\frac{1}{2} \right) + f_{ab} \left(\frac{1}{2} \right) + f_{ba} \left(\frac{-1}{2} \right) + f_{bb} \left(\frac{-1}{2} \right) \\ &= \frac{\Delta P_I}{2} \end{aligned} \quad [34]$$

The symbol $\bar{I}_z$ is used to represent the average over a population of spins with different wavefunctions, distinguishing this statistical average from the quantum-mechanical average for a pure population, $\langle I_z \rangle$. Similarly, the equilibrium S_z -magnetization is given by:

$$\begin{aligned} \bar{S}_z &= f_{aa}\langle S_z \rangle_{aa} + f_{ab}\langle S_z \rangle_{ab} + f_{ba}\langle S_z \rangle_{ba} + f_{bb}\langle S_z \rangle_{bb} \\ &= \frac{\Delta P_S}{2} \end{aligned} \quad [35]$$

Since there are net positive I_z - and S_z -magnetization components for the population, it might be expected that the average value of the magnetization product $I_z S_z$ would also be positive. This, average is calculated as:

$$\begin{aligned} \overline{I_z S_z} &= f_{aa}\langle I_z S_z \rangle_{aa} + f_{ab}\langle I_z S_z \rangle_{ab} + f_{ba}\langle I_z S_z \rangle_{ba} + f_{bb}\langle I_z S_z \rangle_{bb} \\ &= f_{aa} \left(\frac{1}{4} \right) + f_{ab} \left(\frac{-1}{4} \right) + f_{ba} \left(\frac{-1}{4} \right) + f_{bb} \left(\frac{1}{4} \right) \\ &= 0 \end{aligned} \quad [36]$$

Even though the I_z - and S_z -magnetization of the individual spin-pairs in the population are correlated, there is no net correlation for the population, because the populations for which

$\langle I_z S_z \rangle = 1/4$ ($|\alpha\beta\rangle$ and $|\beta\alpha\rangle$) are balanced exactly by the populations with $\langle I_z S_z \rangle = -1/4$ ($|\alpha\alpha\rangle$ and $|\beta\beta\rangle$). This is one of the important distinctions between the equilibrium population and the small excess of the $|\alpha\alpha\rangle$ states. With appropriate pulses and evolution periods, however, net correlations can be generated and then can be converted into net magnetization components that can be observed in an NMR experiment.

Pulses and evolution

To follow the progress of an NMR experiment, we must calculate the changes in the wavefunctions for the four starting populations, calculate the changes in the average observable magnetization components and correlations for each population and, finally, calculate a weighted average. All of these calculations can be carried out in a relatively compact form using the density matrix, and they can be condensed even further using the product operator-formalism. It is instructive, however, to carry the calculations out in a more explicit form for a simple example before introducing the more abstract representations.

In the discussion and equations that follow, the terms for the fractional populations introduced earlier (f_{aa} , f_{ab} , f_{ba} and f_{bb}) are carried forward to represent the populations of the states derived from the starting states indicated by the subscripts. The same subscripts will also be used to identify the average magnetization components or correlations from the populations derived from the corresponding starting states.

If a selective $(\pi/2)_x$ pulse is applied to the S -spin, the average magnetization components for the four populations will be changed as follows:

$$\begin{aligned} \text{Starting state } |\alpha\alpha\rangle : \langle S_z \rangle = 1/2 &\rightarrow \langle S_y \rangle = 1/2 \\ \text{Starting state } |\alpha\beta\rangle : \langle S_z \rangle = -1/2 &\rightarrow \langle S_y \rangle = -1/2 \\ \text{Starting state } |\beta\alpha\rangle : \langle S_z \rangle = 1/2 &\rightarrow \langle S_y \rangle = 1/2 \\ \text{Starting state } |\beta\beta\rangle : \langle S_z \rangle = -1/2 &\rightarrow \langle S_y \rangle = -1/2 \end{aligned}$$

Since the I -magnetization for each subpopulation is unchanged, the total $\langle I_z \rangle$ remains the same, while the new average S_y -magnetization is calculated as:

$$\begin{aligned} \overline{S_y} &= f_{aa}\langle S_y \rangle_{aa} + f_{ab}\langle S_y \rangle_{ab} + f_{ba}\langle S_y \rangle_{ba} + f_{bb}\langle S_y \rangle_{bb} \\ &= f_{aa}\left(\frac{1}{2}\right) + f_{ab}\left(-\frac{1}{2}\right) + f_{ba}\left(\frac{1}{2}\right) + f_{bb}\left(-\frac{1}{2}\right) \\ &= \frac{\Delta P_S}{2} \end{aligned} \quad [37]$$

This simply confirms the expectation that the total S -magnetization for the population is rotated from the z -axis to the y' -axis. The new non-zero product operator averages are:

$$\begin{aligned} \langle I_z S_y \rangle_{aa} &= 1/4 \\ \langle I_z S_y \rangle_{ab} &= -1/4 \\ \langle I_z S_y \rangle_{ba} &= -1/4 \\ \langle I_z S_y \rangle_{bb} &= 1/4 \end{aligned}$$

When these values are weighted by the corresponding fractional populations and added together, the total average correlation, $\overline{I_y S_z}$, is zero, just as $\overline{I_z S_z}$ was zero in the starting state.

If the $(\pi/2)_{x,S}$ pulse is followed by an evolution period, the transverse S_y -magnetization of each subpopulation evolves according to the following relations:

$$\begin{aligned}\langle S_y(t) \rangle_{aa} &= (1/2) \cos(2\pi(\nu_s - J/2)t) \\ \langle S_y(t) \rangle_{ab} &= - (1/2) \cos(2\pi(\nu_s - J/2)t) \\ \langle S_y(t) \rangle_{ba} &= (1/2) \cos(2\pi(\nu_s + J/2)t) \\ \langle S_y(t) \rangle_{bb} &= - (1/2) \cos(2\pi(\nu_s + J/2)t)\end{aligned}\quad [38]$$

Summing over the population-weighted averages gives:

$$\begin{aligned}\overline{S_y}(t) &= (f_{aa} - f_{ab})(1/2) \cos(2\pi(\nu_s - J/2)t) + (f_{ba} - f_{bb})(1/2) \cos(2\pi(\nu_s + J/2)t) \\ &= \frac{\Delta P_s}{4} (\cos(2\pi(\nu_s - J/2)t) + \cos(2\pi(\nu_s + J/2)t))\end{aligned}\quad [39]$$

The S_x -magnetization also evolves, as given by:

$$\overline{S_x}(t) = \frac{\Delta P_s}{4} (\sin(2\pi(\nu_s - J/2)t) + \sin(2\pi(\nu_s + J/2)t)) \quad [40]$$

Now, both frequency components are present, representing the populations for which I_z is either positive or negative. This, is different from what is seen with just the pure $|\alpha\alpha\rangle$ state, for which there is only a single frequency, but it is entirely consistent with what is actually observed in the demonstration experiment shown in Figure 1a.

The average correlations for the four populations also evolve with time. The initial $I_z S_y$ correlations change according to:

$$\begin{aligned}\langle I_z S_y(t) \rangle_{aa} &= (1/4) \cos(2\pi(\nu_s - J/2)t) \\ \langle I_z S_y(t) \rangle_{ab} &= - (1/4) \cos(2\pi(\nu_s - J/2)t) \\ \langle I_z S_y(t) \rangle_{ba} &= - (1/4) \cos(2\pi(\nu_s + J/2)t) \\ \langle I_z S_y(t) \rangle_{bb} &= (1/4) \cos(2\pi(\nu_s + J/2)t)\end{aligned}\quad [41]$$

Summing the contributions of the four populations gives:

$$\begin{aligned}\overline{I_z S_y}(t) &= (f_{aa} - f_{ab})(1/4) \cos(2\pi(\nu_s - J/2)t) + (-f_{ba} + f_{bb})(1/4) \cos(2\pi(\nu_s + J/2)t) \\ &= \frac{\Delta P_s}{8} (\cos(2\pi(\nu_s - J/2)t) - \cos(2\pi(\nu_s + J/2)t))\end{aligned}\quad [42]$$

Although there was no net correlation either at equilibrium or immediately after a $\pi/2$ -pulse ($t = 0$), the various components of the system evolve so as to cyclically generate a net correlation, reflecting the differences in precession frequencies associated with the different subpopulations. For each of the populations, precession of the S -magnetization is also associated with changes in the correlation between S_x and I_z . Summing over the contributions from the four populations gives:

$$\overline{I_z S_x}(t) = \frac{\Delta P_s}{8} (\sin(2\pi(\nu_s - J/2)t) - \sin(2\pi(\nu_s + J/2)t)) \quad [43]$$

The situation is more easily visualized if v_S is set to zero. With this assumption, the observable S_y -magnetization evolves as:

$$\overline{S_y} = \frac{\Delta P_s}{2} \cos(\pi J t) \quad [44]$$

This can be represented by a pair of vectors moving in opposite directions from the y' -axis, as described earlier for both the I - and S -magnetization of a pure population with the wavefunction $|\psi_C\rangle$ and illustrated in Figure 11. Now, however, the two frequency components come from separate populations and the net I -magnetization is still aligned with the z -axis. At the same time that the S_y -magnetization evolves, the net $I_z S_x$ -correlation fluctuates according to:

$$\overline{I_z S_x}(t) = -\frac{\Delta P_s}{4} \sin(\pi J t) \quad [45]$$

Thus, the observable S_y -magnetization interconverts with the $I_x S_z$ -correlation, as illustrated earlier in Figure 12a. With $v_S = 0$, both the average S_x -magnetization and the correlation represented by $I_z S_y$ remain zero.

This example suggests that the evolution of the average magnetization components and correlations for a mixed population can be described using the same diagrams and rules developed for pure populations. The major difference is that the starting equilibrium state does not display any net correlation, even though the spins of the individual molecules are correlated. Following a pulse to rotate one or both of the magnetization vectors into the transverse plane, however, further evolution can convert the observable magnetization into correlations. The magnetization components and correlations can be further manipulated by pulses, which, again, follow the rules implied by the vector diagrams. This treatment is formalized in the following section using the density matrix.

THE DENSITY MATRIX AND OPERATOR BASIS SET

The approach outlined in the previous section, in which each of the four starting states are followed through the various pulses and evolution periods, quickly becomes awkward for analyzing experiments of even modest complexity. A much more efficient approach involves the density matrix, which combines all of the important information about the populations into a compact form. Only a cursory description of the density matrix calculations will be presented here in order to illustrate the principles and the relationship between this treatment and the quantum mechanical operators discussed in the previous sections. Much more complete treatments can be found in the classic review by Fano (4) and in textbooks (22,23). Before discussing the density matrix for a coupled spin-pair, the concepts will be illustrated with the simpler case of a population of isolated spins.

A population of single spins

For a system of single spins, the density matrix is defined as:

$$\rho = \begin{bmatrix} \overline{c_\alpha c_\alpha^*} & \overline{c_\alpha c_\beta^*} \\ \overline{c_\beta c_\alpha^*} & \overline{c_\beta c_\beta^*} \end{bmatrix} \quad [46]$$

The four elements of the matrix are the averages of products of coefficients in the wavefunctions, as defined in Equation [1], for the individual spins in the population. For instance, the term $\overline{c_\alpha c_\alpha^*}$ is calculated by multiplying c_α for each spin by its complex conjugate, c_α^* , and averaging the values over the entire population. Importantly, the products are calculated before the values are averaged.

Provided that the elements are known, the average value of any observable property of the population can be calculated from the density matrix. This calculation is carried out by multiplying the density matrix by a matrix associated with the observation and then calculating the sum of the diagonal terms (the trace) of the resulting matrix. For instance, the matrix for calculating the population average of the z -magnetization is:

$$I_z = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad [47]$$

This matrix is a representation of the $\hat{I}_z$ operator. (Symbols representing matrices are written here using a bold, unitalicized sans serif font.) The average z -magnetization is calculated as:

$$\begin{aligned} \bar{I}_z &= \text{Tr}(I_z \rho) \\ &= \text{Tr} \left(\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \overline{c_\alpha c_\alpha^*} & \overline{c_\alpha c_\beta^*} \\ \overline{c_\beta c_\alpha^*} & \overline{c_\beta c_\beta^*} \end{bmatrix} \right) \\ &= \text{Tr} \left(\frac{1}{2} \begin{bmatrix} \overline{c_\alpha c_\alpha^*} & \overline{c_\alpha c_\beta^*} \\ -\overline{c_\beta c_\alpha^*} & -\overline{c_\beta c_\beta^*} \end{bmatrix} \right) \\ &= \frac{1}{2} (\overline{c_\alpha c_\alpha^*} - \overline{c_\beta c_\beta^*}) \end{aligned} \quad [48]$$

Notice that if all of the spins are in the $|\alpha\rangle$ state, $c_\alpha = c_\alpha^* = 1$ and $c_\beta = c_\beta^* = 0$ for each spin. Therefore, $\overline{c_\alpha c_\alpha^*} = 1$, and $\overline{c_\beta c_\beta^*} = 0$; and $\bar{I}_z = 1/2$, just as expected.

The key, of course, to using this approach is to know the values of the density matrix elements for a given state of the system. At thermal equilibrium, the density matrix is given by:

$$\rho_{eq} = \frac{1}{2} \begin{bmatrix} 1 + \Delta P & 0 \\ 0 & 1 - \Delta P \end{bmatrix} \quad [49]$$

which can also be written as:

$$\rho_{eq} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\Delta P}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad [50]$$

The first matrix on the right-hand side of this equation (excluding the factor of $1/2$) is known as the identity matrix, I . This matrix does not contribute to any observable properties, and can be dropped to simplify subsequent calculations. Notice that the remaining matrix is simply the I_z matrix multiplied by the population difference, ΔP .

When pulses are applied to the sample, the density matrix naturally changes, and it changes further during evolution periods. These changes are calculated by multiplication with

appropriate matrices. For instance, the density matrix following a $\pi/2$ -pulse along the x' -axis is calculated as:

$$\rho(\pi/2) = R_x(\pi/2)\rho_0 R_x^{-1}(\pi/2) \quad [51]$$

where ρ_0 is the density matrix before the pulse and the matrices $R_x(\pi/2)$ and $R_x^{-1}(\pi/2)$ are given by:

$$R_x(\pi/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad [52]$$

and

$$R_x^{-1}(\pi/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \quad [53]$$

As indicated by their labels, these two matrices are inverses of one another, meaning that their product is the identity matrix. The density matrix following a $(\pi/2)_x$ pulse to the equilibrium state is thus calculated as:

$$\begin{aligned} \rho &= R_x(\pi/2)\rho_{eq}R_x^{-1}(\pi/2) \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \frac{\Delta P}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \\ &= \frac{\Delta P}{4} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \\ &= \frac{\Delta P}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{aligned} \quad [54]$$

When the operator matrices are used to calculate the average magnetization components, the magnetization is found to be rotated to the y' -axis, just as expected. Similar matrices are used to calculate the effects of pulses of other durations or along the y' -axis, and a matrix form of the Hamiltonian operator is used to calculate the evolution in a constant magnetic field.

The matrix calculations can be simplified somewhat by representing the density matrix as a sum of basis matrices. The basis set must include four matrices, all of which are orthogonal to one another, meaning that none of them can be written as a linear combination of the other three. There are innumerable sets of matrices that can be used as the basis, but a particularly convenient basis set is found in the matrices representing the operators for the magnetization components, I_x , I_y and I_z , together with the identity matrix. Thus, the density matrix, or any other 2×2 matrix, can be written as:

$$\rho = c_x I_x + c_y I_y + c_z I_z + c_1 I \quad [55]$$

where c_x , c_y , c_z and c_1 are coefficients that represent the relative contributions of the basis matrices to the sum. Since, in general, the elements of a matrix can be complex numbers, the coefficients above can also be complex. It turns out, however, that the coefficients are always real when the linear combination is used to represent a density matrix. In addition,

the identity matrix does not contribute to any observable measurements, and the last term of the sum can be dropped. The virtue of representing the density matrix in this way derives from the fact that the members of the basis set are orthogonal to one another, which requires that the trace of the product of any pair of members is zero. That is, the following relationships hold:

$$\begin{aligned}\text{Tr}(I_x I_y) &= \text{Tr}(I_y I_x) = 0 \\ \text{Tr}(I_x I_z) &= \text{Tr}(I_z I_x) = 0 \\ \text{Tr}(I_y I_z) &= \text{Tr}(I_z I_y) = 0\end{aligned}\quad [56]$$

When the operator matrices are multiplied by themselves, the traces of the products are found to each equal 1/2.

Given these relationships, consider now what happens if we go on to calculate one of the average magnetization components, $\bar{I}_x$ for instance, for an arbitrary population defined by a density matrix written in the form of Equation [55]. Matrix multiplication is distributive over addition, as is the calculation of the trace. The calculation can, therefore, be carried out as follows:

$$\begin{aligned}\bar{I}_x &= \text{Tr}(I_x \rho) \\ &= \text{Tr}\left(I_x (c_x I_x + c_y I_y + c_z I_z)\right) \\ &= \text{Tr}(I_x c_x I_x) + \text{Tr}(I_x c_y I_y) + \text{Tr}(I_x c_z I_z) \\ &= \frac{1}{2} c_x\end{aligned}\quad [57]$$

Thus, the average x -magnetization is simply one half of the coefficient of I_x in the expansion of the density matrix. The analogous relationships for $\bar{I}_y$ and $\bar{I}_z$ also hold. Thus, the average magnetization components can be determined by simple examination of the density matrix, provided that it is written as a linear combination of the operator matrices.

Once the density matrix is written in the form of Equation [55], the matrix multiplications for calculating the effects of pulses and evolution periods can also be carried out on the individual components, thus simplifying the analysis of an experiment. Examination of Equation [50] reveals that the equilibrium density matrix can be written as:

$$\rho_{eq} = \Delta P I_z \quad [58]$$

Application of a $(\pi/2)_x$ -pulse changes the density matrix to:

$$\rho = \frac{\Delta P}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \Delta P I_y \quad [59]$$

Thus, the pulse converts the density matrix from I_z to I_y (each multiplied by a constant representing the equilibrium excess of $|\alpha\rangle$ -spins). Subsequent evolution changes ρ to:

$$\rho = \Delta P (\sin(2\pi\nu t) I_x + \cos(2\pi\nu t) I_y) \quad [60]$$

Note that the coefficients change in the same way as do the average magnetization components and the projections of the magnetization vector in the diagram shown in Figure 3c.

These examples suggest a more formal interpretation of the vector diagrams, based on quantum and statistical mechanics. In this view, the vectors represent the contributions of the operator basis set matrices to the density matrix under different conditions. For the simple single-spin system, it is difficult to justify all of the extra mathematics of the density matrix for describing an NMR experiment, since the simple diagrams are so effective and clear. The treatment of this system should help, however, in visualizing the more complicated case for a coupled spin-pair discussed below.

The two-spin system

Since the two-spin system is described by a wavefunction composed of four basis functions, the corresponding density matrix has four rows and columns, made up of the average products of the wavefunction coefficients:

$$\rho = \begin{bmatrix} \overline{c_{\alpha\alpha}c_{\alpha\alpha}^*} & \overline{c_{\alpha\alpha}c_{\alpha\beta}^*} & \overline{c_{\alpha\alpha}c_{\beta\alpha}^*} & \overline{c_{\alpha\alpha}c_{\beta\beta}^*} \\ \overline{c_{\alpha\beta}c_{\alpha\alpha}^*} & \overline{c_{\alpha\beta}c_{\alpha\beta}^*} & \overline{c_{\alpha\beta}c_{\beta\alpha}^*} & \overline{c_{\alpha\beta}c_{\beta\beta}^*} \\ \overline{c_{\beta\alpha}c_{\alpha\alpha}^*} & \overline{c_{\beta\alpha}c_{\alpha\beta}^*} & \overline{c_{\beta\alpha}c_{\beta\alpha}^*} & \overline{c_{\beta\alpha}c_{\beta\beta}^*} \\ \overline{c_{\beta\beta}c_{\alpha\alpha}^*} & \overline{c_{\beta\beta}c_{\alpha\beta}^*} & \overline{c_{\beta\beta}c_{\beta\alpha}^*} & \overline{c_{\beta\beta}c_{\beta\beta}^*} \end{bmatrix} \quad [61]$$

As for the single-spin system, the magnetization operators can be written as six orthogonal matrices, in this case 4×4 matrices. These six, however, are not sufficient to form a basis set for the density matrix, which requires 16. This, in itself, is an interesting observation since it implies that the observable magnetization components are not sufficient to fully define the system. But, there are also nine matrices that represent the product operators, and when these are added to the six magnetization matrices, together with the identity matrix, a full basis set is formed. Thus, the density matrix can be written as:

$$\begin{aligned} \rho = & c_{I_x} I_x + c_{I_y} I_y + c_{I_z} I_z + c_{S_x} S_x + c_{S_y} S_y + c_{S_z} S_z + c_{I_x S_x} I_x S_x \\ & + c_{I_y S_x} I_y S_x \\ & + c_{I_z S_x} I_z S_x \\ & + c_{I_x S_y} I_x S_y \\ & + c_{I_y S_y} I_y S_y \\ & + c_{I_z S_y} I_z S_y \\ & + c_{I_x S_z} I_x S_z \\ & + c_{I_y S_z} I_y S_z \\ & + c_{I_z S_z} I_z S_z + I \end{aligned} \quad [62]$$

The identity matrix can be omitted from the calculations, and all of the coefficients are real numbers, just as for the single-spin system. Using the same logic as before, the coefficients of the matrices corresponding to the observable magnetization components are proportional to the population averages of the magnetization components, and the coefficients for the product operator terms are proportional to the averages of the corresponding magnetization products, or correlations. However, the proportionalities are different for the two types of coefficients: The average values of the magnetization components are equal to the corresponding coefficients, while the average products are equal to the coefficients divided

by four. Starting with a beginning state, any sequence of pulses and evolution periods can be described by the changes in the coefficients, and, importantly, the changes in the coefficients can be calculated using simple rules based on the diagrams introduced earlier.

GRAPHICALLY-AIDED CALCULATIONS

Numerous articles and textbooks provide sets of equations for calculating the evolution of the operator basis components during pulses and evolution periods (6–8,10,22,23). The treatment presented here is entirely consistent with these rules, and the graphical representations can aid in carrying out the calculations. It should be noted, however, that the most commonly used equations, those originally introduced by Sørensen *et al.* (6), represent a “right-hand rule” for rotations due to precession and pulses, while the treatment here uses the left-hand rule to maintain consistency with most vector representations. The rules presented below can be converted to the more common form by reversing the signs of the sine terms in the expressions for chemical-shift evolution and pulses. The expressions for scalar-coupling evolution are the same in the two representations.

The general approach to using the operator basis set is to treat each term in the expansion of Equation [62] individually, calculating the effects of pulses and evolution periods on the states represented by the operators. Each calculation generally generates one or more new terms, which must again be treated individually. The one exception to this rule is that terms representing the identity matrix can be safely dropped. Also, multiples of the same density-matrix term may be added together. Once the manipulations for all of the steps in the experiment are carried out, the final coefficients represent the net observable magnetization components and correlations.

Pulses

The rules for describing the effect of pulses on the density matrix components representing observable *I*-magnetization components (I_x , I_y and I_z) are shown in Figure 14. As illustrated by the diagrams, all of these effects are easily visualized in terms of a classical vector representation. The corresponding rules for the *S*-magnetization components have exactly the same form, with the symbol *S* replacing *I* in the expressions.

The same rules are used to calculate the effects of pulses on the terms representing correlations, with the result of the calculation for each component replacing the symbol for the affected spin. For instance, the effect of an *I*-pulse of angle *a* along the *x*-axis, applied to the matrix $I_y S_z$, is calculated as:

$$I_y S_z \rightarrow (\cos(a)I_y - \sin(a)I_z)S_z \quad [63]$$

Multiplication of the operators is distributive, so that this can be re-written as:

$$I_y S_z \rightarrow \cos(a)I_y S_z - \sin(a)I_z S_z \quad [64]$$

If this is followed immediately by an *x*-pulse of angle *a* to the *S*-spin, the equivalent rule is applied to the *S*-components of both of the terms generated by the previous step, yielding the following results:

$$\begin{aligned}
 \cos(a)I_y S_z &\rightarrow \cos(a)I_y (\cos(a)S_z + \sin(a)S_y) \\
 &= \cos^2(a)I_y S_z \\
 &\quad + \cos(a)\sin(a)I_y S_y \\
 &\quad - \sin(a)I_z S_z \rightarrow \\
 &\quad - \sin(a)I_z (\cos(a)S_z + \sin(a)S_y) \\
 &= -\sin(a)\cos(a)I_z S_z \\
 &\quad - \sin^2(a)I_z S_y
 \end{aligned}
 \tag{65}$$

All of these terms are added together to give the overall result of the pulses to the two spins:

$$I_y S_z \rightarrow \cos^2(a)I_y S_z + \cos(a)\sin(a)I_y S_y - \sin(a)\cos(a)I_z S_z - \sin^2(a)I_z S_y \tag{66}$$

In this way, the effects of any pulse can be calculated for any of the magnetization or correlation operators. Note, however, that pulses will never convert observable magnetization into a correlation, or vice versa.

Chemical-shift evolution

For purposes of calculation, it is convenient to separate the treatment of precession due to chemical shift differences from that due to scalar coupling. In essence, this means first performing a calculation assuming that $J = 0$ and then taking the result and performing a calculation assuming that ν_I and ν_S are zero. The two calculations can be performed in either order.

The rules for calculating the effects of evolution in the absence of scalar coupling are shown in Figure 15 for the I -magnetization matrices. The analogous rules for the S -magnetization components differ only in the replacement of ν_I with ν_S . As is the case for pulses, the effects of chemical-shift evolution on the magnetization operators are entirely consistent with a simple vector representation.

The effects on the correlation operators can be calculated by first applying the rules for one spin and then the other, even though the precession of the two spins cannot really be separated in time. For instance, time evolution of the $I_y S_x$ operator can be calculated by first applying the rule for I_y :

$$I_y S_x \rightarrow (\cos(2\pi\nu_I t)I_y + \sin(2\pi\nu_I t)I_x)S_x = \cos(2\pi\nu_I t)I_y S_x + \sin(2\pi\nu_I t)I_x S_x \tag{67}$$

and then applying the rule for S_x to each of the resulting terms:

$$\begin{aligned}
 \cos(2\pi\nu_I t)I_y S_x &\rightarrow \cos(2\pi\nu_I t)I_y (\cos(2\pi\nu_S t)S_x - \sin(2\pi\nu_S t)S_y) \\
 &= \cos(2\pi\nu_I t)\cos(2\pi\nu_S t)I_y S_x - \cos(2\pi\nu_I t)\sin(2\pi\nu_S t)I_y S_y \\
 \sin(2\pi\nu_I t)I_x S_x &\rightarrow \sin(2\pi\nu_I t)I_x (\cos(2\pi\nu_S t)S_x - \sin(2\pi\nu_S t)S_y) \\
 &= \sin(2\pi\nu_I t)\cos(2\pi\nu_S t)I_x S_x - \sin(2\pi\nu_I t)\sin(2\pi\nu_S t)I_x S_y
 \end{aligned}
 \tag{68}$$

Adding all of these terms together gives the overall effect of precession on $I_y S_x$:

$$I_y S_x \rightarrow \cos(2\pi\nu_I t) \cos(2\pi\nu_S t) I_y S_x - \cos(2\pi\nu_I t) \sin(2\pi\nu_S t) I_y S_y + \sin(2\pi\nu_I t) \cos(2\pi\nu_S t) I_x S_x - \sin(2\pi\nu_I t) \sin(2\pi\nu_S t) I_x S_y$$

[69]

Although this expression is rather formidable, it can be visualized as the independent precession of the vectors representing the I - and S -components of the correlation, as illustrated for the matrix $I_y S_y$ in Figure 13. Note that evolution in the absence of scalar coupling only converts correlations into other correlations and magnetization components into other magnetization components. Also note that z -components always remain constant and are not generated from chemical-shift evolution of transverse components.

Evolution under the influence of scalar coupling

As discussed in the previous sections, the most distinctive features of a population of coupled spin-pairs emerge when considering the influence of the scalar coupling constant J during evolution periods. The rules describing the effects on the observable I -magnetization components are shown in the top three rows Figure 16. The matrices representing transverse-magnetization components, I_x and I_y , interconvert with the mixed correlation matrices $I_y S_z$ and $I_x S_z$, respectively, while I_z is unchanged. The interconversion of magnetization and correlation terms is visualized as the precession of two I -vectors in opposite directions. It should be noted that a factor of two is present in the terms for $I_y S_z$ and $I_x S_z$, so that, for instance, I_x is converted to $2I_y S_z$ when $t = 1/(2J)$. This factor arises because the magnetization components for individual spins have values of $1/2$ or $-1/2$, while the correlations have values of $1/4$ or $-1/4$. The factor of two maintains consistency in the representation of the density matrix, and most texts include this factor in the definitions of the two-spin matrices, labeling them $2I_x S_z$, $2I_y S_x$ etc.

Figure 16 also shows the rules and corresponding diagrams for evolution beginning with the matrices for two mixed correlations, $I_x S_z$ and $I_y S_z$. These rules can also be derived by substitution of the rules for I_x and I_y . For instance the effect on $I_x S_z$ is calculated as:

$$I_x S_z \rightarrow (\cos(\pi J t) I_x + 2 \sin(\pi J t) I_y S_z) S_z = \cos(\pi J t) I_x S_z + 2 \sin(\pi J t) I_y S_z S_z$$

[70]

The triple product, $I_y S_z S_z$, is a new twist, but is readily dealt with because $S_z S_z$ is equal to the identity matrix multiplied by $1/4$. The same relationship holds for the product of any single-spin matrix and itself. Thus, the rule can be written as:

$$I_x S_z \rightarrow \cos(\pi J t) I_x S_z + \frac{1}{2} \sin(\pi J t) I_y$$

[71]

which is consistent with the vector diagrams. Other operator products frequently arise when calculations are carried out for multi-step experiments, and the following relationships can be used to simplify the results:

$$\begin{aligned} I_x I_y &= -I_y I_x = \frac{i}{2} I_z \\ I_y I_z &= -I_z I_y = \frac{i}{2} I_x \\ I_z I_x &= -I_x I_z = \frac{i}{2} I_y \end{aligned}$$

[72]

The equivalent relationships for the products involving S_x , S_y and S_z also hold.

Together the rules for pulses, chemical-shift evolution and scalar coupling are sufficient to analyze any experiment involving a weakly coupled spin-pair. Although the graphical representations are not necessary, they can help in visualizing the evolution of the system, particularly the interconversion of observable magnetization and correlations.

Analysis of some simple experiments

The experiments described in the INTRODUCTION and illustrated in Figure 1 can be used to demonstrate the approach, which should help remove some of the mystery of those experiments. As usual, the starting point is the density matrix for the system at equilibrium, which can be written as:

$$\rho_{eq} = \frac{\Delta P_I}{2} I_z + \frac{\Delta P_S}{2} S_z \quad [73]$$

This state is illustrated graphically by the two z -magnetization vectors, in the leftmost diagram in Figure 17a. As discussed earlier, there is no *net* correlation between the I_z - and S_z -magnetization components, even though the spins in the individual molecules are correlated. In experiment (a) of Figure 1, the first pulse, applied to the S -spin along the x' -axis converts the density matrix to:

$$\rho = \frac{\Delta P_I}{2} I_z + \frac{\Delta P_S}{2} S_y \quad [74]$$

This corresponds to the rotation of the S_z -magnetization to the y' -axis, as illustrated by the diagram in Figure 17a. Still, there is no net correlation component present. However, the S_y -magnetization can be represented by a pair of double arrows indicating the potential correlation that can evolve with time.

To simplify the analysis, the Larmor frequencies of both spins are assumed to match their respective rotating frame frequencies. Given this assumption, the evolution of the density matrix following the initial pulse is:

$$\rho(t) = \frac{\Delta P_I}{2} I_z + \frac{\Delta P_S}{2} (\cos(\pi J t) S_y - 2 \sin(\pi J t) I_z S_x) \quad [75]$$

During this evolution period, the I_z -magnetization remains constant, while the S_y -magnetization interconverts with an $I_z S_x$ correlation. At any time, the density-matrix components arising from the initial S_z -magnetization can be represented as a linear combination of S_y and $I_z S_x$. If an FID is recorded from the S -spins immediately after the pulse and Fourier transformed, the result is a doublet centered at the rotating frame frequency. After a delay of $\tau = 1/(2J)$ following the pulse, the density matrix is:

$$\rho = \frac{\Delta P_I}{2} I_z - \frac{\Delta P_S}{2} 2 I_z S_x \quad [76]$$

At this point, the S -magnetization has been completely converted to the $I_z S_x$ -correlation, and subsequent evolution of the density matrix is described by:

$$\rho(t) = \frac{\Delta P_I}{2} I_z - \frac{\Delta P_S}{2} (\sin(\pi J t) S_y - 2 \cos(\pi J t) I_z S_x) \quad [77]$$

Now, the phase of the signal generated by the S -spins is shifted by $\pi/2$ radians from that observed when the signal is recorded directly after the pulse, as observed experimentally.

Now, consider what happens if a $(\pi/2)_x$ -pulse is applied to the I -spins after the $\tau = 1/(2J)$ delay, as in experiment (b) of Figure 1. In addition to the rotation of the I_z -magnetization to the y' -axis, the pulse also converts the $I_z S_x$ -correlation into an $I_y S_x$ -correlation as illustrated with vector diagrams in Figure 17c and represented by the density matrix:

$$\rho = \frac{\Delta P_I}{2} I_y - \frac{\Delta P_S}{2} 2 I_y S_x \quad [78]$$

During the data-acquisition period, the density matrix evolves according to:

$$\rho(t) = \frac{\Delta P_I}{2} (\cos(\pi J t) I_y - 2 \sin(\pi J t) I_x S_z) - \frac{\Delta P_S}{2} I_y S_x \quad [79]$$

Although the I_y -magnetization and the $I_x S_z$ -correlation interconvert cyclically with one another, the $I_y S_x$ -correlation, as observed in the rotating frame, does not change with time. Most importantly, this correlation does not interconvert with observable magnetization, thus accounting for the suppression of the S -signal in the spectrum. (The small residual signal in the spectrum shown in Figure 1 is accounted for by a small amount of S_y -magnetization still remaining at the end of the delay period.) Thus, the effect of the $(\pi/2)_x$ -pulse to the I -spin on the S -spin signal is due to a rotation of a correlation component, rather than any effect on the net I -magnetization *per se*.

Finally, consider what happens when a $(\pi/2)_x$ -pulse is also applied to the I -spins at the beginning of the experiment, as in experiment (c) of Figure 1. This pulse will rotate the I_z -magnetization to the y' -axis, and the resulting I_y -magnetization will interconvert with an $I_x S_z$ -correlation, as illustrated in Figure 17d. Only the correlation is present after a delay of $\tau = 1/(2J)$. However, the second $(\pi/2)_{x,I}$ -pulse does not affect this correlation, since the I -components of the correlation are aligned with the direction of the pulse. Thus, the correlation continues to evolve and interconvert with I_y -magnetization. If an FID were to be recorded from the I -spin, the same result would be observed with or without the second $(\pi/2)_{x,I}$ -pulse. On the other hand, a $(\pi/2)_{x,S}$ -pulse would suppress the signal from the I -spin by converting the $I_x S_z$ -correlation into an “invisible” $I_y S_y$ -correlation, just as the $(\pi/2)_{x,I}$ -pulse suppresses the signal from the S -spin in experiments (b) and (c).

The INEPT experiment

As a final example, we will consider one of the most important pulse sequence elements used in heteronuclear NMR experiments, the INEPT sequence (Insensitive Nuclei Enhanced by Polarization Transfer) (24). This pulse sequence rather miraculously enhances the signal from a nucleus with a relatively low gyromagnetic ratio (*e.g.* ^{13}C or ^{15}N) by exploiting the larger equilibrium population difference associated with a scalar-coupled nucleus with a larger gyromagnetic ratio (*e.g.* ^1H). The INEPT sequence can be written as:

$$\begin{array}{l}
 I: \quad (\pi/2)_x \xrightarrow{\tau=1/(4J)} \pi_x \xrightarrow{\tau=1/(4J)} (\pi/2)_y \\
 S: \quad \xrightarrow{\tau=1/(4J)} \pi_x \xrightarrow{\tau=1/(4J)} (\pi/2)_x \rightarrow \text{data acquisition}
 \end{array}$$

The I -spin is assumed to have the larger gyromagnetic ratio. In order to make the analysis general, no assumptions will be made about the precession frequencies of the two spins, so that the effects of both chemical shift and scalar coupling must be considered during each evolution period.

The initial $(\pi/2)_{x,I}$ -pulse rotates the I_z magnetization to the y' -axis and converts the equilibrium density matrix to:

$$\rho = \frac{\Delta P_I}{2} I_y + \frac{\Delta P_S}{2} S_z \quad [80]$$

To keep things from becoming too complicated, it is useful to treat the two components separately, starting with the I_y component.

For the first evolution period, with $t = 1/(4J)$, we first calculate effects of chemical-shift evolution on the I_y component of the density matrix, which evolves to:

$$\frac{\Delta P_I}{2} I_y \rightarrow \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) I_y + \frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) I_x \quad [81]$$

as illustrated using the vector representation in Figure 18a. Again, it is helpful to divide the analysis and treat these two components separately. For the I_y component, the scalar-coupling evolution results in a partial conversion of the observable magnetization to an $I_x S_z$ correlation, as illustrated in Figure 18b and expressed mathematically as:

$$\begin{aligned}
 \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) I_y &\rightarrow \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) \left(\cos\left(\frac{\pi J}{4J}\right) I_y - 2 \sin\left(\frac{\pi J}{4J}\right) I_x S_z \right) \\
 &= \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) \left(\frac{1}{\sqrt{2}} I_y - 2 \frac{1}{\sqrt{2}} I_x S_z \right)
 \end{aligned} \quad [82]$$

The scalar coupling evolution also partially converts the I_x component to an $I_y S_z$ correlation:

$$\frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) I_x \rightarrow \frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) \left(\frac{1}{\sqrt{2}} I_x + 2 \frac{1}{\sqrt{2}} I_y S_z \right) \quad [83]$$

as illustrated in Figure 18c

The first evolution period is followed by π -pulses applied to both spins along the x' -axis, reversing the signs of each of the I_y , I_z , S_y and S_z terms, as represented graphically in Figs. 18b and 18c and mathematically as:

$$\frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) \left(\frac{1}{\sqrt{2}} I_y - 2 \frac{1}{\sqrt{2}} I_x S_z \right) \rightarrow \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) \left(-\frac{1}{\sqrt{2}} I_y + 2 \frac{1}{\sqrt{2}} I_x S_z \right) \quad [84]$$

and

$$\frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) \left(\frac{1}{\sqrt{2}} I_x + 2 \frac{1}{\sqrt{2}} I_y S_z \right) \rightarrow \frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) \left(\frac{1}{\sqrt{2}} I_x + 2 \frac{1}{\sqrt{2}} I_y S_z \right) \quad [85]$$

Notice that there is no net effect on the $I_y S_z$ correlation of Equation [85], since the signs of both components are reversed.

The π -pulses are then followed by a second evolution period of $t = 1/(4J)$, illustrated graphically in Figure 19. Scalar coupling (panels *a* and *c*) leads to the conversion:

$$\frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) \left(-\frac{1}{\sqrt{2}} I_y + 2 \frac{1}{\sqrt{2}} I_x S_z \right) \rightarrow \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) 2 I_x S_z \quad [86]$$

and:

$$\frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) \left(\frac{1}{\sqrt{2}} I_x + 2 \frac{1}{\sqrt{2}} I_y S_z \right) \rightarrow \frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) 2 I_y S_z \quad [87]$$

Chemical shift evolution during the same period converts $I_x S_z$ into a mixture of this correlation and $I_y S_z$ (Figure 19*b*):

$$\frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) 2 I_x S_z \rightarrow \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) \cos\left(\frac{\pi \nu_I}{2J}\right) 2 I_x S_z - \frac{\Delta P_I}{2} \cos\left(\frac{\pi \nu_I}{2J}\right) \sin\left(\frac{\pi \nu_I}{2J}\right) 2 I_y S_z \quad [88]$$

Similarly, $I_y S_z$ is converted to a mixture with $I_x S_z$ (Figure 19*d*):

$$\frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) 2 I_y S_z \rightarrow \frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) \cos\left(\frac{\pi \nu_I}{2J}\right) 2 I_y S_z + \frac{\Delta P_I}{2} \sin\left(\frac{\pi \nu_I}{2J}\right) \sin\left(\frac{\pi \nu_I}{2J}\right) 2 I_x S_z \quad [89]$$

At this point it is convenient to combine the terms in Equations [88] and [89]. The two terms containing $I_y S_z$ sum to zero, and the $I_x S_z$ terms yield:

$$\frac{\Delta P_I}{2} \cos^2\left(\frac{\pi \nu_I}{2J}\right) 2 I_x S_z + \frac{\Delta P_I}{2} \sin^2\left(\frac{\pi \nu_I}{2J}\right) 2 I_x S_z = \Delta P_I I_x S_z \quad [90]$$

The terms reflecting chemical shift evolution, with frequency ν_I , have now canceled out exactly, reflecting the effects of the refocusing pulses applied between the two evolution periods. The final two pulses are applied along the y' -axis to the I -spin and along the x' -axis to the S -spin, converting $I_x S_z$ to $I_z S_y$. Thus, the effects of the full INEPT sequence on the initial I_z component of the density matrix can be summarized as:

$$\frac{\Delta P_I}{2} I_z \xrightarrow{\text{INEPT}} \Delta P_I I_z S_y \quad [91]$$

This simple result is general: Whenever an I_z component is present at the beginning of an INEPT sequence, it will be converted to $2 I_z S_y$ at the end, irrespective of how the I_z was

initially generated or whether other components are present. In the present case, we must also consider how the pulse sequence affects the initial S_z component. Since the initial S -magnetization remains aligned with either the $+z$ - or $-z$ -axis until the very end of the pulse sequence, there are no effects from the evolution periods, and the net effect of the two S -pulses, π_x and $(\pi/2)_x$, is to simply convert S_z to $-S_y$:

$$\frac{\Delta P_s}{2} S_z \xrightarrow{\text{INEPT}} -\frac{\Delta P_s}{2} S_y \quad [92]$$

Next, we calculate the evolution of observable magnetization during the data acquisition period. For this, the effects of both chemical shift and scalar coupling for both the $I_z S_y$ term and the S_y term must be considered. For the $I_z S_y$ term, scalar-coupling evolution leads to the interconversion between $I_z S_y$ and S_x :

$$\Delta P_I I_z S_y \rightarrow \Delta P_I \left(\cos(\pi J t) I_z S_y - \frac{1}{2} \sin(\pi J t) S_x \right) \quad [93]$$

where t is the evolution time. Chemical shift evolution during the data-acquisition period will lead to a further interconversion between $I_z S_y$ and $I_z S_x$, but this can be safely ignored because only observable magnetization components are important now. The S_x component, however, interconverts with S_y due to chemical shift evolution to give the following expression for the magnetization components as a function of time:

$$-\frac{\Delta P_I}{2} \sin(\pi J t) S_x \rightarrow -\frac{\Delta P_I}{2} (\cos(2\pi \nu_s t) \sin(\pi J t) S_x - \sin(2\pi \nu_s t) \sin(\pi J t) S_y) \quad [94]$$

Using trigonometric identities, the S_y term can be rewritten as:

$$\frac{\Delta P_I}{2} \sin(2\pi \nu_s t) \sin(\pi J t) S_x = \frac{\Delta P_I}{4} (\cos(2\pi(\nu_s - J/2)t) - \cos(2\pi(\nu_s + J/2)t)) S_y \quad [95]$$

and the S_x term can be written as:

$$-\frac{\Delta P_I}{2} \cos(2\pi \nu_s t) \sin(\pi J t) S_x = \frac{\Delta P_I}{4} (\sin(2\pi(\nu_s - J/2)t) - \sin(2\pi(\nu_s + J/2)t)) S_x \quad [96]$$

For the $-S_y$ term that is derived from the initial excess S -spin population, scalar coupling during the data-acquisition period gives rise to a mixture of S_y and $I_z S_x$ terms:

$$-\frac{\Delta P_s}{2} S_y \rightarrow -\frac{\Delta P_s}{2} (\cos(\pi J t) S_y - 2 \sin(\pi J t) I_z S_x) \quad [97]$$

Chemical shift evolution causes interconversion between the $I_z S_x$ term and $I_z S_y$, but both of these can be ignored since they are not observable. The S_y term interconverts with S_x according to:

$$-\frac{\Delta P_s}{2} \cos(\pi J t) S_y \rightarrow -\frac{\Delta P_s}{2} (\cos(2\pi \nu_s) \cos(\pi J t) S_y + \sin(2\pi \nu_s) \cos(\pi J t) S_x) \quad [98]$$

The S_y and S_x terms can be rewritten as:

$$-\frac{\Delta P_s}{4} (\cos(2\pi(\nu_s - J/2)t) + \cos(2\pi(\nu_s + J/2)t)) S_y \quad [99]$$

and

$$-\frac{\Delta P_s}{4} (\sin(2\pi(\nu_s - J/2)t) + \sin(2\pi(\nu_s + J/2)t)) S_x \quad [100]$$

Finally, the contributions from the initial I - and S -spin populations are combined to calculate the net evolution of observable magnetization during the data-acquisition period. Addition of Equations [95] and [99] gives the expression for the time-dependent change of the S -magnetization along the y' -axis:

$$\langle S_y(t) \rangle = \frac{\Delta P_I - \Delta P_s}{4} \cos(2\pi(\nu_s - J/2)t) - \frac{\Delta P_I + \Delta P_s}{4} \cos(2\pi(\nu_s + J/2)t) \quad [101]$$

The corresponding expression for the magnetization along the x' -axis is obtained by addition of Equations [96] and [100]

$$\langle S_x(t) \rangle = \frac{\Delta P_I - \Delta P_s}{4} \sin(2\pi(\nu_s - J/2)t) - \frac{\Delta P_I + \Delta P_s}{4} \sin(2\pi(\nu_s + J/2)t) \quad [102]$$

These expressions reveal that there are two frequency components, π radians out of phase with one another, which can be visualized as two precessing vectors, one initially oriented along the $+y$ -axis and precessing with frequency $\nu_s - J/2$, and the other initially oriented along the $-y$ -axis and precessing with frequency $\nu_s + J/2$. After Fourier transformation, the spectrum will contain an antiphase doublet, with one frequency component having an amplitude proportional to the sum of ΔP_I and ΔP_s and the other proportional to $(\Delta P_I - \Delta P_s)$. Had the spectrum been generated by a conventional $\pi/2$ -pulse to the S -spin, the two frequency components would have the same phase, but would have amplitudes proportional to ΔP_s . Since the initial population differences depend on the gyromagnetic ratios, the INEPT sequence leads to an increase in sensitivity of approximately 4-fold when applied to a ^{13}C -nucleus coupled to a proton, and an increase of 10-fold for an ^{15}N -nucleus.

This analysis has been presented in somewhat more detail than typically found in textbooks, in the hopes that it will serve as a model for those learning to carry out product operator calculations on their own. Additional insights into the physics underlying the INEPT sequence can be obtained by considering the four eigen states present at equilibrium, as suggested in one of the Questions.

SUMMARY

The analysis and vector diagrams presented here emphasize the central role of quantum-mechanical correlations in the behavior of scalar-coupled spins. The correlations can be

manipulated by pulses and evolution periods, and they must be explicitly accounted for in order to predict accurately the outcome of NMR experiments with coupled spin-pairs. The product-operator formalism is an efficient means of keeping track of the net magnetization components and the correlations for a population of spin-pairs with different wavefunctions, but, by its nature, this treatment tends to obscure the distinction between quantum-mechanical phenomena and those that represent population differences. The vector representation of the quantum correlations is intended to help clarify the distinction between observable magnetization components and correlations and can be a useful aid in visualizing the various steps in a product-operator calculation.

Supplementary Material

Refer to Web version on PubMed Central for supplementary material.

Acknowledgments

I am indebted to colleagues too numerous to list who have, over the years, tried to answer my naive questions about NMR and patiently listened as I tried to formulate some answers of my own. Special thanks are owed to Dr. Peter F. Flynn for executing the experiments shown in Figure 1, as well as for numerous helpful discussions. I am also particularly grateful to Dr. Michael F. Summers for stimulating discussions that helped clarify the difficulties in describing the behavior of coupled spin-pairs. Research in my laboratory is supported by grants from the U. S. National Institutes of Health and National Science Foundation.

QUESTIONS AND ANSWERS

Questions

1. For an isolated spin 1/2, the wavefunction $|\Psi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + \frac{1}{\sqrt{2}}|\beta\rangle$ is an eigenfunction for the $\hat{I}_x$ operator.
 - a. Draw a vector diagram that represents the average magnetization components for a pure population of spins described by this wavefunction.
 - b. Suppose that you were to prepare a pure population of spins with this wavefunction, in a vapor form, and pass them through a Stern-Gerlach filter such as shown in Figure 2. What pattern of particles would you expect to see leaving the filter?
 - c. Suppose that you were to rotate the Stern-Gerlach filter so that its magnetic field is aligned with the x -axis. What pattern would you expect to see when particles with the wavefunction $\frac{1}{\sqrt{2}}|\alpha\rangle + \frac{1}{\sqrt{2}}|\alpha\rangle$ are passed through this filter? What pattern would you expect if particles with the wavefunction $|\alpha\rangle$ were passed through the rotated filter?
2. For a coupled spin pair, consider the following wavefunctions that were defined in the text:
 - $|\Psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\alpha\rangle + \frac{i}{\sqrt{2}}|\alpha\beta\rangle$
 - $|\Psi_B\rangle = \frac{1}{\sqrt{2}}|\beta\alpha\rangle + \frac{i}{\sqrt{2}}|\beta\beta\rangle$

$$\bullet \quad |\Psi_c\rangle = \frac{1}{2}|\alpha\alpha\rangle + \frac{i}{2}|\alpha\beta\rangle + \frac{i}{2}|\beta\alpha\rangle - \frac{1}{2}|\beta\beta\rangle$$

For each of these, describe the pattern that would be generated if particles with these wavefunctions were passed through a Stern-Gerlach filter such as shown in Figure 4. Assume that the gyromagnetic ratio for the I -spin is about twice that of the S -spin, as in Figure 4b.

3. The wavefunction, $|\Psi_D\rangle = \frac{1}{\sqrt{2}}|\alpha\beta\rangle + \frac{1}{\sqrt{2}}|\beta\alpha\rangle$, was introduced in the text without any explanation of where it might come from. The important properties of spin-pairs with this wavefunction are:

- There is no net average magnetization in any direction
- There are three non-zero correlations: $\langle I_x S_x \rangle = 1/4$, $\langle I_y S_y \rangle = 1/4$ and $\langle I_z S_z \rangle = -1/4$.

Devise a pulse sequence that will convert the wavefunction $|\alpha\alpha\rangle$ to a wavefunction that has these properties. You should be able to solve this problem using the vector diagrams introduced in the text. If, however, you analyze the problem using the *Mathematica* package, you will likely find that the resulting wavefunction does not have exactly the same mathematical form as $|\Psi_D\rangle$, but your result should differ from $|\Psi_D\rangle$ only by a complex constant of multiplication.

4. The INEPT pulse-sequence discussed in the text can also be analyzed, without using the product operator formalism, by following the effects on each of the four eigen states that represent the equilibrium distribution: $|\alpha\alpha\rangle$, $|\alpha\beta\rangle$, $|\beta\alpha\rangle$ and $|\beta\beta\rangle$. For this purpose it is convenient to assume that the Larmor frequencies of the two spins match the corresponding rotating-frame frequencies, so that the refocussing pulses can be eliminated and leaving a simplified version of the experiment:

$$\begin{array}{l} I: \quad (\pi/2)_x \xrightarrow{\tau=1/(2J)} (\pi/2)_y \\ S: \quad \xrightarrow{\tau=1/(2J)} (\pi/2)_x \rightarrow \text{data acquisition} \end{array}$$

For each of the starting states, use vector diagrams to describe the average magnetization components that are present after each of the steps. Then, sum up the net signal expected from the S -spins, weighting each of the contributions according to the initial populations in the equilibrium distribution. How does this signal compare with the one expected from a simple one dimensional experimental experiment in which a $(\pi/2)_x$ -pulse is applied to the S -spins and the S -signal is recorded?

Do the two treatments of the INEPT sequence seem equivalent? Which approach gives *you* more physical insight into the experiment?

Answers

1. a. Since the wavefunction is an eigenfunction for the $\hat{I}_x$ operator, the measurement of I_x must either always yield $1/2$ or always yield $-1/2$, so that the average magnetization lies entirely along the positive or negative x -axis. As it happens, this wavefunction is associated with the eigenvalue $1/2$, so that the average magnetization is aligned with the positive x -axis.

- b. Because the average magnetization is perpendicular to the z -axis, the average outcome of measuring I_z must be zero. If I_z is measured using a Stern-Gerlach filter, half of the particles will be diverted upwards and half downwards.
- c. With the Stern-Gerlach filter rotated so that it measures I_x , all of the

particles with the wavefunction $|\Psi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + \frac{1}{\sqrt{2}}|\alpha\rangle$ will be diverted in the same direction. But, if particles with the $|\alpha\rangle$ wavefunction are passed through this filter, the average value of I_x will be 0, so that half of the particles are diverted in each direction.

2. The key point is that the Stern-Gerlach filter independently measures the z -magnetization component of the two spins, but is not sensitive to the magnetization components perpendicular to the z -axis.

- For $|\Psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\alpha\rangle + \frac{i}{\sqrt{2}}|\alpha\beta\rangle$, the average I -magnetization is aligned with the positive z -axis and the S -magnetization is aligned with the y -axis. Each particle will be deflected upwards according to the positive I_z -component. Since the average S_z -magnetization is zero, measuring S_z will yield $1/2$ and $-1/2$ with equal probability, which, in the Stern-Gerlach filter, means that half of the particles will be subjected to an additional upward deflection and half to a downward deflection. In short, two beams will be generated, corresponding to the two upper beams in Figure 4.

- For $|\Psi_B\rangle = \frac{1}{\sqrt{2}}|\beta\alpha\rangle + \frac{i}{\sqrt{2}}|\beta\beta\rangle$, $\langle I_z \rangle = -1/2$ and $\langle S_z \rangle = 0$. In this case, the I -spin leads to a downward deflection for each particle, while the S -spin contributes an upward or downward deflection with equal probability. This gives rise to the lower two beams in Figure 4.

- For $|\Psi_C\rangle = \frac{1}{2}|\alpha\alpha\rangle + \frac{i}{2}|\alpha\beta\rangle + \frac{i}{2}|\beta\alpha\rangle - \frac{1}{2}|\beta\beta\rangle$, the z -magnetization components for both spins are zero, meaning that upward and downward deflections for both are equally probable. In addition, there is no correlation between I_z and S_z . (You should be able to demonstrate this result using vector diagrams, or it can be calculated using the *Mathematica* notebook.) The Stern-Gerlach filter will separate the particles into all four beams.

3. The following pulse sequence, starting with $|\alpha\alpha\rangle$ and assuming no chemical shift evolution, will generate a wavefunction with the desired properties:

$$\begin{array}{l} I: (\pi/2)_x \xrightarrow{\tau=1/(2J)} (\pi/2)_y \\ S: (\pi/2)_x \xrightarrow{\tau=1/(2J)} \end{array}$$

4. a. The changes in the observable magnetization components due to the simplified INEPT sequence are shown for each of the starting states in Figure 20. Note the following

- For the spin pairs that begin in either the $|\alpha\beta\rangle$ state or the $|\beta\beta\rangle$ state, the net effect of the pulses is the same as if a simple $(\pi/2)_{x,S}$ -pulse were applied.

- For the spin pairs that begin in the $|\alpha\alpha\rangle$ state, the result is the same as if a $(\pi/2)_{x,S}$ -pulse had been applied to the $|\alpha\alpha\rangle$ state.
- For the spin pairs that begin in the $|\beta\alpha\rangle$ state, the result is the same as if a $(\pi/2)_{x,S}$ -pulse had been applied to the $|\alpha\alpha\rangle$ state.

In essence, the effect of the pulse sequence is to exchange the populations that begin in the $|\alpha\alpha\rangle$ and $|\beta\alpha\rangle$ states and then apply a $(\pi/2)_{x,S}$ -pulse to the entire population.

During the data-acquisition period, the S -magnetization will evolve with two frequencies, $\nu_S - J/2$ and $\nu_S + J/2$. For the frequency, $\nu_S + J/2$, the strength of the signal will depend on the populations that began in the $|\alpha\alpha\rangle$ and $|\beta\beta\rangle$ states. Using the definitions from Equation [33], the signal with frequency $\nu_S + J/2$ will have an intensity that is proportional to:

$$f_{aa} - f_{bb} = \frac{1}{2}(\Delta P_I + \Delta P_S) \quad [103]$$

Compare this with the expected intensity when a simple $(\pi/2)_{x,S}$ -pulse is applied to the equilibrium population. In this case the signal intensity for the frequency $\nu_S + J/2$ depends on the initial concentrations of spin pairs in the $|\alpha\alpha\rangle$ and $|\alpha\beta\rangle$ states:

$$f_{ba} - f_{bb} = \frac{1}{2}(\Delta P_S) \quad [104]$$

Using the INEPT sequence, the equilibrium population difference for the I -spin has, in effect, been added to that of the S -spin, thereby increasing the intensity of the signal. For the frequency $\nu_S - J/2$, the signal following a $(\pi/2)_{x,S}$ -pulse to the equilibrium population is proportional to:

$$f_{aa} - f_{ab} = \frac{1}{2}(\Delta P_S) \quad [105]$$

Thus, the two frequency components have the same intensity in the simple experiment. With the INEPT experiment, however, the intensity for the $\nu_S - J/2$ depends on the initial concentrations of the $|\beta\alpha\rangle$ and $|\alpha\beta\rangle$ states, according to:

$$f_{ba} - f_{ab} = \frac{1}{2}(\Delta P_S - \Delta P_I) \quad [106]$$

For this signal the population difference for the I -spin has been subtracted from that of the S -spin, giving rise to a negative signal (assuming that $\Delta P_I > \Delta P_S$). Still, the absolute value of the signal intensity is greater than that obtained in the simple experiment. The net result of the INEPT sequence is an anti-phase doublet with enhanced intensity, and slightly different intensities for the two components.

The results obtained in this way are equivalent to those obtained using the product operator formalism. The expressions above differ from the earlier ones by a factor of 2 only because they do not include the quantum-spin number $1/2$ that is incorporated in the expressions for average magnetization.

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Biography

David Goldenberg received an A.B. degree from Whitman College in 1976 (Chemistry and Mathematics) and a Ph.D. from the Massachusetts Institute of Technology in 1981 (Biology). He then spent four years as a postdoctoral fellow at the Medical Research Laboratory of Molecular Biology, in Cambridge, UK, before moving to the University of

Utah, where he is currently a Professor of Biology. His research interests lie in the areas of protein folding, dynamics and function. In the mid 1990s, it became clear to him that NMR spectroscopy was the way forward in the study of these subjects, and he undertook, rather late in life, the task of learning the fundamentals of magnetic resonance. His own inability to grasp the physical significance of the standard treatments of scalar-coupled spins led to the approach presented in this article. He is currently working on a textbook that incorporates these ideas and is intended as a primer to the more advanced texts currently available.

Figure 1.

Heteronuclear NMR experiments demonstrating some of the special properties of a scalar-coupled spin-pair. The left-most panel in each row represents the pulse sequence, with τ representing an adjustable delay time between the initial pulse and the data acquisition period. Vector diagrams represent the expected observable magnetization components at the time of data acquisition for each experiment, with τ equal to 0 or $1/(2J)$, where J is the scalar coupling constant. The Larmor frequencies of both nuclei are assumed to match the corresponding reference frequencies. Experimental spectra are shown for CHCl_3 , with ^{13}C serving as the observed S -nucleus.

Figure 2.
The Stern-Gerlach experiment.

Figure 3.

Vector representations of the average magnetization for a population of spin-1/2 nuclei. (a) Projections of the average magnetization on the x -, y - and z -axes. (b) Vector representation of a pulse in the rotating frame (c) Precession of average magnetization in a uniform field.

Figure 4. Hypothetical Stern-Gerlach experiments for scalar-coupled pairs of spin-1/2 nuclei. (a) Predicted results for an unpolarized sample of spin-pairs composed of nuclei with equal gyromagnetic ratios. (b) Predicted results for an unpolarized sample if $\gamma_I \approx 2\gamma_S$. (c) Expected behavior of spins with wavefunction $|\Psi_D\rangle = 1/\sqrt{2}|\alpha\beta\rangle - 1/\sqrt{2}|\beta\alpha\rangle$, assuming $\gamma_I \approx 2\gamma_S$.

Figure 5.

Vector representations of the effects of $(\pi/2)_x$ -pulses on the wavefunctions $|\alpha\alpha\rangle$ (a and c) and $|\beta\alpha\rangle$ (b). In (a) and (b), the pulse is selective for the S -spin, while a non-selective pulse is depicted in (c). The resulting wavefunctions $|\Psi_A\rangle$, $|\Psi_B\rangle$ and $|\Psi_C\rangle$ are superpositions of the basis wavefunctions.

Figure 6.

Vector representations of the average magnetization components (a) and correlations (b)

associated with a scalar coupled spin-pair with the wavefunction $|\Psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\alpha\rangle + \frac{i}{\sqrt{2}}|\beta\alpha\rangle$.

Figure 7.

Vector representations of the correlations associated with the wavefunction

$|\Psi_D\rangle = \frac{1}{\sqrt{2}}|\alpha\beta\rangle + \frac{1}{\sqrt{2}}|\beta\alpha\rangle$. All of the average observable magnetization components are zero, meaning that the two possible outcomes, $1/2$ or $-1/2$, for measurements of the magnetization of either spin are expected with equal probability. However, the measurements for the two spins in each pair are correlated, so that $I_x S_x = 1/4$, $I_y S_y = 1/4$ and $I_z S_z = -1/4$.

Figure 8.

Vector diagrams representing the effects of pulses on the correlation initially associated with the $|\alpha\alpha\rangle$ wavefunction. A selective $(\pi/2)_x$ pulse to the S -spin rotates the S -vectors of the correlation diagram from the positive and negative z -axes to the positive and negative y' -axis, illustrating the conversion from an $I_z S_z$ correlation to an $I_y S_z$ correlation. A subsequent $(\pi/2)_x$ pulse to the I -spin converts the correlation to $I_y S_y$.

Figure 9. Vector representations of the precession of magnetization for pure populations of spin-pairs in the $|\Psi_A\rangle$ and $|\Psi_B\rangle$ states.

Figure 10.

Vector representations of the precession of magnetization for a pure population of spin-pairs in the $|\Psi_C\rangle$ state. For clarity, the precessing S - and I -magnetization components are shown separately.

Figure 11. Vector representation of the evolution of S_y -magnetization when ν_S equals the rotating frame frequency.

Figure 12.

Vector representations of the interconversion of observable transverse magnetization and correlations. In (a), ν_S matches the rotating frame frequency, and S_y interconverts only with $I_z S_x$. In (b) the average precession frequency does not match the reference frequency and S_y interconverts with S_x -magnetization as well as both $I_z S_x$ - and $I_z S_y$ -correlations. In both cases, the observable magnetization is converted completely to correlations at intervals of $t = 1/(2J)$.

Figure 13.

Vector representations of the evolution of correlations with two transverse components. Each component precesses at the Larmor frequency of the corresponding spin, and the precession frequencies are independent of the scalar coupling constant. The contributions of the four transverse correlations ($I_x S_x$, $I_x S_y$, $I_y S_x$ and $I_y S_y$) are calculated from Equations [31] and represent the products of the projections of the vectors on the corresponding axes.

Figure 14.

Rules for calculating the effects of pulses on operator-basis terms.

Figure 15.
Rules for calculating the effects of chemical-shift evolution on operator-basis terms.

Figure 16.

Rules for calculating the effects of evolution under the influence of scalar coupling.

Figure 17.

Graphical product-operator analysis of the demonstration experiments shown in Figure 1. (a) Conversion of S_z -magnetization to S_y -magnetization by a $(\pi/2)_x$ -pulse to the S -spin. (b) Evolution of the S_y -magnetization to generate an $I_z S_x$ -correlation. (c) Conversion of the $I_z S_x$ -correlation to an $I_y S_x$ -correlation, which does not change with time if the Larmor frequencies match the reference frequencies for the two nuclei. Conversion of the $I_z S_x$ -correlation into a transverse correlation accounts for the ability of a pulse to the I -spin to eliminate the signal from the S -spin. (d) Evolution of I_y -magnetization to generate an $I_x S_z$ -correlation. This correlation is unaffected by an I -pulse along the x' -axis.

Figure 18.

Graphical product-operator analysis of the first portion of the INEPT sequence. (a) Chemical-shift evolution of the I_y component during the first delay period, following the initial $(\pi/2)_x$ -pulse to the I -spin. (b) Scalar-coupling evolution of the I_y component during the first delay period, followed by the π_x -pulses to the I - and S -spins. (c) Scalar-coupling evolution of the I_x component during the first delay period, followed by the π_x -pulses to the I - and S -spins.

Figure 19.

Graphical product-operator analysis of the second portion of the INEPT sequence. (a) Scalar-coupling evolution of the I_y and $I_x S_z$ components during the second delay period, following the refocusing pulses illustrated in Figure 18(b). (b) Chemical-shift evolution of the $I_x S_z$ component (derived from the scalar-coupling evolution illustrated in panel a) during the second delay period. (c) Scalar-coupling evolution of the I_x and $I_y S_z$ components during the second delay period, following the refocusing pulses illustrated in Figure 18(c). (d) Chemical-shift evolution of the $I_y S_z$ component (derived from the scalar-coupling evolution illustrated in panel c) during the second delay period.

Figure 20. Vector diagrams for the answer to Question 4, showing the changes in the magnetization associated with each of the starting states in the simplified INEPT sequence.