Rayleigh–Lamb wave propagation on a fractional order viscoelastic plate

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A previous study of the authors published in this journal focused on mechanical wave motion in a viscoelastic material representative of biological tissue [Meral *et al.*, J. Acoust. Soc. Am. **126**, 3278–3285 (2009)]. Compression, shear and surface wave motion in and on a viscoelastic halfspace excited by surface and sub-surface sources were considered. It was shown that a fractional order Voigt model, where the rate-dependent damping component that is dependent on the first derivative of time is replaced with a component that is dependent on a fractional derivative of time, resulted in closer agreement with experiment as compared with conventional (integer order) models, such as those of Voigt and Zener. In the present study, this analysis is extended to another configuration and wave type: out-of-plane response of a viscoelastic plate to harmonic anti-symmetric Lamb wave excitation. Theoretical solutions are compared with experimental measurements for a polymeric tissue mimicking phantom material. As in the previous configurations the fractional order modeling assumption improves the match between theory and experiment over a wider frequency range. Experimental complexities in the present study and the reliability of the different approaches for quantifying the shear viscoelastic properties of the material are discussed.

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I. INTRODUCTION

As reviewed in the authors' recent publication,¹ an improved understanding of mechanical wave propagation in viscoelastic materials and structures could fundamentally catalyze technical developments in many areas including medicine, geophysics, infrastructure, and manufacturing. In medical imaging in particular, non-invasive measurement of shear wave motion in soft biological tissues can provide unique spatially localized information about the tissue's material properties. Such information can reflect the development of pathology and in some cases biomechanical integrity. It may also provide a unique way to nondestructively track the morphogenesis of engineered tissues.² Over the past few decades, mechanical wave propagation has been combined with the medical imaging modalities of ultrasound (US) or magnetic resonance imaging (MRI) to establish non-invasive means of visualizing shear wave motion for diagnosis or nondestructive tissue assessment.^{2–6} These dynamic elastography imaging techniques, as they are commonly called, have received much attention because shear moduli have a large dynamic range within soft biological tissues, particularly when compared with the variation of material parameters that provide contrast in conventional

imaging techniques. For soft biological tissues (e.g., comparing muscle with fat), the x-ray attenuation coefficient varies only by a factor of 2 (Ref. 7), while magnetic resonance (MR) relaxation times vary by a factor of 3 (Ref. 8). The shear moduli, on the other hand, can vary by more than a factor of ten.^{9–12}

Estimating material viscoelastic properties based on measured or imaged shear wave motion is called inversion or reconstruction. Reconstruction requires both a viscoelastic model of the tissue as well as the imaged wave patterns. A range of computational techniques or algorithms for reconstruction have been proposed.^{13–19} These techniques often assume a linear constitutive relation for the tissue viscoelasticity, such as a Voigt model, as the basis for interpreting the measurements. Studies, however, have shown that such simple viscoelastic models have limitations in their ability to accurately model dynamic phenomena over multiple time scales and/or with broad spectral content, particularly for biological tissues and tissue mimicking phantoms, and that one way of overcoming such limitations is through the use of fractional order models.^{1,20–25} Fractional order viscoelastic modeling starts with the idea from fractional calculus that the order of the derivative of the strain can be intermediate between zero and one since it is the derivative of the strain that characterizes the material's behavior (assuming a one dimensional stress–strain relation).^{26–29} Such an approach offers a new viscoelastic stress–strain model. When the order of the derivative is zero it represents a Hookean

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solid and when it is of order one it describes a Newtonian fluid. Viscoelastic materials occupy the intermediate range with a fractional order “ α ” between zero and one. Using this approach, it is possible to build a multi-component fractional viscoelastic model by replacing one or more springs and dashpots with “springpots.”²⁹ Such models are linear and have shown the potential to yield new disease and treatment specific parameters that more effectively predict underlying changes in tissue associated with developing pathology, such as liver fibrosis,³⁰ breast cancer,²² or brain degradation.³¹

In our previous studies we investigated surface waves on a half space created by a disk oscillating normal to the surface³² and the surface response due to sub-surface elementary sources, including a vertically oscillating dipole.³³ In a more recent study of these canonical test beds,¹ it was shown that a fractional order Voigt model of the halfspace material viscoelasticity resulted in closer agreement of theory with experiment as compared to conventional (integer order) models, such as those of Voigt and Zener. In the present paper this comparative analysis is extended to consider both analytically and experimentally Rayleigh–Lamb (R–L) wave motion of a thick viscoelastic plate.

Graff³⁴ introduced the R–L equation for an elastic plate. Goodman³⁵ investigated axisymmetric loading and resulting circularly crested waves. Rogers³⁶ utilized R–L waves to develop a technique that measures the elastic properties of metal plates from phase speed. Karp³⁷ investigated the plane strain response of a semi-infinite, elastic strip to harmonic, symmetric, and non-uniform end excitation. The research presented in this paper uses a similar technique as described in Ref. 36, using the phase speed of R–L waves to calculate the mechanical material properties of a viscoelastic plate. Additionally, attenuation of the waves is considered by introducing a complex term into the R–L frequency equation that results from the viscosity of the material. The calculated material properties are frequency-dependent; as in our previous study,¹ viscoelasticity is modeled using a fractional order model as well as integer-order models. Theoretical predictions and experimental measurements are compared and discussed.

II. THEORY

A. R–L waves

The problem considered here is that of a circular rigid rod normal to and penetrating through a viscoelastic plate; the rod vibrates axially and normal to the plate free surfaces. While both symmetric and anti-symmetric R–L waves may propagate in the plate (Fig. 1), the excitation condition described above should excite only the anti-symmetric waves if a welded contact between the rod and plate material exists along the entire length of the embedded rod.

The radius of the rod is very small compared to the plate thickness. The rod effectively acts as a nonhomogeneous boundary condition driving axisymmetric wave propagation in the plate away from the insertion point of the rod. The geometry of the problem is depicted in Fig. 2.

Goodman³⁵ investigated the frequency equation for axisymmetric loading and resulting circularly crested waves; in

FIG. 1. Displacement profiles for (a) symmetric and (b) anti-symmetric R–L waves. Wave profiles through the thickness below indicate amplitude of vertical displacement from equilibrium. Key: —, first mode (a_0); ---, second mode (a_1); - · - ·, third mode (a_2); · · · ·, fourth mode (a_3).

his study, he considered a plate of thickness $2b$, and no-stress boundary conditions at the surfaces,

$$\tau_{zz} = \tau_{zr} = \tau_{z\theta} = 0 \quad \text{at} \quad z = \pm b. \quad (1)$$

Cylindrical coordinates are used to solve this problem, which results in the following equations of motion,

$$(\lambda + \mu) \frac{\partial \Delta}{\partial r} + \mu \nabla^2 u_r - \mu \frac{u_r}{r^2} = \rho \frac{\partial^2 u_r}{\partial t^2}, \quad (2)$$

$$(\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 u_z = \rho \frac{\partial^2 u_z}{\partial t^2}. \quad (3)$$

Here, u_r and u_z are the displacements in radial and axial directions, respectively, ρ is the density of the medium, λ and μ are the Lamé constants of the medium, and Δ is the dilatation, given by

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}. \quad (4)$$

The solutions for the displacements are given for the anti-symmetric wave case as

$$u_r = J_1(\xi r) \left\{ B \cosh \bar{\beta} z - B' \frac{\bar{\beta}}{\xi} \sinh \bar{\beta} z + A \frac{\xi}{\bar{\alpha}^2 - \bar{\beta}^2} \sinh \bar{\alpha} z \right\} e^{i\omega t}, \quad (5)$$

FIG. 2. Infinite viscoelastic plate with a vertical rod inserted for creating anti-symmetric R–L waves of the first mode.

$$u_z = J_0(\xi r) \left\{ B' \cosh \bar{\beta} z - B \frac{\xi}{\bar{\beta}} \sinh \bar{\beta} z - A \frac{\bar{\alpha}}{\bar{\alpha}^2 - \bar{\beta}^2} \cosh \bar{\alpha} z \right\} e^{j\omega t}. \quad (6)$$

Here, A , B , and B' are unknown constants, which can be solved by using the free surface boundary conditions at $z = \pm b$ and a zero stress or zero displacement boundary condition at $z = 0$ depending on the symmetry condition. Also, J_0 and J_1 are Bessel functions (first kind), and $\bar{\alpha}$ and $\bar{\beta}$ are given as

$$\bar{\alpha}^2 = \xi^2 - \omega^2/c_1^2, \quad (7)$$

$$\bar{\beta}^2 = \xi^2 - \omega^2/c_2^2, \quad (8)$$

where ξ is the R-L wave number, ω is the cyclic frequency, and c_1 and c_2 are compression and shear wave speeds, respectively. The frequency equation for anti-symmetric Lamb waves is given by Graff³⁴ as

$$\frac{\tanh \bar{\beta} b}{\tanh \bar{\alpha} b} = \frac{(\xi^2 - \bar{\beta}^2)^2}{4\bar{\alpha}\bar{\beta}\xi^2}. \quad (9a)$$

For the symmetric case it is necessary to interchange $\sinh$ and $\cosh$ terms. This yields the frequency equation for symmetric Lamb waves,

$$\frac{\tanh \bar{\beta} b}{\tanh \bar{\alpha} b} = \frac{4\bar{\alpha}\bar{\beta}\xi^2}{(\xi^2 - \bar{\beta}^2)^2}. \quad (9b)$$

Solutions to this expression can be divided into three types depending on the relationship between the R-L wave number ξ , the compression wave number ω/c_1 , and the shear wave number ω/c_2 ; see the Appendix for the solution of the R-L frequency equation for these various cases. Here, our interest is in the case that $\xi^2 > \omega^2/c_1^2$ and ω^2/c_2^2 , as in this range the R-L waves are dominantly governed by the shear properties of the material. Based on this result, the solutions of the above Eq. (9a) are considered in the zone on the complex wave number domain where $\xi^2 > \omega^2/c_1^2$. Equations (9a) and (9b) have multiple solutions of ξ that correspond to different Lamb modes of the plate vibrations. These modes are shown in Fig. 1 for symmetric (a) and anti-symmetric (b) R-L waves.

It is also assumed that the problem considered here consists of an infinitely large plate without reflections from finite boundaries. The excitation boundary condition of the vertically oscillating rigid rod creates only anti-symmetric Lamb waves in the plate; therefore, this study mainly focuses on the anti-symmetric case of the R-L equation. In order to find values of ξ , satisfying Eq. (9a) a function $F(\xi)$ is defined such that its value is minimized for the solutions of the R-L frequency equation,

$$F(\xi) = \frac{\tanh \bar{\beta} b}{\tanh \bar{\alpha} b} - \frac{(\xi^2 - \bar{\beta}^2)^2}{4\bar{\alpha}\bar{\beta}\xi^2}. \quad (10)$$

Elastic materials with no viscous damping yield real compression and shear wave numbers. However, the dispersive nature of R-L waves may lead to complex valued wave

FIG. 3. Cost function $F(\xi)$ for $\mu_0 = 4.5$ kPa, $\mu_1 = 0$ Pa s, and $\omega/2\pi = 50$ Hz in plots (a) and (b). For plot (c) $\mu_1 = 4$ Pa s, assuming a Voigt model of viscoelasticity. (a) $|F(\xi)|$ for real values of ξ ; zeros correspond to the R-L wave numbers satisfying Eq. (9a): $\xi = 155$ m⁻¹ and 80 m⁻¹ correspond to the first and second anti-symmetric Lamb modes, a_0 and a_1 , respectively. (b) Contour plot of $|F(\xi)|$ in the complex plane (fourth quadrant). A complex valued solution is found at $\xi = 30-125j$, which corresponds to a higher order mode. (c) Added damping results in shifting of the zeros and poles of the equation from the real axis into the fourth quadrant of the complex domain. Arrows point to solutions, minima of $|F(\xi)|$.

numbers for propagating waves and imaginary values for evanescent (non-propagating) waves. Figure 3(a) shows real ξ versus $|F(\xi)|$ for the indicated value of μ . (For this and all plots shown, density $\rho = 1100$ kg/m³ and volume elasticity $\lambda = \lambda_0 = 2.9$ GPa. Volume viscosity is negligible for the material and frequency range of interest.¹) Two solutions of the frequency equation can be seen that correspond to the first and second anti-symmetric Lamb modes at 50 Hz. Since higher order modes are evanescent at this frequency they do not appear in the figure.

Once the wave number is obtained by setting Eq. (10) equal to zero and solving, the resulting displacements in the r (radial) and z (thickness) directions can be calculated from

$$u_r = AJ_1(\xi r) \left(\frac{\xi}{\bar{\alpha}^2 - \bar{\beta}^2} \right) \times \left(\sinh \bar{\alpha} z - \frac{\xi^2 + \bar{\beta}^2}{2\xi^2} \frac{\sinh \bar{\alpha} b}{\sinh \bar{\beta} b} \sinh \bar{\beta} z \right) e^{j\omega t}, \quad (11)$$

$$u_z = -AJ_0(\xi r) \left(\frac{\bar{\alpha}}{\bar{\alpha}^2 - \bar{\beta}^2} \right) \times \left(\cosh \bar{\alpha} z - \frac{2\xi^2}{\xi^2 + \bar{\beta}^2} \frac{\cosh \bar{\alpha} b}{\cosh \bar{\beta} b} \cosh \bar{\beta} z \right) e^{j\omega t}. \quad (12)$$

This derivation is slightly modified to fit the present case. The initial boundary condition of the traction-free surface is not applicable for $r = 0$ due to the excitation at $r = 0$ for all z values. This issue is resolved by replacing the Bessel functions J_0 and J_1 by Hankel Functions³⁸ (Bessel function of third kind) of the second type, $H_n^{(2)}(\xi r) = J_n(\xi r) - jN_n(\xi r)$, zeroth and first order ($n = 0, 1$), where J_n and N_n are the Bessel functions of first and second kind, respectively. This represents outgoing waves for the convention assumed: $e^{+j\omega t}$.

B. Viscoelastic continuum: Fractional approach

Up to this point we have considered a lossless medium without damping, which provides a useful approximation for lightly damped materials, such as metals. However, biological tissues show significant viscous damping behavior. The next step in our analysis of R–L waves is to introduce this viscous damping into the frequency equation. The terms ω/c_1 and ω/c_2 are defined for elastic materials, as compression and shear wave numbers, respectively, which are positive and real quantities. However, for viscoelastic materials, losses occur due to the viscous friction. Most of the time, these dissipative losses are related to the rate of deformation. For example, for a linear viscoelastic Voigt material model, the rate-dependent Lamé “constants” are expressible as $\lambda(t) = \lambda_0 + \lambda_1 \partial/\partial t$ and $\mu(t) = \mu_0 + \mu_1 \partial/\partial t$, where λ_0 , λ_1 , μ_0 , and μ_1 are coefficients of volume compressibility, volume viscosity, shear elasticity, and shear viscosity, respectively. This two element material model is a good benchmark example of shear viscoelastic material modeling. But, it has been observed in the literature^{21,22} that the Voigt model does not represent the dynamic shear behavior of the experimentally measured response to various excitations of biological and other media that have multi-scale and/or heterogeneous structure. It is possible to empirically increase the number of elements in a constitutive model for more accurate dynamic response; but, this leads to more complicated models, with higher numbers of elements connected in various ways, each of which must be experimentally determined. Alternatively, a single element, which exhibits rate-dependent shear properties can be employed. Such an element is known as a springpot, whose behavior lies somewhere between a Hookean solid and Newtonian fluid.³⁹ The shear modulus of a model consisting of a spring and a springpot in parallel is

$$\mu = \mu_0 + \mu_\alpha \frac{\partial^\alpha}{\partial t^\alpha}, \quad 0 < \alpha \leq 1. \quad (13)$$

Hereafter, this material model with the fractional derivative term will be called a fractional order Voigt model. A springpot element is the asymptotical result of a ladder-like fractal arrangement of integer-order elastic and viscous components, and such an arrangement is expected to represent the stress–strain behavior of complex multi-scale cellular and extracellular structure, such as is found in biological tissue.²⁹

The Weyl definition of fractional order derivatives is used in this paper. Consequently, for a harmonic function of $f(t) = e^{j\omega t}$, the fractional derivative has the property that $\partial^\alpha/\partial t^\alpha [e^{j\omega t}] = (j\omega)^\alpha e^{j\omega t}$. Real and imaginary parts of the complex shear modulus for a viscoelastic material model are defined as the storage and loss modulus, and for a fractional Voigt model these are given as

$$\mu_R = \mu_0 + \mu_\alpha \omega^\alpha \cos\left(\frac{\pi}{2}\alpha\right), \quad (14a)$$

$$\mu_I = \mu_\alpha \omega^\alpha \sin\left(\frac{\pi}{2}\alpha\right). \quad (14b)$$

In order to account for the viscoelasticity in the R–L frequency equation, terms ω^2/c_1^2 and ω^2/c_2^2 in Eqs. (7) and (8)

are replaced with squares of complex wave numbers k_1^2 and k_2^2 , where

$$k_1 = \omega \sqrt{\rho/(\lambda + \mu)}, \quad (15a)$$

$$k_2 = \omega \sqrt{\rho/\mu}. \quad (15b)$$

This substitution is appropriate regardless of the material model used, conventional (integer order) or fractional order Voigt and is also valid for elastic materials without damping.

III. SIMULATIONS

A. Solution of the frequency equation and wave numbers

Following the theory discussed above, the frequency equation for anti-symmetric R–L waves propagating on a thick, viscoelastic plate are simulated. First, the R–L wave number, ξ , for anti-symmetric waves is calculated using $F(\xi)$ as described in Sec. II. The absolute value of the function is plotted on the complex wave number domain at two different frequencies for visualizing the effects of the material properties.

Simulations are done at 50 and 450 Hz; an integer-order Voigt model is assumed and various combinations of μ_0 and μ_1 are simulated. Resulting wave number domain (k-space) plots are generated and analyzed in further detail for mode shapes.

The first set of frequency equation simulations is calculated at 50 Hz for μ_0 values of 4.5 and 9 kPa and μ_1 values of 4 and 8 Pa s (Fig. 4). Three complex solutions, which correspond to the first three modes of the equation, are observed for the softer material. These solutions move further away from the real axis with increased damping. Stiffer materials (9 kPa) yield wave numbers with smaller real parts, and the wave number for higher order modes in stiffer media becomes purely imaginary. Since the real part of the wave number is related to the phase speed and propagation, these higher order modes do not propagate.

FIG. 4. Contour plot of $|F(\xi)|$ on the complex wave number domain for $\omega/2\pi = 50$ Hz and different shear viscoelasticity values (Voigt model): (a) $\mu_0 = 4.5$ kPa, $\mu_1 = 4$ Pa s; (b) $\mu_0 = 4.5$ kPa, $\mu_1 = 8$ Pa s; (c) $\mu_0 = 9$ kPa, $\mu_1 = 4$ Pa s; (d) $\mu_0 = 9$ kPa, $\mu_1 = 8$ Pa s. Arrows point to solutions, minima of $|F(\xi)|$.

FIG. 5. Contour plot of $|F(\xi)|$ on the complex wave number domain for $\omega/2\pi = 450$ Hz and different shear viscoelasticity values (Voigt model): (a) $\mu_0 = 4.5$ kPa, $\mu_1 = 4$ Pa s; (b) $\mu_0 = 9$ kPa, $\mu_1 = 8$ Pa s. Arrows point from solutions, minima of $|F(\xi)|$, to the corresponding wave profile across the thickness. Key: real (—) and imaginary (---) parts of the vertical displacement u_z across the thickness of the plate.

A higher frequency (450 Hz) solution of the frequency equation with the same material properties yields more solutions than the lower frequency solution. It is possible to excite and observe higher order modes by increasing the frequency. These different mode shapes across the thickness are verified and visualized by plotting vertical displacement across the thickness of the plate using Eq. (12). The results are plotted in Fig. 5 for shear material constants of (a) $\mu_0 = 4.5$ kPa, $\mu_1 = 4$ Pa s and (b) $\mu_0 = 9$ kPa, $\mu_1 = 8$ Pa s. While the softer and less damped material allows five propagating modes, the stiffer and more damped material allows only three within the same wave number range. The relationship between plate thickness, frequency, and number of possible modes is given by Viktorov⁴⁰ as

$$Na = 1 + \left[\frac{2d}{\lambda_c} \right] + \left[\frac{2d}{\lambda_{sh}} + \frac{1}{2} \right]. \quad (16)$$

Here, λ_c and λ_{sh} are compression and shear wavelengths at a given frequency and brackets indicate the nearest integer number. Using this formulation the number of propagating modes can be calculated as 5 for the softer and 4 for the harder materials (the fourth mode for the harder case is not seen in the figure).

B. Dispersion analysis

The phase speed of a R–L wave carries essential information for solving for the wave numbers, which is necessary to calculate the stresses or displacements. Phase speed information would be enough to calculate the wave number for elastic media; however, for viscoelastic media information regarding viscous damping and attenuation is also necessary. When R–L waves are propagating in multiple modes all of these modes travel at their own phase speeds. Modes with different phase speeds separate from each other as they travel and the resulting wave front loses its sharpness. Dispersion curves, which

FIG. 6. Dispersion curves for the first five anti-symmetric modes (a_0 , a_1 , a_2 , a_3 , and a_4) of the viscoelastic plate. Key: —, fractional Voigt model; ---, integer Voigt model.

represent the phase speed variation of different modes as a function of frequency, are shown in Fig. 6. These curves are generated for integer and fractional order Voigt models.

C. Finite element analysis

A finite element model of the viscoelastic thick plate is built using a commercial finite element package (ANSYS, Canonsburg, PA) for simulating the harmonic response. The model is constructed using two-dimensional (2D), four-node, axisymmetric elements representing a half cross-section of the circular plate. The plate thickness is 33.5 mm, and unlike the formulation derived in Sec. II, its radial dimension is finite, 70 mm. The Young's modulus, Poisson's ratio, and structural damping are defined as frequency-dependent properties. These parameters are related to the Lamé constants as follows:

$$\nu = \lambda_0 / (2\lambda_0 + 2\mu_R), \quad (17)$$

$$E = 2(1 + \nu)\mu_R, \quad (18)$$

$$\beta^\xi = \frac{\mu_I}{\mu_R}. \quad (19)$$

The values for the Lamé constants are taken from our group's previous publication.¹ For volume elasticity, λ_0 , a value of 2.9 GPa is used; volume viscosity is assumed negligible in the frequency range of interest. Values for μ_R and μ_I/ω lie between 30 and 60 kPa, and between 5 and 23 Pa s, respectively, as shown in Fig. 7. Material density, ρ , has the value of 1100 kg/m³. In the finite element method the plate has free boundary conditions except for the excitation displacement condition at $r = 0$, which oscillates vertically at a constant frequency. The harmonic analysis is done for a range of frequencies between 200 and 800 Hz, in 10 Hz increments. Further analysis of finite element analysis results is conducted in MATLAB® (Mathworks, Natick, MA).

In order to be consistent with the experimental study, displacement data from the upper surface of the plate is analyzed. Transverse displacement magnitude and phase as a function of frequency referenced to the excitation point is recorded for 10 points that are spaced in 10 mm increments.

FIG. 7. Complex shear modulus $\mu = \mu_r + j\mu_i$ calculated using the R-L wave phase speed and attenuation data from FE simulation. (a) Real part and (b) imaginary part/ ω . Key: $\circ$ $\circ$ $\circ$, values used to build the FE model; —, values derived from FE simulations.

The phase speed of R-L waves as a function of frequency is calculated based on averaging phase differences between different points to obtain the least square error fit for the entire frequency span shown in Fig. 8(a). For an additional comparison the phase speed is also calculated from the theoretical derivations of Sec. II A. Theoretical phase speed calculations and finite element model phase speeds differ by less than 8%. This difference could be due to the fact that the theoretical calculations assume an infinitely large disk in the radial direction whereas the finite element model has a preset radial dimension.

Phase speed is related to the real part of the R-L wave number, $c_{ph} = \omega / \text{Re}(\xi)$; so, direct calculation of the real part is possible. The imaginary part of the R-L wave number affects attenuation; it is calculated by fitting a Hankel function to the attenuation data. A Hankel function with a complex argument is defined in Sec. II A. Attenuation from simulations and Hankel function fits are presented in Fig. 8(b).

R-L wave numbers are the solutions for Eq. (10). However, as discussed in Sec. III A, this equation has multiple solutions that correspond to different R-L modes. When complex R-L wave numbers are known, one can use these values to solve for unknown shear wave values from the R-L frequency equation. However, it is also important to know which R-L mode corresponds to a particular wave number. Finite element (FE) analysis is also used to identify the dominant mode of vibration for the frequency range of interest, which is found to be mostly the first and second modes of anti-symmetric R-L waves in this case. These findings differ from the results of Eq. (16), which predicts the possible number of modes as three up to 500 Hz, four between 500 and 700 Hz, and five above 700 Hz. However, Eq. (16) does not account for the excitation boundary condition, which drives purely the first mode R-L waves. Assuming only the first anti-symmetric mode of vibrations propagates on the plate, shear wave numbers (k_2) are solved using Eq. (10). Using the shear wave number definition

FIG. 8. R-L wave phase speed and attenuation versus frequency (Hz). (a) The solid line is the phase speed based on the FE simulation, and $\circ$ $\circ$ $\circ$ denotes the theoretical phase speed calculation. (b) Attenuation versus radial distance at 210 and 610 Hz. (Attenuation increases with increased frequency.) Solid lines are finite element analysis attenuation results and dashed lines are best fit Hankel functions with complex ξ arguments.

given in Eq. (15) the shear modulus can be calculated as $\mu = \left(\frac{\omega}{k_2}\right)^2 \rho$.

IV. EXPERIMENT

Rayleigh-Lamb wave experiments were conducted as depicted in Fig. 9 using a silicone polymer, CF-11 (NuSil Technology, Carpinteria, CA), which had a density of 1100 kg/m³ calculated through mass-volume measurements of small test specimens. While in liquid form, the material is poured into the container and then cured at room temperature. The cured specimen is 33.4 mm thick and has a 14 cm diameter. This disk is mounted on a large plexiglass ring in order to create top and bottom free surfaces. An acrylic needle, diameter 1.6 mm, driven by a mechanical shaker (Model F5B, Wilcoxon Research, Gaithersburg, MD) that is supported by a separate structure, is positioned at the center of the plexiglass

FIG. 9. Experimental schematic for laser Doppler vibrometer measurement of vertical surface motion of propagating R-L waves caused by a vertically oscillating rod.

ring and phantom. The contact between the needle and the polymer is ensured by friction. The shaker is driven with a signal from the built-in source of a scanning laser Doppler vibrometer system (PSV-400, Polytec, Irvine, CA). Force and acceleration of the needle are measured with an impedance head (Model Z11, Wilcoxon Research), and the out-of-plane velocity at discrete points on the surface is measured using the LDV system. The surface of the semi-translucent phantom material is colored with a silver marker to increase reflectivity and to aid in vibrometer measurements. The first measurement point is 5 mm apart from the center of the needle; data are taken from 10 consecutive points, which are 1.75 mm apart from each other. Measurement signals are recorded and the frequency response function (FRF) between the output (vertical velocity of the surface points) and input (motion input of the disk) are calculated using the LDV system. Further analysis of the data is conducted in MATLAB.

V. RESULTS AND DISCUSSION

A. Differences between and sources of error in theory, computation, and experiment

There are differences between the theoretical, computational (FE), and experimental systems. First of all, the theoretical treatment assumes a plate of infinite dimension; however, the viscoelastic plate that is used in the FE analysis and in the experiments has a finite diameter, resulting in reflections and standing wave patterns that can be significant, especially at lower frequencies; standing wave patterns were clearly evident in the FE simulations. In the FE analysis a free boundary at the outer edge of the disk is assumed. However, in the experimental setup the disk sits on a larger ring close to its outer edge. This boundary condition may interrupt the motion of the first anti-symmetric mode and transform it into other modes. Also, the excitation condition used in theoretical and FE simulations assumes welded (no-slip) contact between the rod and viscoelastic media, which facilitates the excitation of the first mode of R–L waves only. However, in the actual setup the contact is based on friction and may not be a uniform no-slip condition through the entire thickness of the plate. This may also result in waves other than the first mode. The presence of anti-symmetric R–L waves of higher order modes, which travel at faster phase speeds (see Fig. 6), can significantly alter estimations of shear viscoelasticity (increased storage and decreased loss moduli) that are based on the assumption that only the lowest order anti-symmetric mode is present.

B. Computational (finite element) results

Finite element-based results are presented, including an application of a material property calculation algorithm based on the observed wave speed and attenuation. Material properties for the FE simulation were frequency-dependent and defined at seven different frequencies; intermediate values between these frequencies were interpolated by the FE software. Amplitude and phase information of the complex frequency response data are used to calculate the surface wave speed and the rate of attenuation at different frequencies from the remaining points. The real part of the surface

FIG. 10. CF-11 complex shear modulus $\mu = \mu_R + j\mu_I$ as a function of frequency for anti-symmetric R–L wave study. (a) Real part and (b) imaginary part/ ω . Key: $\circ \circ \circ$, experimentally derived values; —, best fit Voigt model; — · —, best fit $\alpha = 0.6$ fractional order Voigt model; — — —, best fit SLS model.

wave number, $\text{Re}(\zeta)$, is obtained from the surface wave phase speed. Radial spreading and viscous effects result in wave amplitude attenuation, accounted for in a Hankel function with complex arguments. An estimate of complex valued ζ is then obtained via a least squared error fit to the FE-simulated or experimental data. This is then used to calculate the complex shear wave number, k_2 , which is directly related to the shear modulus. Thus, a calculation of the real and imaginary parts of complex shear modulus, μ_R and μ_I , is possible from k_2 . Results of this inversion method are shown in Fig. 7 for the FE-simulated data. Actual values and calculated values differ by less than 10%. This residual error is likely due to the difference of the phase speeds based on theory and FE simulations as discussed in Sec. III C.

C. Experimental results

Experimentally obtained phase speed and attenuation data are used, as described in the previous paragraph in regard to FE calculations, to obtain material properties based on R–L wave propagation. Results of this inversion are shown in Fig. 10. Three different viscoelastic material models of similar complexity were used to fit the experimentally obtained material properties. These models are the conventional Voigt model, the

TABLE I. CF-11 best fit viscoelastic constants for Rayleigh–Lamb wave experiments for Voigt, fractional order Voigt and SLS models. (For the fractional order Voigt and SLS models, μ_0 is the value obtained from the static indentation test. For the Voigt model, μ_0 is the average value of real part of complex μ in the frequency range of interest.)

	$\alpha = 1$	$\alpha = 0.6$	SLS, $\alpha = 1$
μ_0 (Pa)	75 500	21 000	21 000
μ_α (Pa s $^\alpha$)	13	786	43
μ_ω (Pa)	—	—	70 000

standard linear solid (SLS) model (also known as the Zener model¹), and the fractional order Voigt model. A value for shear elasticity μ_0 , which is a common parameter to all three models, was obtained from a static indentation test¹ and is provided in Table I. This value is used in the SLS and fractional order Voigt models for the static elasticity, and the remaining parameters are based on a best fit to the dynamic test data. The shear elasticity for the conventional Voigt model, μ_0 , which is a constant, is calculated differently by averaging the elasticity values (μ_R) over the frequency range of 100–1000 Hz. Similarly, the shear viscosity of the conventional Voigt model, μ_1 , also a constant, is calculated by determining the average value of the imaginary part of the shear modulus, μ_I , divided by frequency ω . For the SLS model, the two remaining parameters, μ_ω and μ_α are obtained by minimizing the least square error fit to the experimentally determined values of μ_R and μ_I over the frequency range of interest. The values of μ_ω and μ_α yielding the best overall fit are given in Table I. For the fractional order Voigt model, α is assumed to be equal to 0.6, which was shown to be the best fit for modeling the behavior of CF-11 in two other experimental configurations.¹ With α chosen, μ_α is optimized to minimize the least squares error fit to the experimentally determined real and imaginary parts of the shear modulus, μ_R and μ_I over the frequency range of interest. The resulting value of μ_α is given in Table I. Figure 10 shows the increasing trend of the storage modulus μ_R with frequency, which the conventional Voigt model fails to represent. Also plotted in Fig. 10 is the ratio of μ_I/ω , which decreases with frequency, though it should be a constant according to the conventional Voigt model.

The optimized SLS and fractional order Voigt models do have the frequency-dependent trends in storage and loss modulus that are observed experimentally, as shown in Fig. 10. It appears that the SLS model matches μ_I much more closely whereas the fractional order Voigt model is usually a better match to μ_R for the considered frequency range. An error function between the experimental data and fitted material model is defined as the squared difference between the complex material property values at the measured frequencies,

$$\text{Error} = \sum_{\omega} (\mu_{\text{exp}} - \mu_{\text{model}})^2, \quad (20)$$

where μ_{exp} are the experimental shear moduli values and μ_{model} are the values from the fitted material model. The error values are 285×10^9 for the SLS model and 199×10^9 for the fractional order Voigt model. Of course, it is possible to increase the number of parameters in any of these models in order to improve the fit to experiment. Note here that the fractional order Voigt model, with effectively only one unknown parameter optimized based on the R–L wave propagation data (as α was set equal to 0.6 based on other dynamic measurements), provided a slightly better fit than the SLS model, which had two unknown parameters optimized based on the same experimental data.

D. Comparisons with shear viscoelasticity estimates using other wave types

The elasticity and viscosity values calculated from the R–L wave experiments are compared with values obtained from our group's previous studies¹ of surface waves on the

FIG. 11. CF-11 complex shear modulus $\mu = \mu_R + j\mu_I$ as a function of frequency for different experimental studies. (a) Real part, (b) imaginary part/ ω , and (c) ratio of real part to imaginary part (storage modulus to loss modulus: μ_R/μ_I). Key: — + —, surface (Rayleigh) waves due to surface excitation; — × —, surface response due to a sub-surface acoustic source (Ref. 1); —*—, anti-symmetric R–L waves. (Ref. 1)

same media. Data from different experiments, presented in Fig. 11 (— + —: surface response due to a surface source, — × —: surface response due buried dipole source, and —*—: R–L waves), differ from each other, which may be attributable in part to unavoidable small variations in material properties for each sample of the phantom material. The experiments are consistent in that they all show a similar increasing trend for elasticity, μ_R , and a decreasing trend for the ratio of viscosity over frequency, μ_I/ω . However, note in Fig. 11(c), there are significant differences in the trend of the ratio of μ_R/μ_I with respect to frequency, which can also be used to determine the most appropriate model type. Asymptotically, as frequency increases this ratio will decrease for the integer Voigt model (inversely proportional to ω), approach a constant value, $\tan(\alpha\frac{\pi}{2})$ for the fractional order Voigt model, and increase for the SLS model (asymptotically proportional to ω). Referring to the figure, while for the surface wave and dipole experiments this ratio seems closer to a constant, for the R–L experiment the ratio seems more proportional to ω . One possible explanation for this is that, at the higher end of the frequency range shown in Fig. 11, higher order R–L waves are actually being excited, which will inflate the values of μ_R and decrease the values of μ_I that are calculated by assuming only the lowest order R–L wave

was present. In terms of the reliability of the experimental predictions, values calculated using the surface wave response to a surface source seem the most consistent ($- + -$), with less fluctuation as a function of frequency.

VI. CONCLUSION

Viscoelastic material properties were extracted from measurements of Lamb wave propagation and attenuation on a thick plate fabricated from a tissue mimicking phantom material. Like in prior studies of our group focused on other wave types that depend on shear viscoelastic properties, results reported here show that a simple “fractional order” Voigt model improves the match of theory to experiment over a wider frequency range when compared with the conventional “integer order” Voigt model and, to a lesser degree, when compared with the SLS model, which has the same number of independent parameters as the fractional order Voigt model. Due to the likely existence of higher order modes in the Lamb wave experiment that will skew shear viscoelasticity estimates, the authors have more confidence in the shear viscoelastic values estimated from another experimental configuration, that of the surface wave response due to a source on the surface.¹ Improved model accuracy may be possible by incorporating additional fractional elements; a more thorough investigation of the fundamentals governing viscoelastic behavior at multiple scales in such materials may serve to motivate and direct such an investigation in the future.

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APPENDIX

The following summarizes the solution of the Rayleigh–Lamb equation in various regions,

$$\text{Region I: } \xi^2 > \omega^2/c_2^2, \quad \omega^2/c_1^2. \quad (\text{A1})$$

Then $\bar{\alpha}$ and $\bar{\beta}$ are replaced by $j\bar{\alpha}'$ and $j\bar{\beta}'$, where

$$\bar{\alpha}'^2 = \xi^2 - \omega^2/c_1^2, \quad (\text{A2})$$

$$\bar{\beta}'^2 = \xi^2 - \omega^2/c_2^2. \quad (\text{A3})$$

The anti-symmetric R–L wave equation becomes

$$\frac{\tanh \bar{\beta}' b}{\tanh \bar{\alpha}' b} = \frac{(\xi^2 - \bar{\beta}'^2)^2}{4\bar{\alpha}' \bar{\beta}' \xi^2}. \quad (\text{A4})$$

$$\text{Region II, } \omega/c_2 > \xi > \omega/c_1. \quad (\text{A5})$$

This time only $\bar{\alpha}$ is replaced by $j\bar{\alpha}'$,

$$\bar{\alpha}'^2 = \xi^2 - \omega^2/c_1^2. \quad (\text{A6})$$

The wave equation becomes

$$\frac{\tanh \bar{\beta} b}{\tanh \bar{\alpha}' b} = \frac{(\xi^2 - \bar{\beta}^2)^2}{4\bar{\alpha}' \bar{\beta} \xi^2}. \quad (\text{A7})$$

$$\text{Region III, } \omega/c_1 > \xi. \quad (\text{A8})$$

The wave equation has the form

$$\frac{\tanh \bar{\beta} b}{\tanh \bar{\alpha} b} = \frac{(\xi^2 - \bar{\beta}^2)^2}{4\bar{\alpha} \bar{\beta} \xi^2}. \quad (\text{A9})$$

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