Statistical model of clutter suppression in tissue harmonic imaging

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A statistical model is developed for the suppression of clutter in tissue harmonic imaging (THI). Tissue heterogeneity is modeled as a random phase screen that is characterized by its correlation length and variance. With the autocorrelation function taken to be Gaussian and for small variance, statistical solutions are derived for the mean intensities at the fundamental and second-harmonic frequencies in the field of a focused sound beam that propagates through the phase screen. The statistical solutions are verified by comparison with ensemble averaging of direct numerical simulations. The model demonstrates that THI reduces the aberration clutter appearing in the focal region regardless of the depth of the aberrating layer, with suppression of the clutter most effective when the layer is close to the source. The model is also applied to the reverberation clutter that is transmitted forward along the axis of the beam. As with aberration clutter, suppression of such reverberation clutter by THI is most pronounced when the tissue heterogeneity is located close to the source.

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I. INTRODUCTION

Tissue harmonic imaging (THI) is a modality used in medical ultrasound that relies on nonlinear propagation effects, particularly second-harmonic generation, to reduce the overall level of clutter due to phase aberration artifacts and reverberation noise.¹ Publications on THI began with a trio of papers in 1997 by Averkiou *et al.*,² Ward *et al.*,³ and Christopher.⁴ These and more recent papers^{5–12} (see also the list of citations in Ref. 8) report numerical and experimental results illustrating the benefits of THI and aiming to identify the underlying physical mechanisms. Of particular significance is an analytical investigation by Bradley¹³ that isolated and explained the physical mechanisms responsible for THI's impressive suppression of clutter.

Bradley's approach was to employ a perturbation method in which the acoustic Mach number and the variation in tissue heterogeneity are taken to be the small ordering parameters. Clutter in the field of the second harmonic appears in this model at third order. Aberration clutter was found to result from the second-harmonic generation due to nonlinear interaction of the undeformed component of the fundamental field, which appears at first order, and the deformed component of the fundamental field, which appears at second order. These two interacting components correspond to the incident and scattered fields in the Born approximation. Bradley showed that because the interacting components propagate in different directions, suppression of aberration clutter results from the shortened nonlinear interaction region and phase mismatch

that are inherent to noncollinear interactions of directional sound beams. Reverberation clutter was identified as linear scattering of the second-order undeformed component of the second harmonic. It was shown that suppression of this reverberation clutter is especially pronounced when the dominant heterogeneity is located close to the source because of the growth of the undeformed component of the second harmonic beyond the region of substantial heterogeneity.

The deterministic model developed by Bradley to explain these physical processes was based on elementary representations of tissue heterogeneity. A sinusoidal phase screen was used to model aberration, representing a single spatial frequency component in isolation from the continuous distribution associated with a random phase distribution. A generic scattered field function was used to model reverberation.

Varslot *et al.*¹² subsequently used multiple random phase screens to approximate an extended heterogeneous region in a numerical investigation of THI. Eight equally spaced phase screens were used to represent a heterogeneous layer occupying the first third of the propagation path from the source to the focal plane. Results are presented for propagation through a single realization of the layer. They concluded that aberrations of the second harmonic are produced at all depths, and in particular beyond the heterogeneous layer due to the aberrated fundamental. This conclusion is consistent with Bradley's model of aberration clutter.

In the present work, a heterogeneous layer of tissue is modeled as a single random phase screen that is characterized by the correlation length and variance of the heterogeneity. The phase screen can be at any location between the source and focal planes. With the autocorrelation function for the tissue heterogeneity taken to be Gaussian and for small random phase variations, statistical solutions are derived for the mean intensities of the fundamental and second-harmonic fields.

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The statistical solutions are verified in comparison with ensemble averaging of direct numerical simulations. Figures of merit similar to those introduced by Bradley are used to quantify the suppression of clutter due to aberration and reverberation individually.

Before proceeding to the analysis, we address the validity of using a phase screen to model propagation through a heterogeneous layer of tissue. Mast *et al.*^{14,15} performed a thorough investigation of this problem, comparing approximations based on both amplitude and phase screens with two-dimensional (2D) numerical simulations of propagation through anatomically realistic models of the abdominal wall. They concluded that while phase screens provide reasonable agreement with numerically simulated arrival times, amplitude screens were found deficient due to scattering that occurs within the layers.

Although a complete model of propagation through a heterogeneous layer of tissue requires both amplitude and phase variations to be taken into account, in the present paper only phase variations are considered. The justification is that the analysis presented here is restricted to propagation through a layer with weak heterogeneity. As discussed further in Sec. III, under this condition the differences in scattering due to amplitude and phase variations are negligible in the paraxial region of a sound beam. While the abdominal wall may not be weakly heterogeneous, the underlying physical mechanisms in THI nevertheless apply to the wave field perturbations that are taken into account here.

Selected results from the present investigation of aberration clutter have been published previously by the authors.¹⁶

II. FIELD INTEGRALS

Nonlinear propagation of a sound beam along the z axis in a homogeneous medium is modeled using the following augmented Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation:¹⁷

$$\frac{\partial p}{\partial z} = \frac{c_0}{2} \int_{-\infty}^{\tau} (\nabla_{\perp}^2 p) d\tau' + \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p^2}{\partial \tau} + L_{\tau}(p). \quad (1)$$

This equation is used to describe the field in the medium on both sides of the phase screen, which is inserted along the path of the beam. Here, p is the acoustic pressure, $\tau = t - z/c_0$ is the retarded time, c_0 is a reference sound speed, $\nabla_{\perp}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in the plane perpendicular to the beam axis, and ρ_0 and β are the density and coefficient of nonlinearity, respectively, of the medium. Frequency dependent attenuation and dispersion are taken into account by the linear operator L_{τ} . For a thermoviscous medium one may write $L_{\tau} = (\delta/2c_0^3)\partial^2/\partial \tau^2$, where δ is the diffusivity of sound. For tissue, L_{τ} is typically characterized in the frequency domain by an attenuation coefficient obeying a power law for the frequency dependence, together with its corresponding dispersion law.¹⁸

A solution of Eq. (1) is sought in the form of the simple perturbation $p = p_1 + p_2$, where p_1 represents the pressure at the fundamental angular frequency ω , p_2 the second harmonic at frequency 2ω , and $|p_2| \ll |p_1|$ is assumed throughout the nonlinear interaction region. With

$$p_n(x, y, z, \tau) = \text{Re}\{q_n(x, y, z)e^{jn\omega\tau}\} \\ = \frac{1}{2}q_n(x, y, z)e^{jn\omega\tau} + \text{c.c.} \quad (2)$$

($n = 1, 2$), L_{τ} is defined by the relations¹⁷

$$L_{\tau}(e^{jn\omega\tau}) = -\tilde{\alpha}_n e^{jn\omega\tau}, \quad \tilde{\alpha}_n = \alpha_n + jn\omega(c_n^{-1} - c_0^{-1}), \quad (3)$$

where α_n and c_n are the attenuation coefficient and phase speed, respectively, at frequency $n\omega$. Equation (1) then yields the following two equations to be solved in succession:¹⁹

$$\frac{\partial q_1}{\partial z} + \frac{j}{2k} \nabla_{\perp}^2 q_1 + \tilde{\alpha}_1 q_1 = 0, \quad (4)$$

$$\frac{\partial q_2}{\partial z} + \frac{j}{4k} \nabla_{\perp}^2 q_2 + \tilde{\alpha}_2 q_2 = \frac{\beta k}{2\rho_0 c_0^2} q_1^2, \quad (5)$$

where $k = \omega/c_0$ is the wavenumber at the fundamental frequency.

For a source that radiates only at frequency ω the boundary conditions on Eqs. (4) and (5) are expressed as

$$q_1(\mathbf{r}, 0) = q_{10}(\mathbf{r}), \quad q_2(\mathbf{r}, 0) = 0, \quad (6)$$

where $\mathbf{r} = (x, y)$. In the case of a phase screen located immediately in front of the source, which is the geometry used in Sec. IV to model aberration clutter, the random phase variation is included in $q_{10}(\mathbf{r})$. For the more general cases of aberration and reverberation clutter due to forward scatter from a phase screen at an arbitrary location between the source and focal planes, as considered in Secs. V and VI, the solutions are obtained via straightforward transformations of those satisfying the boundary conditions in Eq. (6).

The integral solutions of Eqs. (4) and (5) that satisfy Eq. (6) are^{20,21}

$$q_1(\mathbf{r}, z) = \frac{jke^{-\tilde{\alpha}_1 z}}{2\pi z} \int_{\mathbf{r}'} q_{10}(\mathbf{r}') \exp\left(-\frac{jk|\mathbf{r} - \mathbf{r}'|^2}{2z}\right) d\mathbf{r}', \quad (7)$$

$$q_2(\mathbf{r}, z) = -\frac{\beta k^3 e^{-\tilde{\alpha}_2 z - jkr^2/z}}{8\pi^2 \rho_0 c_0^2 z} \int_{\mathbf{r}_c} \int_{\mathbf{r}_d} q_{10}\left(\mathbf{r}_c + \frac{\mathbf{r}_d}{2}\right) q_{10}\left(\mathbf{r}_c - \frac{\mathbf{r}_d}{2}\right) \\ \times H(r_d, z) \exp\left(-\frac{jkr_c^2}{z} + \frac{j2\mathbf{k}\mathbf{r} \cdot \mathbf{r}_c}{z}\right) d\mathbf{r}_c d\mathbf{r}_d, \quad (8)$$

where $r_{c,d} = |\mathbf{r}_{c,d}|$ and

$$H(r_d, z) = \int_0^z \exp\left(-\frac{jkr_d^2}{4z'} + (\tilde{\alpha}_2 - 2\tilde{\alpha}_1)z'\right) \frac{dz'}{z'}. \quad (9)$$

Note that the integral for $q_2(\mathbf{r}, z)$ depends only the source function $q_{10}(\mathbf{r})$, i.e., it does not depend explicitly on the solution for $q_1(\mathbf{r}, z)$, even though the second harmonic is generated by the cumulative distortion of the beam as it propagates. This formulation results from the fact that the propagation path is through a homogeneous medium, and it facilitates subsequent calculations.

Note also the manner in which attenuation and dispersion, which are embodied by the coefficients $\tilde{\alpha}_n$, enter into the solutions. It can be seen that the beam profile (dependence on $\mathbf{r}$ in a plane at distance z from the source) of the fundamental is unaffected, and that of the second harmonic is affected only through the dependence of $H(r_d, z)$ on the combination $(\tilde{\alpha}_2 - 2\tilde{\alpha}_1)$. Since the dependence of attenuation on frequency in soft tissue is close to linear (with $\alpha_2 - 2\alpha_1 = 0$ if the relation is exactly linear), and the difference between the sound speeds of the fundamental and second harmonic is very small (less than $\sim 0.1\%$ for soft tissue at frequencies of several megahertz¹⁸), the effect of the factor $(\tilde{\alpha}_2 - 2\tilde{\alpha}_1)z'$ in Eq. (9) is negligible at frequencies and over propagation distances relevant to imaging in the human body. For the purposes of the present investigation, it is therefore reasonable to omit the factor $(\tilde{\alpha}_2 - 2\tilde{\alpha}_1)z'$, doing which yields

$$q_2(\mathbf{r}, z) = -\frac{\beta k^3 e^{-\tilde{\alpha}_2 z - jkr^2/z}}{8\pi^2 \rho_0 c_0^2 z} \int_{\mathbf{r}_c} \int_{\mathbf{r}_d} q_{10}\left(\mathbf{r}_c + \frac{\mathbf{r}_d}{2}\right) q_{10}\left(\mathbf{r}_c - \frac{\mathbf{r}_d}{2}\right) \times E_1\left(\frac{jkr_d^2}{4z}\right) \exp\left(-\frac{jkr_c^2}{z} + \frac{j2k\mathbf{r} \cdot \mathbf{r}_c}{z}\right) d\mathbf{r}_c d\mathbf{r}_d, \quad (10)$$

where

$$E_1(\chi) = \int_1^\infty \frac{e^{-\chi t}}{t} dt \quad (11)$$

is the exponential integral. A lossless axisymmetric form of Eq. (10) was derived previously by Berntsen *et al.*²²

Ultimately of interest is the mean intensity of the fundamental and second harmonic,

$$\langle I_n(\mathbf{r}, z) \rangle = \frac{\langle |q_n(\mathbf{r}, z)|^2 \rangle}{2\rho_0 c_0} = \frac{\langle q_n(\mathbf{r}, z) q_n^*(\mathbf{r}, z) \rangle}{2\rho_0 c_0}, \quad (12)$$

where the asterisk denotes the complex conjugate, the brackets $\langle \cdot \rangle$ denote the expected value, and $n = 1, 2$. From Eqs. (7) and (10), the integral solutions for the mean intensities are

$$\langle I_1(\mathbf{r}, z) \rangle = \frac{k^2 e^{-2\alpha_1 z}}{8\pi^2 \rho_0 c_0 z^2} \int_{\mathbf{r}_a} \int_{\mathbf{r}_b} \langle q_{10}(\mathbf{r}_a) q_{10}^*(\mathbf{r}_b) \rangle \times \exp\left(-\frac{jk|\mathbf{r} - \mathbf{r}_a|^2}{2z} + \frac{jk|\mathbf{r} - \mathbf{r}_b|^2}{2z}\right) d\mathbf{r}_a d\mathbf{r}_b, \quad (13)$$

$$\begin{aligned} \langle I_2(\mathbf{r}, z) \rangle &= \frac{\beta^2 k^6 e^{-2\alpha_2 z}}{128\pi^4 \rho_0^3 c_0^5 z^2} \int_{\mathbf{r}_{c1}} \int_{\mathbf{r}_{d1}} \int_{\mathbf{r}_{c2}} \int_{\mathbf{r}_{d2}} E_1\left(\frac{jkr_{d1}^2}{4z}\right) \\ &\times E_1\left(-\frac{jkr_{d2}^2}{4z}\right) \left\langle q_{10}\left(\mathbf{r}_{c1} + \frac{\mathbf{r}_{d1}}{2}\right) q_{10}\left(\mathbf{r}_{c1} - \frac{\mathbf{r}_{d1}}{2}\right) \right. \\ &\quad \times q_{10}^*\left(\mathbf{r}_{c2} + \frac{\mathbf{r}_{d2}}{2}\right) q_{10}^*\left(\mathbf{r}_{c2} - \frac{\mathbf{r}_{d2}}{2}\right) \rangle \\ &\times \exp\left(-\frac{jk(r_{c1}^2 - r_{c2}^2)}{z} + \frac{j2k\mathbf{r} \cdot (\mathbf{r}_{c1} - \mathbf{r}_{c2})}{z}\right) \\ &\times d\mathbf{r}_{c1} d\mathbf{r}_{d1} d\mathbf{r}_{c2} d\mathbf{r}_{d2}. \end{aligned} \quad (14)$$

The expected value operators apply only to the products of the source function q_{10} and its complex conjugate because all other quantities are deterministic.

III. PHASE SCREEN

A heterogeneous layer of tissue having small but finite thickness Δz is modeled as a plane phase screen with infinitesimal thickness oriented perpendicular to the axis of a beam propagating in the z direction. The phase screen modifies transmission at angular frequency ω by multiplying the complex pressure by a transmission function $T(\mathbf{r}) = e^{j\phi(\mathbf{r})}$ where ϕ is a random function of $\mathbf{r}$. The phase aberration ϕ in the plane $z = z_{ph}$ is expressed as²³

$$\phi(\mathbf{r}) = k \int_{z_{ph}}^{z_{ph} + \Delta z} [\langle n(\mathbf{r}, z) \rangle - n(\mathbf{r}, z)] dz, \quad (15)$$

where $n(\mathbf{r}, z) = c_0/c(\mathbf{r}, z)$ is the index of refraction and $c(\mathbf{r}, z)$ the sound speed in the layer.

The mean and variance of ϕ at frequency ω are, respectively,

$$\langle \phi(\mathbf{r}) \rangle = 0, \quad \langle \phi^2(\mathbf{r}) \rangle = \sigma^2. \quad (16)$$

The phase aberration is assumed to be an isotropic random field variable with a Gaussian autocorrelation function,

$$R_\phi(\mathbf{r}_1, \mathbf{r}_2) = \langle \phi(\mathbf{r}_1) \phi(\mathbf{r}_2) \rangle = \sigma^2 e^{-|\mathbf{r}_1 - \mathbf{r}_2|^2/l^2}, \quad (17)$$

where l is the correlation length of the heterogeneity. Use of Eq. (17) does not necessarily mean that ϕ possesses Gaussian statistics. In fact, it will not be required to identify a specific probability density function in the analysis that follows. However, random media in general²³ and tissue heterogeneity in particular^{24,25} often possess autocorrelation functions that are approximately Gaussian, and this assumption facilitates obtaining analytical results.

The phase screen thus represents a thin layer that wrinkles wavefronts transmitted through it due to the random variations in sound speed within the layer. Variations in sound speed can be caused by variations in either compressibility or density. Compact regions of different compressibility result in monopole scattering, and regions of different density produce dipole scattering.²⁶ The axis of the dipole scattering coincides approximately with the beam axis, and therefore the angular distribution of the dipole field is relatively uniform within the paraxial region of the beam. Consequently, scattered fields resulting from variations in compressibility and density are indistinguishable within the parabolic approximation underlying Eq. (1). It is therefore sufficient to model weak forward scattering by heterogeneities using a phase screen that accounts for only variations in sound speed.

IV. ABERRATION CLUTTER WITH $z_{ph} = 0$

Aberration clutter is associated with forward scatter by the phase screen, and when the phase screen is in the source plane, $z_{ph} = 0$, the fields of the fundamental and second harmonic are given by Eqs. (13) and (14) without modification. The more complicated case of aberration clutter involving forward scatter produced by a phase screen at an arbitrary location $0 < z_{ph} < d$ between the source and focal planes is

analyzed in Sec. V. The case $z_{\text{ph}} = 0$ is also of practical significance because heterogeneity giving rise to aberration clutter is often most pronounced within the body wall very close to the source. Additionally, it is demonstrated in Sec. V B that aberration clutter in the focal plane is greatest when the phase screen is next to the source, while this is also found to be the situation in which THI is most effective in suppressing the clutter.

Most of the integrals can be evaluated analytically for the case of a Gaussian source and small random phase variations. The following boundary condition is therefore used to describe a focused, axisymmetric source with Gaussian amplitude shading, including the effect of the phase screen:

$$q_{10}(\mathbf{r}) = p_0 \exp\left(-\left(1 - jG\right)\frac{r^2}{a^2} + j\phi(\mathbf{r})\right), \quad (18)$$

where p_0 is the peak acoustic pressure amplitude, a is the effective source radius, $G = ka^2/2d$ is the focusing gain, and d is the focal length.¹⁹

A. Fundamental mode

To evaluate Eq. (13) following substitution of Eq. (18), begin by considering the term

$$\begin{aligned} \langle q_{10}(\mathbf{r}_a)q_{10}^*(\mathbf{r}_b) \rangle \\ = p_0^2 e^{-(r_a^2 + r_b^2)/a^2 + jG(r_a^2 - r_b^2)/a^2} \langle e^{j[\phi(\mathbf{r}_a) - \phi(\mathbf{r}_b)]} \rangle. \end{aligned} \quad (19)$$

The expected value on the right-hand side can be evaluated by restricting consideration to small random phase variations. Taylor series expansion of $e^{j[\phi(\mathbf{r}_a) - \phi(\mathbf{r}_b)]}$ about $\phi = 0$, retaining terms through quadratic order in ϕ , and making use of Eqs. (16) and (17) yields

$$\langle e^{j[\phi(\mathbf{r}_a) - \phi(\mathbf{r}_b)]} \rangle = 1 - \sigma^2 + \sigma^2 e^{-|\mathbf{r}_a - \mathbf{r}_b|^2/l^2}. \quad (20)$$

Equation (13) can now be integrated to obtain, at leading order in the beam deformation due to phase aberration,

$$\langle I_{1,\omega} \rangle = (1 - \sigma^2)I_{1,\omega}^{\text{ch}} + \sigma^2 I_{1,\omega}^{\text{sc}}, \quad (21)$$

where

$$I_{1,\omega}^{\text{ch}} = \frac{G^2 I_{01} e^{-2A_1 \zeta}}{\zeta^2 + G^2(1 - \zeta)^2} \exp\left(-\frac{2G^2 \rho^2}{\zeta^2 + G^2(1 - \zeta)^2}\right), \quad (22)$$

$$\begin{aligned} I_{1,\omega}^{\text{sc}} = \frac{G^2 I_{01} e^{-2A_1 \zeta}}{(1 + 2v^2)\zeta^2 + G^2(1 - \zeta)^2} \\ \times \exp\left(-\frac{2G^2 \rho^2}{(1 + 2v^2)\zeta^2 + G^2(1 - \zeta)^2}\right) \end{aligned} \quad (23)$$

describe the undeformed (coherent) and deformed (scattered) components of the field, respectively, and $I_{01} = p_0^2/2\rho_0 c_0$ is the axial intensity of the beam at the source. The subscript ω was appended to the symbols for the intensity to distinguish

this solution from the one developed below for linear radiation by the source at frequency 2ω . It proved convenient to introduce the dimensionless coordinates (ρ, ζ) , the inhomogeneity parameter v , and the attenuation coefficient A_n ($n = 1, 2$):

$$\rho = r/a, \quad \zeta = z/d, \quad v = a/l, \quad A_n = \alpha_n d. \quad (24)$$

Without attenuation and focusing and for small σ^2 , Eqs. (21)–(23) can be shown²¹ to agree with the result obtained by expanding a solution derived by Clarke²⁷ for the mean intensity of a laser beam that propagates through turbulent atmosphere.

Because the source is axisymmetric and the random phase is isotropic, the mean intensity $\langle I_{1,\omega} \rangle$ is symmetric about the beam axis. For $v = 0$ (infinite inhomogeneity length scale, $l = \infty$, corresponding to no, or at least constant, phase aberration), Eqs. (22) and (23) reveal that $I_{1,\omega}^{\text{sc}} = I_{1,\omega}^{\text{ch}}$, and therefore $\langle I_{1,\omega} \rangle = I_{1,\omega}^{\text{ch}}$ from Eq. (21). The quantity $I_{1,\omega}^{\text{ch}}$ is the intensity of the undeformed beam that would be observed in the absence of the phase screen, and hence it is referred to as the coherent field. Its coefficient $(1 - \sigma^2)$ accounts for energy transfer from the coherent field to the scattered field, denoted by $I_{1,\omega}^{\text{sc}}$. Integration of the intensities over any plane $\zeta > 0$ shows that the ratio of the power of the scattered field to that of the coherent field is $\sigma^2/(1 - \sigma^2)$, which is approximately σ^2 from the assumption $\sigma^2 \ll 1$. In the absence of attenuation, energy is conserved by Eq. (21).

An ordering scheme may be introduced to classify the relative magnitudes of the different field components. The standard deviation σ is the characteristic amplitude of the phase aberration, and $p_0/\rho_0 c_0^2$ is the characteristic acoustic Mach number at the source, both of which are presumed to be small dimensionless quantities in the preceding analysis. These correspond to the small quantities used by Bradley¹³ in his ordering scheme for equations describing the acoustic pressure in the medium. The corresponding quantities that describe the acoustic intensity in the medium are therefore σ^2 and $I_{01}/\rho_0 c_0^3$. With each of these latter two quantities classified as first order, the coherent field $I_{1,\omega}^{\text{ch}}$ is identified as first order, the scattered field $\sigma^2 I_{1,\omega}^{\text{sc}}$ as second order, and the correction to the coherent field $-\sigma^2 I_{1,\omega}^{\text{ch}}$ also as second order.

Equations (22) and (23) reveal that both the coherent and scattered beams have Gaussian amplitude profiles, with the width of the scattered beam being greater than that of the coherent beam. The larger the value of v , the wider the scattered beam. Larger v corresponds to smaller l , meaning that the phase aberration is spatially less coherent, which results in broadening of the beam. As v increases, the axial value of the scattered field decreases.

In the focal plane ($\zeta = 1$) Eqs. (22) and (23) reduce to

$$I_{1,\omega}^{\text{ch}}(\rho, 1) = G^2 I_{01} \exp(-2A_1 - 2G^2 \rho^2), \quad (25)$$

$$I_{1,\omega}^{\text{sc}}(\rho, 1) = \frac{G^2 I_{01}}{1 + 2v^2} \exp\left(-2A_1 - \frac{2G^2 \rho^2}{1 + 2v^2}\right). \quad (26)$$

Relative to the coherent field, the beamwidth of the scattered field in the focal plane is wider by a factor of $\sqrt{1 + 2v^2}$, and the axial intensity is reduced by a factor of $(1 + 2v^2)$.

FIG. 1. Solid lines are normalized mean intensity $\langle I_{1,\omega} \rangle$ in the focal region for $G=10$ and different values of ν and σ . Dashed lines are $I_{1,\omega}^{\text{ch}}$, which is the result for no phase screen.

Figure 1 shows plots of $\langle I_{1,\omega} \rangle$ (solid lines) in the focal plane, normalized to 0 dB on the axis, for $G=10$ and two values of each σ and ν . For comparison, given a source of radius $a=1$ cm that radiates at 1.5 MHz (and such that imaging is performed at 3 MHz), measurements on human abdominal wall tissue presented by Hinkelman *et al.*²⁸ in their Table IV yield values of ν that vary from 0.8 to 2.7 and values of σ that range from 0.3 to 0.6. The dashed lines in Fig. 1 are the results for no phase screen, i.e., $I_{1,\omega}^{\text{ch}}$. Because the mean intensity corresponds to an ensemble average, there is an abrupt transition from the coherent center of the beam to the scattered field that eventually dominates off axis. The transition point in the focal plane, obtained by setting $I_{1,\omega}^{\text{ch}}(\rho, 1) = \sigma^2 I_{1,\omega}^{\text{sc}}(\rho, 1)$, is given by

$$\rho_{\text{tr}} = \frac{1}{2\nu G} \sqrt{(1+2\nu^2) \ln[(1+2\nu^2)/\sigma^2]}. \quad (27)$$

As seen from Eq. (21), increasing the standard deviation σ merely increases the contribution from $I_{1,\omega}^{\text{sc}}$ in relation to $I_{1,\omega}^{\text{ch}}$. However, changes in ν alter the structure of the scattered field, increasing its beamwidth as ν is increased. This is to be expected since increasing ν corresponds to broadening the angular spectrum of the scattered field.

This discussion of linear theory concludes with the solution for direct radiation by the source at the second harmonic, which is used below as a basis for comparison with THI because it corresponds to fundamental mode imaging with the same source (same radius a and focal length d) and at the same frequency (2ω). The solution is obtained by replacing σ with 2σ in Eq. (21), and G with $2G$ in Eqs. (22) and (23), because σ and G are each proportional to ω . The result is

$$\langle I_{1,2\omega} \rangle = (1-4\sigma^2)I_{1,2\omega}^{\text{ch}} + 4\sigma^2 I_{1,2\omega}^{\text{sc}}, \quad (28)$$

where

$$I_{1,2\omega}^{\text{ch}} = \frac{4G^2 I_{01} e^{-2A_2 \zeta}}{\zeta^2 + 4G^2(1-\zeta)^2} \exp\left(-\frac{8G^2 \rho^2}{\zeta^2 + 4G^2(1-\zeta)^2}\right), \quad (29)$$

$$I_{1,2\omega}^{\text{sc}} = \frac{4G^2 I_{01} e^{-2A_2 \zeta}}{(1+2\nu^2)\zeta^2 + 4G^2(1-\zeta)^2} \times \exp\left(-\frac{8G^2 \rho^2}{(1+2\nu^2)\zeta^2 + 4G^2(1-\zeta)^2}\right). \quad (30)$$

B. Second harmonic

Evaluation of Eq. (14) for the second harmonic following substitution of Eq. (18) is performed in a similar fashion. The auxiliary notation $\mathbf{r}_\alpha = \mathbf{r}_{c1} + \frac{1}{2}\mathbf{r}_{d1}$, $\mathbf{r}_\beta = \mathbf{r}_{c1} - \frac{1}{2}\mathbf{r}_{d1}$, $\mathbf{r}_\gamma = \mathbf{r}_{c2} + \frac{1}{2}\mathbf{r}_{d2}$, and $\mathbf{r}_\delta = \mathbf{r}_{c2} - \frac{1}{2}\mathbf{r}_{d2}$ is introduced to more compactly express the expected value of the phase terms, through quadratic order in ϕ as in Eq. (20) for the fundamental mode:

$$\begin{aligned} \left\langle e^{j[\phi(\mathbf{r}_\alpha) + \phi(\mathbf{r}_\beta) - \phi(\mathbf{r}_\gamma) - \phi(\mathbf{r}_\delta)]} \right\rangle &= 1 - 2\sigma^2 - \sigma^2(e^{-r_{d1}^2/L^2} + e^{-r_{d2}^2/L^2}) \\ &\quad + \sigma^2(e^{-|\mathbf{r}_\alpha - \mathbf{r}_\gamma|^2/L^2} + e^{-|\mathbf{r}_\beta - \mathbf{r}_\delta|^2/L^2}) \\ &\quad + e^{-|\mathbf{r}_\beta - \mathbf{r}_\gamma|^2/L^2} + e^{-|\mathbf{r}_\alpha - \mathbf{r}_\delta|^2/L^2}. \end{aligned} \quad (31)$$

Following considerable manipulation of the integrals,²¹ Eq. (14) reduces to

$$\langle I_2 \rangle = I_2^{\text{ch1}} - 2\sigma^2(I_2^{\text{ch1}} + I_2^{\text{ch2}}) + 4\sigma^2 I_2^{\text{ab}}, \quad (32)$$

where

$$I_2^{\text{ch1}} = F(\rho, \zeta) \frac{|\ln[1 - (1+j/G)\zeta]|^2}{(1+G^2)[\zeta^2 + G^2(1-\zeta)^2]}, \quad (33)$$

$$\begin{aligned} I_2^{\text{ch2}} &= F(\rho, \zeta) \\ &\times \text{Re} \left\{ \frac{\ln[1 - j\zeta(1-jG+2\nu^2)/G] \ln[1 + j\zeta(1+jG)/G]}{(1+jG)(1+2\nu^2-jG)[\zeta^2 + G^2(1-\zeta)^2]} \right\}, \end{aligned} \quad (34)$$

$$\begin{aligned} I_2^{\text{ab}} &= F(\rho, \zeta) \int_1^\infty \int_1^\infty \exp\left\{ \frac{4\nu^2 G^2 \rho^2 \zeta^2}{[\zeta^2 + G^2(1-\zeta)^2]M(\xi_1, \xi_2)} \right\} \\ &\times \frac{d\xi_1 d\xi_2}{\xi_1 \xi_2 \eta_1(\xi_1) \eta_2(\xi_2) M(\xi_1, \xi_2)}, \end{aligned} \quad (35)$$

and

$$F = I_{02} G^2 \exp\left(-2A_2 \zeta - \frac{4G^2 \rho^2}{\zeta^2 + G^2(1-\zeta)^2}\right), \quad (36)$$

$$\begin{aligned} M &= \nu^2 \zeta^2 + [\zeta^2 + G^2(1-\zeta)^2] \\ &\times \left[1 + \frac{\nu^2}{2} \left(\frac{1}{\eta_1(\xi_1)} + \frac{1}{\eta_2(\xi_2)} \right) \right], \end{aligned} \quad (37)$$

$$\eta_1 = 1 - jG \left(1 - \frac{\xi_1}{\zeta} \right), \quad \eta_2 = 1 + jG \left(1 - \frac{\xi_2}{\zeta} \right). \quad (38)$$

The reference intensity introduced in Eq. (36) is defined by $I_{02} = \beta^2 I_{01}^2 k^4 a^4 / 8 \rho_0 c_0^3$. While appearing somewhat complicated, the result requires only two numerical integrations, as compared with the eight-dimensional integral in Eq. (14). Like Eqs. (33) and (34), Eq. (35) is also a real expression, as can be seen by noting that the integrals over ξ_1 and ξ_2 are complex conjugates of one another.

As with the solution for the fundamental, separate expressions are obtained for the undeformed and deformed components of the field. Setting $v = 0$ yields $I_2^{\text{ch1}} = I_2^{\text{ch2}} = I_2^{\text{ab}}$ and thus $\langle I_2 \rangle = I_2^{\text{ch1}}$. The beamwidths of I_2^{ch1} and I_2^{ch2} are the same and independent of v , and therefore both are designated as coherent field components. As v is increased, the beamwidth of the field component resulting from the phase aberration, I_2^{ab} , becomes increasingly wider than that of the coherent field, as is the case with the linear solution for the scattered field. The increase in beamwidth is associated with the integral in Eq. (35). Whereas the coherent beam patterns are Gaussian, the beam pattern associated with the aberration field is not.

Although perhaps a subtle distinction to make, it was deemed useful for interpretation to label the deformed component of the fundamental field with superscript “sc” to designate scattering, and the deformed component of the second-harmonic field with superscript “ab” to designate aberration. The field described by I_1^{sc} is truly “scattered” by the phase screen in the traditional sense of the word, but the field described by I_2^{ab} is not. The latter is the deformed component of the second harmonic that is generated by nonlinear propagation of the total fundamental field, which involves both $I_{1,\omega}^{\text{ch}}$ and $I_{1,\omega}^{\text{sc}}$. Taken together, these two components correspond to a beam whose wavefronts are wrinkled due to propagation through the phase screen. As shown by Bradley,¹³ it is the mutual nonlinear interaction of the individual field components represented by $I_{1,\omega}^{\text{ch}}$ and $I_{1,\omega}^{\text{sc}}$ that generates the leading-order deformation due to the phase aberration.

According to the ordering procedure discussed after Eq. (24) in relation to Eq. (21), the orders of the terms in Eq. (32) are as follows. The undeformed field component I_2^{ch1} that exists in the absence of the phase screen is second order, the aberration field component $4\sigma^2 I_2^{\text{ab}}$ is third order, and the correction to the undeformed field $-2\sigma^2(I_2^{\text{ch1}} + I_2^{\text{ch2}})$ is also third order. As in linear theory, the phase screen reduces the energy in the coherent field.

Some simplification is achieved in the focal plane, where

$$I_2^{\text{ch1}}(\rho, 1) = \frac{|\ln(j/G)|^2}{1 + G^{-2}} I_{02} e^{-2A_2 - 4G^2 \rho^2}, \quad (39)$$

$$I_2^{\text{ch2}}(\rho, 1) = \text{Re} \left\{ \frac{G^2 \ln(j/G) \ln[-j(1 + 2v^2)/G]}{(1 + jG)(1 + 2v^2 - jG)} \right\} \times I_{02} e^{-2A_2 - 4G^2 \rho^2}, \quad (40)$$

$$I_2^{\text{ab}}(\rho, 1) = G^2 I_{02} e^{-2A_2 - 4G^2 \rho^2} \times \int_1^\infty \int_1^\infty \frac{\exp[4v^2 G^2 \rho^2 / M(\xi_1, \xi_2)] d\xi_1 d\xi_2}{[1 - jG(1 - \xi_1)][1 + jG(1 - \xi_2)] \xi_1 \xi_2 M(\xi_1, \xi_2)}, \quad (41)$$

FIG. 2. Solid lines are normalized mean intensity $\langle I_2 \rangle$ in the focal region for $G = 10$ and different values of v and σ . Dashed lines are I_2^{ch1} , which is the result for no phase screen.

and

$$M = 1 + v^2 + \frac{v^2}{2} \left(\frac{1}{1 - jG(1 - \xi_1)} + \frac{1}{1 + jG(1 - \xi_2)} \right). \quad (42)$$

The solid lines in Fig. 2 show $\langle I_2 \rangle$ in the focal plane for the same parameters as in Fig. 1, and the dashed lines are again the results when there is no phase aberration. Although the scattered field decays more rapidly with distance from the beam axis, the dependence on v is qualitatively the same.

C. Model validation

The validity of the formulas derived in Secs. IV A and IV B is assessed in comparison with ensemble-averaged numerical realizations of propagation through a random phase screen. The phase screens were generated as follows.³⁰ First, the following filter function $H(k_x, k_y)$ is introduced:

$$H(k_x, k_y) = \sqrt{\mathcal{F}\{R_\phi(\mathbf{r}_1 - \mathbf{r}_2)\}} = \sqrt{\mathcal{F}\{R_\phi(\mathbf{r})\}}, \quad (43)$$

where $\mathcal{F}$ is the 2D spatial Fourier transform operator, and $k_{x,y}$ are the spatial angular frequencies. For a Gaussian autocorrelation function as in Eq. (17), $\mathcal{F}\{R_\phi(\mathbf{r})\}$ may be calculated analytically if desired. Next, a 2D random noise function $\eta(\mathbf{r})$ having zero mean and standard deviation σ is taken to be the input to this filter, such that the output spectrum is

$$\hat{\phi}(k_x, k_y) = H(k_x, k_y) \mathcal{F}\{\eta(\mathbf{r})\}. \quad (44)$$

Finally, the inverse spatial Fourier transform is taken to obtain a single realization of the random phase distribution:

$$\phi(\mathbf{r}) = \mathcal{F}^{-1}\{\hat{\phi}(k_x, k_y)\}. \quad (45)$$

Two such realizations of $\phi(\mathbf{r})$, with standard deviation $\sigma = 0.2$ and the correlation function in Eq. (17), were used to

FIG. 3. Numerical simulations of the field in the focal plane for $G = 10$, $\sigma = 0.2$, and $v = 0$ (no phase screen, first column), $v = 2$ (second column), and $v = 5$ (third column). First row shows the circle corresponding to source radius $r = a$ and two realizations of the phase shading in the source plane, and subsequent rows show $I_{1,\omega}$ (second row), $I_{1,2\omega}$ (third row), and I_2 (fourth row).

generate the comparisons presented in Fig. 3. The top row shows the phase distributions for $v = 2$ (middle) and $v = 5$ (right). The circles inscribed on the plots in the top row have radius $r = a$ and are introduced to help provide a sense of the correlation length parameter $v = a/l$: $l = \infty$ for $v = 0$, $l = 0.5a$ for $v = 2$, and $l = 0.2a$ for $v = 5$. The value of the phase is indicated by the gray scale bars adjacent to the plots.

The next step is to calculate the fundamental and second-harmonic fields due to radiation through the simulated phase screens. This was accomplished numerically using the angular spectrum approach described by Landsberger and Hamilton.²⁹ Results for the Gaussian source function in Eq. (18) with $G = 10$ are shown in the second through fourth rows of Fig. 3. The second row corresponds to linear radiation at frequency ω , the third row to linear radiation at frequency 2ω , and the fourth row to nonlinear generation of the second harmonic.

The first column of Fig. 3 shows the focal beam patterns in the absence of a phase screen ($v = 0$), the second shows them for $v = 2$, and the third for $v = 5$. The second row of Fig. 3 shows clearly the increasing clutter at source frequency ω with increasing v . The third row shows that merely doubling the source frequency reduces the clutter. The fourth row shows

that, particularly for $v = 5$ (third column), the second harmonic generated nonlinearly even further reduces the clutter.

The numerical results in Fig. 3 provide the data required to perform ensemble averages for comparison with the statistical solutions. The mean value for each value of ρ was calculated by averaging the pressure around the circle in the focal plane defined by that radius. This procedure is statistically equivalent to averaging beam patterns along (say) the x axes of field distributions in the focal plane resulting from different realizations of the phase screen. The equivalence of the two averaging procedures is due to the fact that each field distribution is a stationary random process in space, and it is therefore mean ergodic.³⁰ Estimating the mean by performing a spatial average over one realization is thus equivalent to performing an ensemble average of many realizations. This is convenient in the present case because one needs to compute only a single field distribution to estimate the mean beam pattern, rather than having to perform many field calculations.

Ensemble averages obtained numerically from the data in Fig. 3 are presented as dotted lines in Fig. 4, and the statistical solutions as solid lines. Dashed lines indicate the result with no phase screen. Standard deviations of the individual

FIG. 4. Comparison of statistical solutions (solid lines) with numerical averages obtained from Fig. 3 (dotted lines). Dashed lines are solutions for no phase screen. The error bars in (d) and (h) indicate the standard deviations of the individual realizations that were averaged to produce the dotted lines in (c) and (g), respectively.

realizations that were averaged to produce Figs. 4(c) and 4(g) are shown in Figs. 4(d) and 4(h), respectively. Although the error bars can be large in the regions dominated by scattering, the averages computed for these cases converge to the statistical solutions down to at least 50 dB below the level on the axis. The agreement deteriorates gradually with increasing σ .

D. Clutter suppression

The success of THI in suppression of aberration clutter is a direct consequence of the “wings” seen in the third row of Fig. 4 being narrower than the wings seen in the second row. A quantitative measure of the suppression may be obtained by calculating the difference in the levels of the deformed field components associated with the wings in the second and third rows. This measure is labeled the clutter suppression level (CSL), which in terms of Eqs. (28) and (32) for $\langle I_{1,2\omega} \rangle$ and $\langle I_2 \rangle$, respectively, evaluated at the focus is expressed as

$$\text{CSL}(\rho) = 10 \log_{10} \frac{4\sigma^2 I_2^{\text{ab}}(\rho, 1) / \langle I_2(0, 1) \rangle}{4\sigma^2 I_{1,2\omega}^{\text{sc}}(\rho, 1) / \langle I_{1,2\omega}(0, 1) \rangle} \text{ dB}. \quad (46)$$

After the factors of $4\sigma^2$ are canceled, and since the random phase variations are assumed small ($\sigma^2 \ll 1$) with $\langle I_2 \rangle = I_2^{\text{ch}} + O(\sigma^2)$ and $\langle I_{1,2\omega} \rangle = I_{1,2\omega}^{\text{ch}} + O(\sigma^2)$, Eq. (46) is well approximated by

$$\text{CSL}(\rho) = 10 \log_{10} \frac{I_2^{\text{ab}}(\rho, 1) / I_2^{\text{ch}}(0, 1)}{I_{1,2\omega}^{\text{sc}}(\rho, 1) / I_{1,2\omega}^{\text{ch}}(0, 1)} \text{ dB}, \quad (47)$$

which depends only on G and v , not on σ . One may recast this expression as

$$\text{CSL}(\rho) = 10 \log_{10} \frac{\text{SCR}_F(\rho, 1)}{\text{SCR}_H(\rho, 1)} \text{ dB}, \quad (48)$$

where

$$\text{SCR}_F(\rho, 1) = \frac{I_{1,2\omega}^{\text{ch}}(0, 1)}{4\sigma^2 I_{1,2\omega}^{\text{sc}}(\rho, 1)}, \quad (49)$$

$$\text{SCR}_H(\rho, 1) = \frac{I_2^{\text{ch}}(0, 1)}{4\sigma^2 I_2^{\text{ab}}(\rho, 1)} \quad (50)$$

are signal-to-clutter ratios for fundamental and harmonic mode imaging. The form of Eq. (47) is thus consistent with Bradley’s definition of clutter suppression by THI.¹³

Equation (47) is plotted in Fig. 5 for different values of G and v . It can be seen that the benefit of THI improves as v increases up to $v=5$, above which value there is little improvement (the dashed lines correspond to $v=\infty$). The curves for $v=2$ and $v=5$ in Fig. 5(b) correspond to the results in the second and third rows of Fig. 4. For $v=5$, THI suppresses aberration clutter by approximately 10 dB at $\rho=0.25$ and 20 dB at $\rho=0.5$. For $v=2$ the suppression is less pronounced, approximately 4 dB at $\rho=0.2$ and 8 dB at $\rho=0.4$. The upturn in CSL observed in Fig. 5(b) for $v=2$ near $\rho=0.4$, and the positive values of CSL for $\rho > 0.8$, are of little practical significance because, as seen from Figs. 4(b) and 4(c), the level of the deformed field in this region is more than 60 dB below the level of the coherent, undeformed sound field on the axis. Similar statements apply to the upturn in Fig. 5(a) for $v=2$ near $\rho=1$.

The improvement in CSL with increasing v is consistent with Bradley’s observation that greater clutter suppression is achieved as the spatial frequency of the phase modulation increases.¹³ However, whereas Bradley’s model predicts

FIG. 5. Solid lines are CSL evaluated for different values of G and v . Dashed lines are limiting values for $v=\infty$.

clutter suppression to increase continuously and substantially with increasing spatial frequency, CSL ceases to improve appreciably for $v \gtrsim 5$. The difference is due to the use of a random phase aberration in the present model rather than the single spatial frequency in Bradley's model. While a phase screen with larger v (smaller correlation length l) generates higher spatial frequencies in the transmitted beam, the angular power spectrum remains Gaussian and centered at $(k_x, k_y) = (0, 0)$. Although the power spectrum becomes broader, a significant amount of energy in the fundamental is still transmitted through the phase screen in the general direction of the z axis, where generation of the second harmonic is most efficient. Mathematically, the limiting form of CSL is associated with v becoming sufficiently large ($v^2 \gg 1$) that the expression for M in Eq. (42) becomes proportional to v^2 . In this case, both $I_2^{\text{ab}}(\rho, 1)$ and $I_{1,2\omega}^{\text{sc}}(\rho, 1)$ are proportional to v^{-2} , and therefore the ratio of these terms in Eq. (47) is independent of v .

V. ABERRATION CLUTTER WITH $0 < z_{\text{ph}} < d$

For $z_{\text{ph}} > 0$ the ensemble average required to calculate the mean intensity in the region after the phase screen must take into account contributions to the pressure field of the second harmonic generated nonlinearly before as well as after the phase screen. It is helpful to consider first and independently the contribution due to generation of the second harmonic before the phase screen. This is done in Sec. V A. As mentioned in Sec. VI A, the results obtained in Sec. V A may be applied directly to the reverberation clutter in cases where the reverberant field propagates forward along the axis of the beam. In Sec. V B, the complete solution is developed for the aberration clutter in the focal plane, taking into account the generation of the second harmonic on both sides of the phase screen.

A. Effects of nonlinearity before phase screen

Considered here is the clutter produced by scattering of the second harmonic that is generated between the source and phase screen. The mean intensity of this scattered component is labeled I_2^{sc} . If there was no second harmonic generated beyond the phase screen, then I_2^{sc} would account for all beam deformations at frequency 2ω . This would be the case if the actual heterogeneous layer being modeled introduced substantial transmission loss at frequency ω , e.g., due to backscatter. As discussed in Sec. VI in connection with the reverberation clutter, it would also be the case if the entire signal were to become weak prior to passing through the phase screen. It is therefore useful to define CSL as in Eq. (47) but with I_2^{sc} replacing I_2^{ab} :

$$\text{CSL}(\rho) = 10 \log_{10} \frac{I_2^{\text{sc}}(\rho, 1)/I_2^{\text{ch1}}(0, 1)}{I_{1,2\omega}^{\text{sc}}(\rho, 1)/I_{1,2\omega}^{\text{ch}}(0, 1)} \text{ dB}. \quad (51)$$

The expression for $I_{1,2\omega}^{\text{sc}}(\rho, 1)$ is derived as follows. Needed first is the pressure incident on the phase screen produced by radiation in the fundamental mode at frequency 2ω . The source condition for this case, in place of Eq. (18), is

$$q_{10,2\omega}(r, 0) = p_0 \exp[-(1 - j2G)r^2/a^2]. \quad (52)$$

Substitution in Eq. (7) with $z = z_{\text{ph}}$ and the replacements $k \rightarrow 2k$, $\tilde{\alpha}_1 \rightarrow \tilde{\alpha}_2$, gives the pressure on the near side of the phase screen,¹⁹ and then multiplication by $e^{j2\phi(\mathbf{r})}$ provides the required boundary condition on the far side of the phase screen:

$$q_{10,2\omega}(\mathbf{r}, z_{\text{ph}}) = \frac{p_0 e^{-\tilde{\alpha}_2 z_{\text{ph}} + j2\phi(\mathbf{r})}}{1 - (1 + j/2G)z_{\text{ph}}/d} \times \exp\left(-\frac{(1 - j2G)r^2/a^2}{1 - (1 + j/2G)z_{\text{ph}}/d}\right). \quad (53)$$

Next, substitution of Eq. (53) in Eq. (13) with $z = d - z_{\text{ph}}$ and the replacements $k \rightarrow 2k$, $\alpha_1 \rightarrow \alpha_2$, yields in dimensionless form

$$I_{1,2\omega}^{\text{sc}}(\rho, 1) = \frac{4G^2 I_{01} e^{-2A_2}}{1 + 2v^2(1 - \zeta_{\text{ph}})^2} \times \exp\left(-\frac{8G^2 \rho^2}{1 + 2v^2(1 - \zeta_{\text{ph}})^2}\right) \quad (54)$$

for the mean intensity of the scattered field in the focal plane [following multiplication by $4\sigma^2$, as in Eq. (28)], where

$$\zeta_{\text{ph}} = \frac{z_{\text{ph}}}{d} \quad (55)$$

is the location of the phase screen expressed as a fraction of the focal length. Since $I_{1,2\omega}^{\text{ch}}(0, 1) = 4G^2 I_{01} e^{-2A_2}$ from Eq. (29), the denominator in Eq. (51) becomes

$$\frac{I_{1,2\omega}^{\text{sc}}(\rho, 1)}{I_{1,2\omega}^{\text{ch}}(0, 1)} = \frac{1}{1 + 2v^2(1 - \zeta_{\text{ph}})^2} \times \exp\left(-\frac{8G^2 \rho^2}{1 + 2v^2(1 - \zeta_{\text{ph}})^2}\right). \quad (56)$$

As the phase screen is moved toward the source ($\zeta_{\text{ph}} \rightarrow 0$), the axial value of $I_{1,2\omega}^{\text{sc}}$ decreases but the beamwidth increases and at $\zeta_{\text{ph}} = 0$ Eq. (54) reduces to Eq. (30) evaluated at $\zeta = 1$. With the screen positioned at the focal plane ($\zeta_{\text{ph}} = 1$) the field exhibits no dependence on inhomogeneity (i.e., v), and the undeformed field in the focal plane is recovered, given by Eq. (29) evaluated at $\zeta = 1$. Integration of Eq. (54) shows that the power of the scattered field is independent of ζ_{ph} (as well as G and v). From Eqs. (28) and (29) it is found that the ratio of the power of the scattered field to that of the coherent field is $4\sigma^2$.

Calculation of the scattered field in the focal plane due to a nonlinearly generated second harmonic incident on the phase screen proceeds in a similar fashion. The source condition is now Eq. (18) with $\phi = 0$ for an undeformed wave radiated at frequency ω . The undeformed, nonlinearly generated, second-harmonic field incident on the phase screen is determined by substituting the source condition in Eq. (10) with $z = z_{\text{ph}}$.¹⁹ Multiplication of the result by $e^{j2\phi(\mathbf{r})}$ yields

$$q_{20}(\mathbf{r}, z_{\text{ph}}) = \frac{jP_2 e^{-\tilde{\alpha}_2 z_{\text{ph}} + j2\phi(\mathbf{r})}}{(1-jG)} \frac{\ln[1 - (1+j/G)z_{\text{ph}}/d]}{1 - (1+j/G)z_{\text{ph}}/d} \times \exp\left(-\frac{2(1-jG)r^2/a^2}{1 - (1+j/G)z_{\text{ph}}/d}\right) \quad (57)$$

for the required boundary condition on the phase screen, where $P_2 = \beta p_0^2 k^2 d^2 / 4p_0 c_0^2$. The scattered field is third order, and therefore subsequent propagation of this field may be calculated using linear theory. Substitution of Eq. (57) for q_{10} in Eq. (13) with $z = d - z_{\text{ph}}$ and the replacements $k \rightarrow 2k$, $\alpha_1 \rightarrow \alpha_2$, gives

$$I_2^{\text{sc}}(\rho, 1) = \frac{|\ln(1 - \zeta_{\text{ph}} - j\zeta_{\text{ph}}/G)|^2}{(1 + G^{-2})[1 + v^2(1 - \zeta_{\text{ph}})^2]} I_{02} e^{-2A_2} \times \exp\left(-\frac{4G^2 \rho^2}{1 + v^2(1 - \zeta_{\text{ph}})^2}\right) \quad (58)$$

for the mean intensity of the scattered field in the focal plane. Since the coherent field I_2^{ch1} in the focal plane is obtained by setting $\zeta_{\text{ph}} = 1$, which recovers Eq. (39), one obtains $I_2^{\text{ch1}}(0, 1) = I_{02} e^{-2A_2} |\ln(jG)|^2 / (1 + G^{-2})$, and the numerator in Eq. (51) becomes

$$\frac{I_2^{\text{sc}}(\rho, 1)}{I_2^{\text{ch1}}(0, 1)} = \frac{|\ln(1 - \zeta_{\text{ph}} - j\zeta_{\text{ph}}/G)|^2}{|\ln(jG)|^2 [1 + v^2(1 - \zeta_{\text{ph}})^2]} \times \exp\left(-\frac{4G^2 \rho^2}{1 + v^2(1 - \zeta_{\text{ph}})^2}\right). \quad (59)$$

As the phase screen is moved toward the source the intensity decreases monotonically toward zero because the amplitude of the nonlinearly generated second harmonic incident on the phase screen goes to zero.

Substitution of Eqs. (56) and (59) in Eq. (51) yields

$$\text{CSL}(\rho) = \text{CSL}(0) + \frac{17.4G^2 \rho^2}{1 + 3v^2(1 - \zeta_{\text{ph}})^2 + 2v^4(1 - \zeta_{\text{ph}})^4} \text{ dB}, \quad (60)$$

where

$$\text{CSL}(0) = 20 \log_{10} \frac{|\ln[(1 - \zeta_{\text{ph}}) - j\zeta_{\text{ph}}/G]|}{|\ln(jG)|} + 10 \log_{10} \frac{1 + 2v^2(1 - \zeta_{\text{ph}})^2}{1 + v^2(1 - \zeta_{\text{ph}})^2} \text{ dB}, \quad (61)$$

and it is seen that $\text{CSL}(0) = 0$ for $\zeta_{\text{ph}} = 1$. The solid lines in Fig. 6 are plots of $\text{CSL}(0)$ for $v = 5$ and several values of G . The dependence on v , not shown, is very weak along the beam axis. Although clutter suppression along the axis is of mainly academic interest because the level of the clutter is many decibels below that of the coherent signal, in combination with Eq. (60) the value of $\text{CSL}(0)$ quantifies clutter suppression throughout the beam.

Figure 6 shows that suppression improves continuously as the phase screen is moved closer to the source, which is in

FIG. 6. Contributions to $\text{CSL}(0)$ from the second harmonic generated before the phase screen as a function of phase screen location ζ_{ph} for $v = 5$ and several values of G .

agreement with Fig. 2(a) of Bradley,¹³ whose interpretation warrants repeating: “Physically, this effect is due to the cumulative growth with range that is characteristic of nonlinear propagation distortion. After the second-harmonic field scatters from the [phase screen], the scattered component of the field propagates linearly, but the unscattered component continues to accumulate amplitude with range. In other words, the ‘signal’ gets boosted more and more with range, but the ‘noise’ does not. Propagation distortion represents a channel by which energy is injected into the signal (courtesy of the fundamental field) and not into the noise.” While Bradley’s remarks were made in the context of reverberation clutter, and whereas the purpose of Eq. (60) in the present section is to describe a component of aberration clutter, it is shown in Sec. VI A that the suppression of reverberation clutter that propagates forward along the beam axis is also described by Eq. (60).

Equation (60) indicates that CSL increases quadratically with ρ . The increase occurs because nonlinearity is amplitude dependent—the intensity of the second harmonic is proportional to the square of the source intensity—and therefore the boost received by the signal is diminished with distance from the beam axis, where the intensity is highest. Also, as seen from Eqs. (54) and (58), the beamwidth of $I_2^{\text{sc}}(\rho, 1)$ is broader than that of $I_{1,2\omega}^{\text{sc}}(\rho, 1)$ regardless of where the phase screen is positioned between the source and focal planes. Further discussion of the dependence on ρ is postponed to Sec. V B in connection with Fig. 7(c), when all contributions to aberration clutter are taken into account, and to Sec. VI A in connection with reverberation clutter.

B. Complete solution

Contributions due to nonlinear generation of the second harmonic on both sides of the phase screen are now taken into account. In the pressure field after the phase screen there are two components that originated before the phase screen. These pressures are labeled q_2^{ch} and q_2^{sc} and correspond to the mean intensities I_2^{ch1} and I_2^{sc} derived in Sec. V A. The pressures represent a single realization, not an ensemble average, and are collectively labeled

FIG. 7. (a) Dependence of W_2 (solid line) on phase screen location for $G=5$ and $v=5$. Dashed lines separate out contributions to W_2 from the second harmonic generated before ($W_2^<$) versus after ($W_2^>$) the phase screen. (b) Corresponding dependence of $CSL(0)$ (solid line) on phase screen location, with dashed lines for $CSL^<$ and $CSL^>$. (c) Dependence of CSL on distance off axis.

$$q_2^< = q_2^{ch} + q_2^{sc}. \quad (62)$$

After the phase screen there are two components of the fundamental pressure field that contribute to generation of a second harmonic, corresponding to Eqs. (22) and (23). The total primary field after the phase screen is thus

$$q_1^> = q_1^{ch} + q_1^{sc}, \quad (63)$$

which gives rise to three nonlinear interactions that generate a second harmonic: $(q_1^{ch})^2$ generates the second-order term q_2^{ch} , $q_1^{ch}q_1^{sc}$ generates the third-order term q_2^{ab} , and $(q_1^{sc})^2$ generates a term at fourth order that is discarded because it is beyond our level of approximation. For the entire field of the second harmonic generated nonlinearly after the phase screen it is therefore sufficient to write

$$q_2^> = q_2^{ch} + q_2^{ab}. \quad (64)$$

The total field of the second harmonic, with contributions from both before and after the phase screen, is the combination of $q_2^<$ and $q_2^>$. Since the common term q_2^{ch} in Eqs. (62) and (64) represents the same component, the total field of the second harmonic through third order is

$$q_2 = q_2^{ch} + q_2^{sc} + q_2^{ab}. \quad (65)$$

The desired mean intensity is therefore

$$\begin{aligned} \langle I_2 \rangle &= \frac{\langle |q_2^{ch} + q_2^{sc} + q_2^{ab}|^2 \rangle}{2\rho_0 c_0} \\ &= \frac{\langle |q_2^{ch}|^2 \rangle}{2\rho_0 c_0} + \frac{\langle |q_2^{sc}|^2 \rangle}{2\rho_0 c_0} + \frac{\langle |q_2^{ab}|^2 \rangle}{2\rho_0 c_0} \\ &\quad + \left[\frac{\langle (q_2^{ch})(q_2^{sc})^* \rangle}{2\rho_0 c_0} + \frac{\langle (q_2^{ch})(q_2^{ab})^* \rangle}{2\rho_0 c_0} \right. \\ &\quad \left. + \frac{\langle (q_2^{sc})(q_2^{ab})^* \rangle}{2\rho_0 c_0} + \text{c.c.} \right]. \end{aligned} \quad (67)$$

Expressions are now required for the complex pressures q_2^{ch} , q_2^{sc} , and q_2^{ab} . For simplicity, results are presented only for the focal plane. The undeformed component q_2^{ch} is given by Eq. (57) with $\phi=0$ and z_{ph} replaced by d . The scattered component q_2^{sc} is obtained by first replacing $e^{j2\phi(\mathbf{r})}$ by $(e^{j2\phi(\mathbf{r})} - 1)$ in Eq. (57) to remove the coherent component from the

calculation, replacing q_{10} in Eq. (7) with the modified Eq. (57), and then making the additional replacements $k \rightarrow 2k$, $\tilde{\alpha}_1 \rightarrow \tilde{\alpha}_2$, and $z \rightarrow d - z_{ph}$ in Eq. (7). The aberration component q_2^{ab} , resulting from the nonlinear interaction of q_1^{ch} and q_1^{sc} beyond the phase screen, is obtained from Eq. (14) after q_{10} and q_{10}^* are replaced by the appropriate boundary conditions for q_1^{ch} and $(q_1^{sc})^*$, respectively, and with $z \rightarrow d - z_{ph}$.

For consistency with Eq. (32), after the expected values in Eq. (67) are calculated, the individual terms are expressed as

$$\frac{\langle |q_2^{ch}|^2 \rangle}{2\rho_0 c_0} = I_2^{ch1}, \quad (68)$$

$$\frac{\langle |q_2^{sc}|^2 \rangle}{2\rho_0 c_0} = 4\sigma^2 I_2^{sc}, \quad (69)$$

$$\frac{\langle |q_2^{ab}|^2 \rangle}{2\rho_0 c_0} = 4\sigma^2 I_2^{ab1}, \quad (70)$$

$$\frac{\langle (q_2^{sc})(q_2^{ab})^* \rangle}{2\rho_0 c_0} + \text{c.c.} = 4\sigma^2 I_2^{ab2}, \quad (71)$$

$$\frac{\langle (q_2^{ch})(q_2^{ab})^* \rangle}{2\rho_0 c_0} + \text{c.c.} = -2\sigma^2 (I_2^{ch1} + I_2^{ch2}), \quad (72)$$

$$\frac{\langle (q_2^{ch})(q_2^{sc})^* \rangle}{2\rho_0 c_0} + \text{c.c.} = -2\sigma^2 I_2^{ch3}, \quad (73)$$

such that the mean intensity becomes

$$\begin{aligned} \langle I_2 \rangle &= I_2^{ch1} - 2\sigma^2 (I_2^{ch1} + I_2^{ch2} + I_2^{ch3}) \\ &\quad + 4\sigma^2 (I_2^{ab1} + I_2^{ab2} + I_2^{sc}). \end{aligned} \quad (74)$$

The results of all the calculations are (see also Ref. 21)

$$I_2^{ch1} = \frac{|\ln(j/G)|^2}{1 + G^{-2}} I_{02} e^{-2A_2 - 4G^2 \rho^2}, \quad (75)$$

$$\begin{aligned} I_2^{ch2} &= \text{Re} \left\{ -(\ln Z) \ln(j/G) \right. \\ &\quad \left. + \frac{\ln(j/G) \ln[1/jGZ - 2jv^2(1 - \zeta_{ph})/G]}{1 + 2v^2 Z/(1 - jG)} \right\} \\ &\quad \times \frac{I_{02} e^{-2A_2 - 4G^2 \rho^2}}{1 + G^{-2}}, \end{aligned} \quad (76)$$

$$I_2^{\text{ch3}} = \text{Re}\{(\ln Z) \ln(j/G)\} \frac{2I_{02}e^{-2A_2-4G^2\rho^2}}{1+G^{-2}}, \quad (77)$$

$$I_2^{\text{ab1}} = \frac{G^2 I_{02} e^{-2A_2-4G^2\rho^2}}{v^4 |Z|^2} \times \int_1^\infty \int_1^\infty \frac{\exp[4v^2 G^2 \rho^2 (1-\zeta_{\text{ph}})^2 U_1/V_1]}{t_1 t_2 V_1} dt_1 dt_2, \quad (78)$$

$$I_2^{\text{ab2}} = 2I_{02}e^{-2A_2-4G^2\rho^2} \times \text{Re}\left\{ \frac{G^2 \ln(Z^*)}{(1+jG)Z} \int_1^\infty \frac{\exp[4v^2 G^2 \rho^2 (1-\zeta_{\text{ph}})^2 U_2/V_2]}{t V_2} dt \right\}, \quad (79)$$

$$I_2^{\text{sc}} = \frac{|\ln Z|^2 I_{02} e^{-2A_2}}{(1+G^{-2})[1+v^2(1-\zeta_{\text{ph}})^2]} \times \exp\left(-\frac{4G^2 \rho^2}{1+v^2(1-\zeta_{\text{ph}})^2}\right). \quad (80)$$

The following additional symbols have been introduced in the above expressions: $Z = 1 - \zeta_{\text{ph}} - j\zeta_{\text{ph}}/G$, and

$$U_1 = \left(\frac{1+jG}{v^2 Z^*} - \frac{jGt_2}{v^2(1-\zeta_{\text{ph}})} \right) \left(\frac{1-jG}{v^2 Z} - \frac{jGt_1}{v^2(1-\zeta_{\text{ph}})} \right), \quad (81)$$

$$V_1 = \frac{1}{v^2 |Z|^2} + \frac{jG(t_1 - t_2)}{2v^2(1-\zeta_{\text{ph}})} + \left[1 + v^2(1-\zeta_{\text{ph}})^2 \right] U_1, \quad (82)$$

$$U_2 = \frac{jGt}{1-\zeta_{\text{ph}}} + \frac{1-jG}{Z}, \quad (83)$$

$$V_2 = \frac{v^2}{2} + \left[1 + v^2(1-\zeta_{\text{ph}})^2 \right] U_2. \quad (84)$$

The expressions for I_2^{ch1} and I_2^{sc} are the same as Eqs. (39) and (58), respectively. For $\zeta_{\text{ph}} = 0$, since $\ln Z = 0$, the quantities I_2^{sc} , I_2^{ab2} , and I_2^{ch3} all vanish, and the expressions for I_2^{ch2} and I_2^{ab1} reduce to Eqs. (40) and (41), respectively.

The contributions from I_2^{ab1} , I_2^{ab2} , and I_2^{sc} in the third term in Eq. (74) are responsible for deformation of the beam profile. The intensity represented by I_2^{sc} is due to generation of the second harmonic before the phase screen, and the intensities represented by I_2^{ab1} and I_2^{ab2} are due to generation after the phase screen. This distinction permits calculation of how much power associated with clutter in the focal plane is due to generation of the second harmonic before versus after the phase screen. The total clutter power is expressed in dimensionless form as

$$W_2(\zeta_{\text{ph}}) = W_2^<(\zeta_{\text{ph}}) + W_2^>(\zeta_{\text{ph}}), \quad (85)$$

where

$$W_2^<(\zeta_{\text{ph}}) = \frac{\int_0^\infty I_2^{\text{sc}} \rho d\rho}{\left(\int_0^\infty I_2^{\text{sc}} \rho d\rho \right)_{\zeta_{\text{ph}}=1}} \quad (86)$$

accounts for clutter power resulting from generation of the second harmonic before the phase screen, and

$$W_2^>(\zeta_{\text{ph}}) = \frac{\int_0^\infty (I_2^{\text{ab1}} + I_2^{\text{ab2}}) \rho d\rho}{\left(\int_0^\infty I_2^{\text{sc}} \rho d\rho \right)_{\zeta_{\text{ph}}=1}} \quad (87)$$

from after the screen. The normalization introduced is such that $W_2(1) = 1$.

Results for $G = 5$ and $v = 5$ are presented in Fig. 7(a). For $0 \leq \zeta_{\text{ph}} \lesssim 0.7$ it is the effect of aberration on generation of the second harmonic after the phase screen that contributes most to clutter in the focal plane. For $0.8 \lesssim \zeta_{\text{ph}} \leq 1$ it is scattering of the second harmonic generated before the phase screen that dominates. As the phase screen is moved closer to the focal plane, the length of the path is reduced along which second-harmonic clutter is generated by the primary wave after the phase screen. However, $W_2^>$ does not decrease monotonically as ζ_{ph} is increased. The initial increase in $W_2^>$ occurs because focusing causes the amplitude of the aberrations in the primary wave to increase with ζ_{ph} . For $\zeta_{\text{ph}} \gtrsim 0.7$, the reduction in the propagation path length ultimately wins out, and $W_2^> \rightarrow 0$ as $\zeta_{\text{ph}} \rightarrow 1$. The peak value of $W_2^>$, and the value of ζ_{ph} for which this is obtained, depends on the inhomogeneity scale v . For smaller v (weaker inhomogeneity) the maximum value of $W_2^>$ is reduced, and it is obtained for a smaller value of ζ_{ph} . The monotonic increase in $W_2^<$ with increasing ζ_{ph} follows from the monotonic growth of the second harmonic between the source and phase screen.

The definition of CSL consistent with Eq. (47) is

$$\text{CSL}(\rho) = 10 \log_{10} \frac{[I_2^{\text{ab1}}(\rho) + I_2^{\text{ab2}}(\rho) + I_2^{\text{sc}}(\rho)]/I_2^{\text{ch1}}(0)}{I_{1,2\omega}^{\text{sc}}(\rho)/I_{1,2\omega}^{\text{ch}}(0)} \text{ dB}, \quad (88)$$

in which it is again understood that all quantities are evaluated in the focal plane. In the denominator, $I_{1,2\omega}^{\text{ch}}$ is given by Eq. (29), and $I_{1,2\omega}^{\text{sc}}$ by Eq. (54). A quantity CSL_T is now defined that is a measure of total clutter power in the focal plane relative to the total power in the coherent field. The quantity is obtained from Eq. (88) by integrating the individual intensity components as follows:

$$\text{CSL}_T = 10 \log_{10} \frac{\int_0^\infty (I_2^{\text{ab1}} + I_2^{\text{ab2}} + I_2^{\text{sc}}) \rho d\rho / \int_0^\infty I_2^{\text{ch1}} \rho d\rho}{\int_0^\infty I_{1,2\omega}^{\text{sc}} \rho d\rho / \int_0^\infty I_{1,2\omega}^{\text{ch}} \rho d\rho} \text{ dB}. \quad (89)$$

Equation (89) is plotted in Fig. 7(b) following the same convention as in Fig. 7(a), where the solid line is CSL_T , the dashed line labeled $\text{CSL}_T^<$ is obtained by setting $I_2^{\text{ab1}} = I_2^{\text{ab2}} = 0$ in Eq. (89), and the dashed line labeled $\text{CSL}_T^>$ is obtained by setting $I_2^{\text{sc}} = 0$. It is seen that on the basis of total power, THI suppresses clutter by somewhat less than 10 dB when the phase screen is adjacent to the source, and the suppression it provides approaches zero as the phase screen is moved closer to the focal plane. It happens that the curves in Fig. 7(b) are identical to their counterparts in Fig. 7(a) when the latter are expressed in decibels.

FIG. 8. Comparison of statistical solutions $\langle I_{1,2\omega} \rangle$ (left column) and $\langle I_2 \rangle$ (right column) at different phase screen locations for $G=5$ and $\nu=5$.

For imaging, of greater interest are the locations in the beam where THI most successfully suppresses the deformed components of the field. Equation (88) is thus plotted in Fig. 7(c) for several values of ζ_{ph} . Clutter suppression off the axis is seen to improve as the phase screen is moved closer to the source. The tendency of the curves to turn increasingly upward as ζ_{ph} is increased is due to the increasing contribution from $CSL^<$, which as seen from Eq. (60) increases quadratically with ρ . While the curve for $\zeta_{ph}=0.75$ indicates that with THI the clutter level increases continuously with distance from the axis, investigation of the beam patterns reveals that THI nevertheless reduces the broadening of the beam.

The beam patterns are shown in Fig. 8, with the fundamental at frequency 2ω shown in the left column and the second harmonic generated nonlinearly in the right column. As in Fig. 1, the dashed lines are the corresponding beam patterns in the absence of the phase screen. It is clear that for $\zeta_{ph}=0.25$ and $\zeta_{ph}=0.5$ (first and second rows) the clutter levels are suppressed by THI (right column), corresponding to the negative values of CSL in Fig. 7(c). The case $\zeta_{ph}=0.75$ discussed in the preceding paragraph is presented in the third row of Fig. 8. The level of clutter as measured by comparing beam patterns at specified values of ρ , corresponding to the definition of CSL, is indeed higher in the beam pattern for $\langle I_2 \rangle$, but the beam broadening is in fact less for $\langle I_2 \rangle$. The positive values of CSL in this particular case are therefore an inaccurate measure of clutter suppression, a consequence of comparing beam patterns that change relatively little with the introduction of the phase screen. When the changes are more substantial, as for $\zeta_{ph}=0.25$ and $\zeta_{ph}=0.5$, reduction of beam broadening with THI is always accompanied by negative values of CSL, and these are the cases of greatest interest.

Regardless of where the phase screen is positioned between the source and focal planes, THI is found to reduce the broadening of the beam in the focal plane caused by aberration. The clutter suppression by THI is more pronounced as the phase screen is moved close to the source, precisely where the phase screen introduces the greatest deformation of the beam in fundamental mode imaging.

VI. REVERBERATION CLUTTER

Reverberant fields that propagate forward along the beam axis are considered first, in Sec. VI A, because they can be analyzed directly using results obtained in Sec. V. Since the present theoretical framework does not lend itself well to modeling reverberant fields that result from scattering in other directions, in Sec. VI B we consider an alternative model of clutter suppression developed by Bradley¹³ for reverberant fields that propagate in any direction. The underlying assumption in Bradley's model is that the spatial distributions of reverberant fields produced in fundamental mode imaging and THI are sufficiently similar to be considered the same. Since the results in Sec. VI A are not restricted in this way, they may be used to test the validity of this assumption at least for the case of forward propagation of the reverberant field, and this is done in Sec. VI B.

A. Forward reverberant field propagation

For reverberant fields that ultimately propagate forward along the beam axis, it is helpful to envision a reverberation layer of thickness Δz encountered by an acoustic pulse at normal incidence. The interfaces of the layer are supposed parallel and characterized by a reflection coefficient R and transmission coefficient T . First assume weak reflection at each interface, such that $|R| \ll 1$ and $T \simeq 1$. The pulse transmitted directly through the layer will be followed by a series of reverberations that are replicas of the incident pulse and lag behind at distances $2\Delta z$, $4\Delta z$, etc., and which are reduced in amplitude by factors of order R^2 , R^4 , etc., respectively. If the first interface is the surface of the skin, it may have a reflection coefficient of $O(1)$, in which case the successive reverberations are reduced in amplitude by factors of order R , R^2 , etc. Either way, the effect of these reverberations on the image of a feature at depth z is that their returns are displayed as increasingly weaker replicas of the feature at depths $z + \Delta z$, $z + 2\Delta z$, etc. In reality the reverberation layer is nowhere near this idealized, being formed by body wall layers and other tissue heterogeneities. In addition to introducing time delays, reverberation produces deformed wavefronts, and in combination these two effects generate clutter throughout the image of the field beyond the reverberation layer.

Determination of how to model suppression of the resulting reverberation clutter by THI may be based on the relation of the pulse length L to the reverberation layer thickness Δz . For a layer of thickness $\Delta z < L/2$, the first reverberation to exit the layer at least partially overlaps the direct pulse. For $\Delta z \ll L$, reverberation is indistinguishable from aberration because its effect is to add a deformed field component to the direct pulse, which interacts nonlinearly with the direct pulse to generate a deformed second-harmonic

component. Analysis of this case is therefore covered in Secs. IV and V B.

Alternatively, for $\Delta z > L/2$ the reverberation does not overlap the direct pulse. Because the reverberation emanating from the layer at the source frequency is at most of order $|R| \ll 1$ in magnitude relative to the direct pulse, any second harmonic generated by the reverberation is insignificant. Therefore, the only component of the second harmonic that contributes to reverberation clutter in this case is the one generated before the reverberation layer, and which subsequently passes through the layer following one or more reverberations. Because the reverberation emanating from the layer at the second harmonic is deformed, an appropriate model of this contribution to the reverberation clutter is the one developed in Sec. V A, with CSL given by Eq. (60).

It is the latter case, $\Delta z > L/2$, that is associated here with reverberation clutter. For a body wall thickness of $\Delta z = 1$ cm (body wall thickness is often larger), and a pulse consisting of 10 cycles at 1.5 MHz, for which the pulse length is $L = 1$ cm (typically fewer cycles, and therefore shorter pulses, are used in practice, and with higher imaging frequencies the pulses are even shorter), the condition $\Delta z > L/2$ is satisfied. Since the body wall is effectively in contact with the transducer, and with the focal length taken to be $d = 8$ cm, Eq. (60) with $\zeta_{ph} \simeq \Delta z/d \ll 1$ describes suppression of the reverberation clutter with THI. As seen in Fig. 6 the suppression on the axis in the focal plane, denoted by $CSL(0)$, is dramatic for $\zeta_{ph} \ll 1$.

It is suppression of the clutter off the axis that is generally most important, yet it is seen from Eq. (60) that the level of suppression provided by THI deteriorates quadratically with distance from the beam axis. To assess this effect we now take $v = 5$ as in Fig. 6 and $\zeta_{ph} = 0.25$, e.g., corresponding to a body wall thickness of 2 cm and a focal length of 8 cm. The second term in Eq. (60) becomes $0.04G^2\rho^2$ dB, or $4\rho^2$ dB for $G = 10$ as in Fig. 4. At $\rho = 0.5$, which in Fig. 4 is seen to be quite far off the axis in relation to the undeformed beam pattern, the second term in Eq. (60) adds only 1 dB to $CSL(0)$, which is negligible in comparison with the large negative values of $CSL(0)$ observed in Fig. 6 for $\zeta_{ph} = 0.25$. For smaller values of either ζ_{ph} or G the second term in Eq. (60) is even smaller. For reverberation layers close to the source, emanating reverberant fields that propagate forward along the beam axis, THI thus provides substantial suppression of reverberation clutter.

B. Arbitrary reverberant fields

Scattering of the beam off to the side and backward produces the main contributions to reverberation clutter. Scattering off ribs is common in echocardiography, and if the scattered sound is scattered yet again by a strong reflector such as lung adjacent to the heart, it can reverse its path back to the rib and then to the transducer. Two important differences with the case of forward scatter from a phase screen, as considered in Sec. VI A, may be noted. One is that the scattering is accompanied by a reduction in amplitude, and the other is that diffraction of the scattered wave will be consid-

erably different. The unknown nature of this diffraction prevents analysis within the present theoretical framework.

However, if the objective is to compare the relative level of reverberation clutter produced by fundamental mode imaging versus THI, and if the spatial distributions of the reverberant fields produced by these two imaging modalities are the same, then it is unnecessary to calculate the reverberant field distributions when evaluating CSL using Eq. (51). The reason is that the ratio of $I_2^{sc}/I_{1,2\omega}^{sc}$ becomes independent of location in the reverberant fields because the spatial dependencies cancel out. This was the key observation made by Bradley¹³ in developing his model of reverberation clutter suppression.

The assumption of identical reverberant fields requires the beam profiles incident on the initial scatterer to be identical in the two imaging modalities. While this is generally not the case [e.g., compare Eqs. (53) and (57) for a transducer with Gaussian amplitude shading], it may be a reasonable assumption for certain transducer apodizations, particularly those that are close to uniform amplitude distributions. A second requirement is that propagation of the reverberant fields is linear and therefore no further generation of a second harmonic occurs, which should ordinarily be a safe assumption given the typical reduction in the amplitude of the primary wave at frequency ω when it too is scattered. Bradley thus accounts for the scattered fields at frequency 2ω by simply multiplying the field incident on the first scatterer by the same “scattered field function” $\gamma(\mathbf{r})$. It is irrelevant what path the reverberant fields follow on their way back to the transducer.

Development of CSL assuming identical spatial distributions of the reverberant fields proceeds as follows. Setting $r = 0$ and then replacing $e^{j2\phi(\mathbf{r})}$ by $\gamma(\mathbf{r})$ in Eqs. (53) and (57) yields, in place of Eqs. (54) and (58),

$$I_{1,2\omega}^{sc}(\rho) = \frac{\langle |\gamma|^2 \rangle I_{01} e^{-2A_2 \zeta_{ph}}}{(1 - \zeta_{ph})^2 + (\zeta_{ph}/2G)^2}, \quad (90)$$

$$I_2^{sc}(\rho) = \frac{|\ln[(1 - \zeta_{ph}) - j\zeta_{ph}/G]|^2}{(1 + G^2)[(1 - \zeta_{ph})^2 + (\zeta_{ph}/G)^2]} \langle |\gamma|^2 \rangle I_{02} e^{-2A_2 \zeta_{ph}}, \quad (91)$$

where here $\zeta_{ph} = z_{ph}/d$ denotes the location of the first scatterer. Substitution in Eq. (51) then yields, in place of Eq. (60),

$$CSL_B = 20 \log_{10} \frac{|\ln[(1 - \zeta_{ph}) - j\zeta_{ph}/G]|}{|\ln(jG)|} + 10 \log_{10} \frac{4(1 - \zeta_{ph})^2 + (\zeta_{ph}/G)^2}{(1 - \zeta_{ph})^2 + (\zeta_{ph}/G)^2} \text{ dB}. \quad (92)$$

For consistency, as in the derivation of Eq. (60), the undeformed fields have been compared in the focal plane. Equation (92) differs from Eq. (61) only in the second term. Like Eq. (61), $CSL_B = 0$ for $\zeta_{ph} = 1$. Unlike Eq. (60), Eq. (92) does not depend on ρ , a consequence of ignoring differences in the spatial distributions of the reverberant fields.

FIG. 9. Comparison of CSL_B with $CSL(0)$, with the latter given by Eq. (61), for $G = 5$ and $\nu = 5$.

Since Eq. (92) accounts for reverberant fields that propagate forward along the beam axis, as well as fields scattered to the side and backward, comparison with Eq. (61) helps assess the consequence of assuming that the spatial distributions of the reverberant fields are identical, because this assumption was not used to derive Eq. (61). The comparison is made in Fig. 9 for $G = 5$. The solid curve is Eq. (92), and the dashed curve is Eq. (61) for $CSL(0)$ with $\nu = 5$, which is the same as the uppermost curve in Fig. 6. The results are seen to be qualitatively the same, and they differ quantitatively by at most about 3 dB. While the comparison is made only for weak forward scattering, at least to this extent Bradley's approximation appears justified.

Most apparent from Fig. 9 is the large suppression of reverberation clutter produced in the vicinity of the source, from 10 to 20 dB.

VII. CONCLUSION

The statistical model accounts for the main physical effects involved in THI, which are nonlinearity, diffraction, and heterogeneity. The advantage of the statistical model is that it avoids having to average many numerical simulations run for different realizations of the random heterogeneity in order to discern trends. The principal limitation of the model is confinement of the heterogeneity to a single thin layer that is interpreted as a phase screen. While the model can be augmented as described elsewhere²¹ to account for two or more phase screens, the solutions rapidly become unwieldy, and ensemble averaging of purely numerical simulations becomes the more practical approach. Purely numerical simulations are also required to model radiation from sources with apodizations other than Gaussian. However, in addition to being relatively simple to model statistically, harmonic generation that accompanies propagation of a nonlinear sound beam radiated from a Gaussian source and transmitted through a single phase screen provides physical insight into the ability of THI to suppress image clutter.

The statistical model describes suppression of aberration clutter by THI that is produced when a focused sound beam encounters a thin heterogeneous layer along its propagation

path. It is found that while aberration clutter in the focal region increases as depth of the aberrating layer decreases, suppression of the clutter by THI is itself most pronounced when the phase aberrations are introduced close to the source. The amount of suppression also increases as the correlation length of the heterogeneity decreases, but not without limit. No further suppression is obtained once the correlation length becomes small compared to the source radius.

The phase screen model permits statistical analysis of reverberation clutter for reverberant fields that are transmitted forward along the beam axis. In this case, when reverberation clutter in the focal region is produced by reflective layers near the source, its suppression by THI is dramatic. Suppression of reverberation clutter produced by scattering in any direction, off to the side and backward in addition to forward, was analyzed using a deterministic model proposed by Bradley.¹³ It too predicts very strong suppression by THI when the reverberations occur near the source. The model is based on the assumption that the beam profiles incident on the first scattering site, both in the fundamental and harmonic mode, are identical. This assumption was tested in comparison with the statistical model for forward propagating reverberation fields, which is not restricted to identical beam profiles, and the assumption is found to be valid at least in this particular case.

In general, the degree to which both aberration and reverberation clutter is mitigated by THI appears to be comparable when the heterogeneity causing the clutter is close to the source, where body wall can introduce strong aberration and ribs can initiate strong reverberation. In this case the model predicts that suppression of both aberration and reverberation clutter can range from 10 to 20 dB. While the physical mechanisms responsible for THI's suppression of both forms of clutter are very different, the effectiveness of each is reduced as the depth of the heterogeneous region is increased. Ensemble averages of numerical simulations based on realistic anatomical models are required for more quantitative evaluation of the physical mechanisms underlying THI than can be deduced from the statistical model developed here.

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